Internal Entanglement and External Correlations of Any Form Limit Each Other

S. Camalet
Laboratoire de Physique Théorique de la Matière Condensée, LPTMC, Sorbonne Université, CNRS, F-75005 Paris, France

(Received 8 April 2018; published 7 August 2018)

We show a relation between entanglement and correlations of any form. The internal entanglement of a bipartite system, and its correlations with another system, limit each other. A measure of correlations, of any nature, cannot increase under local operations. Examples are the entanglement monotones, the mutual information that quantifies total correlations, and the Henderson-Vedral measure of classical correlations. External correlations, evaluated by such a measure, set a tight upper bound on the internal entanglement that decreases as they increase, and so does quantum discord.

DOI: 10.1103/PhysRevLett.121.060504

Quantum entanglement is a useful resource for many tasks, such as cryptographic key distribution [1], state teleportation [2], or clock synchronization [3], to cite just a few. In more precise terms, it is a quantum resource that cannot be generated by local operations and classical communication [4–6]. The corresponding so-called free states, for which the resource vanishes, are the separable states, which are the mixtures of product states. Accordingly, entanglement is quantified by measures, termed entanglement monotones, which are non-negative functions of quantum states that vanish for separable states and are nonincreasing under state transformations involving only local operations and classical communication.

Two real systems, whose entanglement is of interest, are never completely isolated from the surroundings. Consequently, a third system, which cannot be fully controlled, always comes into play. Using Hamiltonian models describing the influence of more or less realistic environments, different dynamic behaviors of the entanglement have been found, depending, e.g., on whether the environment is in thermal equilibrium or not. For instance, an initial entanglement can vanish in finite time [7], or, on the contrary, entanglement can develop transiently [8,9] or even be steady [10–12].

The impact of the surroundings on entanglement can also be approached by studying how entanglement is distributed between three systems in an arbitrary state. The amounts of entanglement between one of them and each of the two other ones constrain each other. This behavior, known as entanglement monogamy, has first been shown for three two-level systems and expressed as an inequality involving a particular entanglement monotone [13]. This inequality does not hold, in general, for familiar monotones such as the entanglement of formation or the regularized relative entropy of entanglement. For these two measures, inequalities involving Hilbert space dimensions explicitly must be considered [14]. Relations have also been found between the amounts of entanglement for the three bipartitions of a tripartite system [15].

Recently, another restriction on the distribution of entanglement between three systems has been shown [16]. It is better understood by considering a finite-dimensional bipartite system, say A, and any other system, say B, which can be seen as the environment of A. It has been found that the internal entanglement, between the two subsystems of A, and the external entanglement, between A and B, limit each other. This relation is expressed by an inequality involving entanglement monotones and the Hilbert space dimensions of the subsystems of A. One may wonder whether this is a specific property of entanglement or whether a similar relation exists between internal entanglement and external correlations of any kind.

In this Letter, we address this issue by using measures of external correlations, which we term correlation monotones. Such a measure C is a non-negative function of the state ρ shared by A and B that vanishes for product states and is nonincreasing under local operations, which do not affect either A or B. These are basic requirements for a measure of correlations, since correlations, whatever their nature, cannot increase when A and B evolve independently. Our main result relies essentially on them. To be more specific, our derivation does not require that C is a strict correlation monotone, but only that it is invariant under unitary local operations and nonincreasing under operations performed on B. Examples of correlation monotones are the entanglement monotones, the mutual information, commonly used to quantify total correlations, and the Henderson-Vedral (HV) measure of classical correlations [17]. Quantum discords, on the other hand, are not correlation monotones [18]. However, the original quantum discord [19] as measured by A satisfies the above mentioned properties [20,21], and so our approach applies to it.

We show in the following that, for an arbitrary finite-dimensional bipartite system A and any system B, under an

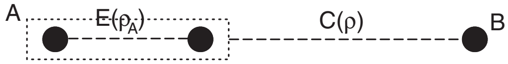

FIG. 1. Schematic representation of the systems and correlations considered.

assumption of continuity usually fulfilled, $C(\rho)$ and the internal entanglement of A are related to each other. More precisely, $C(\rho)$ determines a tight upper bound on $E(\rho_A)$, where ρ_A is the reduced density operator for A and E is any convex entanglement monotone that decreases as $C(\rho)$ increases (see Figs. 1 and 2). As we will see, for familiar correlation monotones, this bound vanishes when $C(\rho)$ equals its maximum value, set by the Hilbert space dimension of A. Moreover, since our result holds when C is the HV measure, it implies that, even when the external correlations are purely classical, they have a detrimental influence on internal entanglement.

In the following, $\boldsymbol{\lambda}(\omega)$ refers to the vector made up of the nonzero eigenvalues of the quantum state ω, in decreasing order. It is a probability vector; i.e., its components are positive and sum to unity. If ω is a density operator on the Hilbert space $\mathcal{H}_d$ of dimension d, $\boldsymbol{\lambda}(\omega)$ belongs to the set $\mathcal{E}_d$ of probability vectors of no more than d components. We call entropy any non-negative function of the probability vectors $\boldsymbol{p}$, which is nondecreasing with disorder, in the sense of majorization [22], and vanishes for $\boldsymbol{p} = 1$ [23]. Any entropy has a largest value on $\mathcal{E}_d$, reached for the equally distributed vector $(1/d, \ldots, 1/d)$, which is majorized by any $\boldsymbol{p} \in \mathcal{E}_d$ and possibly also for other vectors.

To derive our main result, we use the following three Lemmas. The proofs of the first and third are given in the Supplemental Material [24]. The second is proved in Ref. [16].

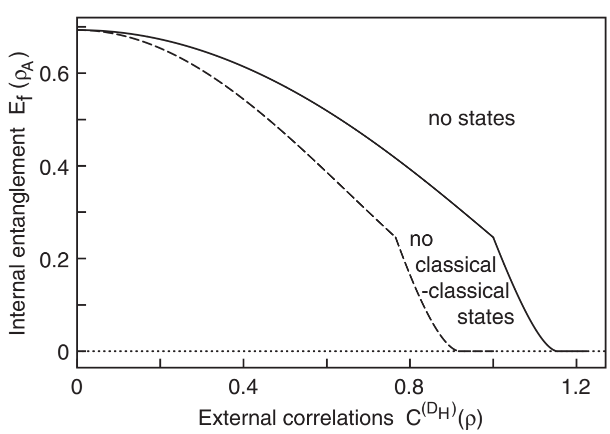

FIG. 2. Maximum internal entanglement as a function of external correlations, for a system A consisting of two two-level systems, the entanglement of formation E_f, and the measure of total correlations $C^{(D_H)}$ (solid line). The maximum entanglement $E_f(\rho_A)$ for classical-classical states ρ is given by the dashed line. This line is also the maximum value of $E_f(\rho_A)$ as a function of $C^{(D_B)}(\rho)$ for all states ρ.

Lemma 1.—For any correlation monotone C, there is a function f of the probability vectors with $f(1) = 0$, such that, for any global state ρ,

$$C(\rho) \leq f(\boldsymbol{\lambda}(\rho_A)), \tag{1}$$

with equality when ρ is pure.

We denote by c_d the supremum of f on $\mathcal{E}_d$. Because of Eq. (1), $C(\rho)$ cannot exceed c_d when the Hilbert space of A is $\mathcal{H}_d$. When C is an entanglement monotone, f is necessarily an entropy [30]. It is the Shannon entropy h for many familiar entanglement monotones and for the HV measure [6,17,31–33]. For robustness and negativity, f is a function of the Rényi entropy [34–37]. From the Araki-Lieb inequality $S(\rho) \geq |S(\rho_B) - S(\rho_A)|$, where S is the von Neumann entropy [38], it follows that $f = 2h$ for the mutual information $S(\rho_A) + S(\rho_B) - S(\rho)$. As mentioned in the Introduction, the quantum discord as measured by A, though not a correlation monotone, has the required properties to satisfy Lemma 1 (see the proof). The corresponding function f is h [21]. When f is an entropy, C coincides with an entanglement monotone for pure states [6,39]. For all the correlation monotones mentioned above, f equals c_d for $(1/d, \ldots, 1/d)$ and for no other vector of $\mathcal{E}_d$. This means that, on the set of the pure states $|\psi\rangle$ of $\mathcal{H}_d \otimes \mathcal{H}_{d'}$, where $d' \geq d$, the maximally entangled states are the only ones for which $C(|\psi\rangle\langle\psi|)$ is maximum.

Lemma 2.—For any convex entanglement monotone E, and integers $d_1 \geq 2$ and $d_2 \geq d_1$, there are a positive number e_{d_1} and an entropy s_{d_1,d_2} such that the states ρ_A on $\mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2}$ satisfy

$$E(\rho_A) \leq e_{d_1} - s_{d_1,d_2}(\boldsymbol{\lambda}(\rho_A)) \tag{2}$$

and such that, for any $\boldsymbol{p} \in \mathcal{E}_{d_1 \times d_2}$ and $\eta > 0$, there is ρ_A for which $\boldsymbol{\lambda}(\rho_A) = \boldsymbol{p}$ and $e_{d_1} - s_{d_1,d_2}(\boldsymbol{p}) - E(\rho_A) < \eta$.

This Lemma expresses quantitatively how the mixedness of a quantum state limits its amount of entanglement [40]. In Ref. [16], e_{d_1} is obtained as the largest value of $E(\rho_A)$ for pure states ρ_A. Thus, it depends only on d_1 [5]. Inequality (2) shows that it is the maximum of $E(\rho_A)$ on the set of all the density operators ρ_A on $\mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2}$. Contrary to e_{d_1}, the entropy s_{d_1,d_2} can depend on both d_1 and d_2 (see the Supplemental Material).

Lemma 3.—For any positive integer d, entropy s, and nonnegative continuous function f of the probability vectors with $f(1) = 0$, there is a nondecreasing function g_d on $I = [0, c_d]$, where c_d is the maximum of f on $\mathcal{E}_d$, such that $g_d(0) = 0$, $g_d \circ f \leq s$ on $\mathcal{E}_d$, and, for any $x \in I$ and $\eta > 0$, there is $\boldsymbol{p} \in \mathcal{E}_d$ for which $f(\boldsymbol{p}) = x$ and $s(\boldsymbol{p}) - g_d(x) < \eta$.

If $f(1/d, \ldots, 1/d) = c_d$ and $f(\boldsymbol{p}) < c_d$ for any other $\boldsymbol{p} \in \mathcal{E}_d$, then $g_d(c_d) = s(1/d, \ldots, 1/d)$.

Using this Lemma with the function f given by Lemma 1, and the entropy s_{d_1,d_2} given by Lemma 2, and defining

$\xi_{d_1,d_2} = e_{d_1} - g_d$, with $d = d_1 d_2$, we have the following result.

Theorem.—Let $\mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2}$, with $d_2 \geq d_1$, be the Hilbert space of system A, and $d = d_1 d_2$.

For a convex entanglement monotone E, and a correlation monotone C such that f is continuous, $C(\rho)$ and $E(\rho_A)$ obey, for any global state ρ,

$$E(\rho_A) \leq \xi_{d_1,d_2}(C(\rho)), \tag{3}$$

where ξ_{d_1,d_2} is a nonincreasing function on $[0, c_d]$ with $\xi_{d_1,d_2}(0) = e_{d_1}$. For any amount of correlations $x \leq c_d$, there are states ρ such that $C(\rho) = x$ and the two sides of inequality (3) are as close to each other as we wish.

If $f(1/d, \ldots, 1/d) = c_d$ and $f(\boldsymbol{p}) < c_d$ for any other $\boldsymbol{p} \in \mathcal{E}_d$, then $\xi_{d_1,d_2}(c_d) = 0$.

Inequality (3) can be rewritten, in a more familiar form, as $E(A_1 : A_2) \leq \xi_{d_1,d_2}[C(A_1 A_2 : B)]$, where A_1 and A_2 are the two subsystems of A (see Fig. 1) [16]. For any $x \in [0, c_d]$ and small η, Lemmas 2 and 3 ensure that there is a local state ρ_A such that $\xi_{d_1,d_2}(x) - E(\rho_A) < \eta$ and $f(\boldsymbol{\lambda}(\rho_A)) = x$. Because of Lemma 1, all the pure states ρ for which the reduced density operator for A is ρ_A are such that $C(\rho) = x$. For such global states ρ, $E(\rho_A) \simeq \xi_{d_1,d_2}(C(\rho))$, and an increase of the correlations between A an B means a reduction of the internal entanglement of A, and reciprocally. In general, the external correlations and the local entanglement limit each other (see Fig. 2). For any amount of correlations $x \leq c_d$, there is no state ρ such that $C(\rho) = x$ and $E(\rho_A)$ exceeds $\xi_{d_1,d_2}(x)$. Similarly, for any amount of entanglement $y \leq e_{d_1}$, there is no state ρ such that $E(\rho_A) = y$ and $C(\rho)$ is larger than the bound given by Eq. (3). On the contrary, there are no positive lower bounds for $E(\rho_A)$ for a given $C(\rho)$, and for $C(\rho)$, for a given $E(\rho_A)$, whatever are the monotones E and C [41].

For more than two systems, say $A, B_1, B_2, \ldots$, different bounds on the entanglement $E(\rho_A)$ can be obtained via Eq. (3), depending on which systems B_n are taken into account. Let us first observe that only the systems sharing a state with genuine multipartite correlations matter [42]. Indeed, if the global state is of the form $\rho = \tilde{\rho} \otimes \hat{\rho}$, where $\tilde{\rho}$ is the state of A and some systems B_n, and $\hat{\rho}$ is the state of the other systems, then $C(\rho) = C(\tilde{\rho})$, where C measures the amount of correlations between A and the considered systems B_n, since ρ and $\tilde{\rho}$ can be transformed into each other by local operations. For a global state ρ with genuine multipartite correlations, as tracing out a system B_n is a local operation and ξ_{d_1,d_2} is a nonincreasing function, the lowest bound on $E(\rho_A)$ is given by Eq. (3) with the state ρ of all the systems.

We now consider specific cases for which the boundary given by Eq. (3) can be determined explicitly. A measure of total correlations can be defined as a minimal distance to the set of product states, i.e., $C^{(D)}(\rho) = \inf_{\delta_A,\delta_B} D(\rho, \delta_A \otimes \delta_B)$, where the infimum is taken over all the density operators of A and B, and D fulfills $D[\Lambda(\omega), \Lambda(\omega')] \leq D(\omega, \omega')$ for any quantum operation Λ. Some possible choices for D are the relative entropy, the Bures distance D_B, or the Hellinger distance D_H [43–45]. For the relative entropy, the above definition gives the mutual information [44]. For the monotones $C^{(D_B)}$ and $C^{(D_H)}$, an explicit expression for f can be obtained (see the Supplemental Material). For the entanglement of formation E_f, the entropy $s_{2,2}$ is known [46]. Using these results, we find

$$\xi_{2,2}^{E_f,D_B}(x) = u\left(x^2 - \frac{x^4}{4}\right), \qquad \xi_{2,2}^{E_f,D_H}(x) = u\left(\frac{x^2}{2}\right), \tag{4}$$

where x varies from 0 to 1 for $C^{(D_B)}$ and from 0 to $\sqrt{3/2}$ for $C^{(D_H)}$. The expression of u is given in the Supplemental Material.

Figure 2 displays these two functions. They both vanish on a finite interval. As a consequence, for any state ρ such that $C(\rho)$ exceeds a threshold value, the local entanglement $E(\rho_A)$ necessarily vanishes, whereas for any amount of correlations x below this threshold, there are states ρ such that $C(\rho) = x$ and ρ_A is entangled. The existence of this threshold also implies that $C(\rho)$ is at a finite distance from the maximum value c_d as soon as $E(\rho_A)$ is not zero. As shown in the Supplemental Material, this feature is not specific to the particular cases considered above. Moreover, the threshold is the same for all the monotones E vanishing only for separable states.

As seen above, for some correlation monotones, $C(\rho) = c_d$ ensures the vanishing of $E(\rho_A)$. On the contrary, for any monotones C and E, and dimension d_1, there are states ρ for which $E(\rho_A) = e_{d_1}$ and $C(\rho)$ is as high as we wish, provided d_2 is large enough. They are pure states ρ such that the reduced density operator $\rho_A = \sum_i p_i |\phi_i\rangle\langle\phi_i|$ is a mixed maximally entangled state [47]. That is to say, the eigenvectors of ρ_A are of the form $|\phi_i\rangle = \sum_{j=1}^{d_1} |j\rangle_1 |ij\rangle_2 / \sqrt{d_1}$, where $|j\rangle_1$ are orthonormal states of $\mathcal{H}_{d_1}$, and $|ij\rangle_2$ of $\mathcal{H}_{d_2}$, i.e., ${}_2\langle ij|i'j'\rangle_2 = \delta_{i,i'}\delta_{j,j'}$. As ρ is pure, $C(\rho) = f(\boldsymbol{p})$, and, provided d_2/d_1 is large enough, there is $\boldsymbol{p}$ such that $C(\rho) \geq x$, where x is any amount of correlations. For any entanglement monotone E, $E(\rho_A) = E(|\phi_1\rangle\langle\phi_1|) = e_{d_1}$, since ρ_A and $|\phi_1\rangle\langle\phi_1|$ can be transformed into each other by local operations that do not affect one subsystem of A [48]. Note that, though $E(\rho_A) = e_{d_1}$ does not imply $C(\rho) = 0$ in general, this is true for the entanglement of formation E_f and $d_2 < 2d_1$, since, for such dimensions, the only states ρ_A for which $E_f(\rho_A)$ is maximum are pure [47].

The above Theorem applies to many kinds of external correlations, as discussed below Lemma 1. When C is an entanglement monotone, it generalizes previously obtained results [16]. As mentioned above, C can also be a measure of total correlations or the HV measure of classical correlations. For this last correlation monotone, Eq. (1) is an equality for some classical-classical states

$\rho = \sum_{i,j} p_{ij} |i\rangle_{AA}\langle i| \otimes |j\rangle_{BB}\langle j|$, where $|i\rangle_A$ and $|i\rangle_B$ are orthonormal states of A and B, respectively, and p_{ij} are probabilities summing to unity [49,50]. They are the strictly correlated classical-classical states, i.e., such that $p_{ij} = p_i \delta_{i,j}$ [17]. Consequently, there are not only pure states but also classical-classical states close to the boundary given by Eq. (3), for any amount of correlations. Moreover, since $\xi_{d_1,d_2}(c_d) = 0$, this shows that, even when external correlations are purely classical, the maximum accessible local entanglement decreases to zero as they increase.

In general, it can be proved that the classical-classical states ρ obey Eq. (1) with f replaced by a function $\tilde{f} \leq f$, such that $C(\rho) = \tilde{f}(\boldsymbol{\lambda}(\rho_A))$ when ρ is strictly correlated (see the Supplemental Material). Provided $\tilde{f}$ is continuous, it follows that, for a classical-classical state ρ, $E(\rho_A)$ and $C(\rho)$ satisfy Eq. (3) with ξ_{d_1,d_2} replaced by an *a priori* different function ζ_{d_1,d_2}. When C is an entanglement monotone, this is meaningless, since $C(\rho) = 0$ for all classical-classical states ρ. As seen above, for the HV measure, $\zeta_{d_1,d_2} = \xi_{d_1,d_2}$. For other correlation monotones, they obviously fulfill $\zeta_{d_1,d_2} \leq \xi_{d_1,d_2}$. For the measure of total correlations $C^{(D_H)}$ and the entanglement of formation E_f, we find $\zeta_{2,2}^{E_f,D_H} = \xi_{2,2}^{E_f,D_B}$ (see the Supplemental Material and Fig. 2). For the mutual information, $\tilde{f}$ is the Shannon entropy h. For this correlation monotone, inequality (1) with h in place of f, and hence $E(\rho_A) \leq \zeta_{d_1,d_2}(C(\rho))$, is actually valid for all separable states ρ, as $S(\rho_B) \leq S(\rho)$ for any separable state ρ [51], and since $f = 2\tilde{f} = 2h$, $\zeta_{d_1,d_2}(x) = \xi_{d_1,d_2}(2x)$, where $x \in [0, \ln d]$, for any entanglement monotone E.

We finally discuss the relations of other local properties to external correlations. A first natural question is whether E can be replaced by any correlation monotone in inequality (3). Lemma 2 is not specific to entanglement monotones. It only requires that E is convex [16]. Many familiar entanglement monotones are convex, though this is not a basic requirement for such a measure [6]. For other correlation monotones, imposing convexity can lead to some difficulties. A convex correlation monotone is necessarily zero for all separable states. The measures of total correlations considered above do not vanish for all separable states, by construction, and are hence not convex. Consequently, the above derivation of Eq. (3) does not apply if E is replaced by any one of these measures. Entanglement is not the only quantum resource for which there are measures that vanish only for free states and are convex. Other examples are the nonuniformity, which can be quantified by $\ln d - S(\rho_A)$ for a system A of Hilbert space dimension d [52], and the coherence, which can be quantified by $-\sum_i p_i \ln p_i - S(\rho_\mathrm{A})$, where $p_i = \langle i|\rho_\mathrm{A}|i\rangle$ and $\{|i\rangle\}_i$ is the basis with respect to which the incoherent states are defined [53]. In both these cases, inequality (3) is satisfied with the above corresponding measure in place of E, $\ln d - x$ in place of $\xi_{d_1,d_2}(x)$, and any correlation monotone C for which $f = h$ [16]. A relation of the form of Eq. (3) can also be obtained for contextuality quantifiers [30,54].

In summary, we have shown that internal entanglement and external correlations limit each other, whatever the nature of the correlations. For a given amount of external correlations $C(\rho)$, the internal entanglement $E(\rho_A)$ can approach but not exceed a value that decreases with increasing $C(\rho)$, and reciprocally. For familiar correlation monotones, $E(\rho_A)$ vanishes when the correlations are maximal. The entanglement can even be suppressed for lower values of $C(\rho)$. In two particular cases, we have determined explicitly the tight upper bound on $E(\rho_A)$ set by $C(\rho)$ and found that the entanglement vanishes when the amount of correlations is above a threshold value. Such a threshold also exists for other entanglement and correlation monotones. On the contrary, a maximum internal entanglement does not always ensure that the external correlations vanish, due to the existence of mixed maximally entangled states [47]. If E is the entanglement of formation, e.g., this is only true if none of the subsystems of A has a Hilbert space dimension larger, or equal, than twice that of the other one. As we have seen, the generalization of our result to other internal correlations is not obvious with the approach we have used. But it may be correct, and it would be of interest to determine whether this is indeed so.

[1] A. Ekert, Quantum Cryptography Based on Bell's Theorem, Phys. Rev. Lett. **67**, 661 (1991).

[2] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, Teleporting an Unknown Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels, Phys. Rev. Lett. **70**, 1895 (1993).

[3] R. Jozsa, D. S. Abrams, J. P. Dowling, and C. P. Williams, Quantum Clock Synchronization Based on Shared Prior Entanglement, Phys. Rev. Lett. **85**, 2010 (2000).

[4] G. Vidal, Entanglement monotones, J. Mod. Opt. **47**, 355 (2000).

[5] M. B. Plenio and S. Virmani, An introduction to entanglement measures, Quantum Inf. Comput. **7**, 1 (2007).

[6] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, Rev. Mod. Phys. **81**, 865 (2009).

[7] T. Yu and J. H. Eberly, Finite-Time Disentanglement via Spontaneous Emission, Phys. Rev. Lett. **93**, 140404 (2004).

[8] D. Braun, Creation of Entanglement by Interaction with a Common Heat Bath, Phys. Rev. Lett. **89**, 277901 (2002).

[9] Z. Ficek and R. Tanaś, Dark periods and revivals of entanglement in a two-qubit system, Phys. Rev. A **74**, 024304 (2006).

[10] O. Çakir, A. A. Klyachko, and A. S. Shumovsky, Steady-state entanglement of two atoms created by classical driving field, Phys. Rev. A **71**, 034303 (2005).

[11] S. Camalet, Non-equilibrium entangled steady state of two independent two-level systems, Eur. Phys. J. B **84**, 467 (2011).

[12] S. Camalet, Entanglement on macroscopic scales in a resonant-laser-field-excited atomic ensemble, Phys. Rev. A **91**, 032312 (2015).

[13] V. Coffman, J. Kundu, and W. K. Wootters, Distributed entanglement, Phys. Rev. A **61**, 052306 (2000).

[14] C. Lancien, S. Di Martino, M. Huber, M. Piani, G. Adesso, and A. Winter, Should Entanglement Measures be Monogamous or Faithful?, Phys. Rev. Lett. **117**, 060501 (2016).

[15] X.-N. Zhu and S.-M. Fei, Generalized monogamy relations of concurrence for N-qubit systems, Phys. Rev. A **92**, 062345 (2015).

[16] S. Camalet, Monogamy Inequality for Any Local Quantum Resource and Entanglement, Phys. Rev. Lett. **119**, 110503 (2017).

[17] L. Henderson and V. Vedral, Classical, quantum and total correlations, J. Phys. A **34**, 6899 (2001).

[18] B. Dakić, V. Vedral, and Č. Brukner, Necessary and Sufficient Condition for Nonzero Quantum Discord, Phys. Rev. Lett. **105**, 190502 (2010).

[19] H. Ollivier and W. H. Zurek, Quantum Discord: A Measure of the Quantumness of Correlations, Phys. Rev. Lett. **88**, 017901 (2001).

[20] A. Streltsov, H. Kampermann, and D. Bruß, Linking Quantum Discord to Entanglement in a Measurement, Phys. Rev. Lett. **106**, 160401 (2011).

[21] K. Modi, A. Brodutch, H. Cable, T. Paterek, and V. Vedral, The classical-quantum boundary for correlations: Discord and related measures, Rev. Mod. Phys. **84**, 1655 (2012).

[22] A. W. Marshall, I. Olkin, and B. C. Arnold, *Inequalities: Theory of Majorization and its Applications*, Springer Series in Statistics 2nd ed. (Springer, New York, 2011).

[23] A. I. Khinchin, *Mathematical Foundations of Information Theory* (Dover, New York, 1957).

[24] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.121.060504, which includes Refs. [25–29], for the proofs.

[25] N. Johnston, Separability from spectrum for qubit-qudit states, Phys. Rev. A **88**, 062330 (2013).

[26] N. Canosa, R. Rossignoli, and M. Portesi, Majorization properties of generalized thermal distributions, Physica (Amsterdam) **368A**, 435 (2006).

[27] G. L. Gilardoni, On Pinsker's and Vajda's type inequalities for Csiszár's f-divergences, IEEE Trans. Inf. Theory **56**, 5377 (2010).

[28] T. van Erven and P. Harremoës, Rényi divergence and kullback-leibler divergence, IEEE Trans. Inf. Theory **60**, 3797 (2014).

[29] L. Gurvits and H. Barnum, Largest separable balls around the maximally mixed bipartite quantum state, Phys. Rev. A **66**, 062311 (2002).

[30] S. Camalet, Monogamy inequality for entanglement and local contextuality, Phys. Rev. A **95**, 062329 (2017).

[31] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Concentrating partial entanglement by local operations, Phys. Rev. A **53**, 2046 (1996).

[32] V. Vedral and M. B. Plenio, Entanglement measures and purification procedures, Phys. Rev. A **57**, 1619 (1998).

[33] L. Zhang, J. Wu, and S.-M. Fei, Universal upper bound for the Holevo information induced by a quantum operation, Phys. Lett. A **376**, 3588 (2012).

[34] G. Vidal and R. Tarrach, Robustness of entanglement, Phys. Rev. A **59**, 141 (1999).

[35] G. Vidal and R. F. Werner, Computable measure of entanglement, Phys. Rev. A **65**, 032314 (2002).

[36] A. W. Harrow and M. A. Nielsen, How robust is a quantum gate in the presence of noise?, Phys. Rev. A **68**, 012308 (2003).

[37] M. Steiner, Generalized robustness of entanglement, Phys. Rev. A **67**, 054305 (2003).

[38] H. Araki and E. H. Lieb, Entropy inequalities, Commun. Math. Phys. **18**, 160 (1970).

[39] M. A. Nielsen, Conditions for a Class of Entanglement Transformations, Phys. Rev. Lett. **83**, 436 (1999).

[40] K. Życzkowski, P. Horodecki, A. Sanpera, and M. Lewenstein, Volume of the set of separable states, Phys. Rev. A **58**, 883 (1998).

[41] For any ρ, $C(\tilde{\rho}) = C(\rho)$ for all $\tilde{\rho} = U\rho U^\dagger$, where U is any unitary operator of A, whereas there is such a U changing ρ_A into the separable state $\tilde{\rho}_A = \sum_i \lambda_i(\rho_A)|i\rangle\langle i|$, where $|i\rangle$ are product states of $\mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2}$, for which $E(\tilde{\rho}_A) = 0$. For any ρ_A, $C(\rho) = 0$ for all $\rho = \rho_A \otimes \rho_B$.

[42] C. H. Bennett, A. Grudka, M. Horodecki, P. Horodecki, and R. Horodecki, Postulates for measures of genuine multipartite correlations, Phys. Rev. A **83**, 012312 (2011).

[43] K. M. R. Audenaert, J. Calsamiglia, R. Munoz-Tapia, E. Bagan, L. Masanes, A. Acin, and F. Verstraete, Discriminating States: The Quantum Chernoff Bound, Phys. Rev. Lett. **98**, 160501 (2007).

[44] K. Modi, T. Paterek, W. Son, V. Vedral, and M. Williamson, Unified View of Quantum and Classical Correlations, Phys. Rev. Lett. **104**, 080501 (2010).

[45] T. R. Bromley, M. Cianciaruso, R. Lo Franco, and G. Adesso, Unifying approach to the quantification of bipartite correlations by Bures distance, J. Phys. A **47**, 405302 (2014).

[46] F. Verstraete, K. Audenaert, and B. De Moor, Maximally entangled mixed states of two qubits, Phys. Rev. A **64**, 012316 (2001).

[47] Z.-G. Li, M.-J. Zhao, S.-M. Fei, H. Fan, and W. M. Liu, Mixed maximally entangled states, Quantum Inf. Comput. **12**, 0063 (2012).

[48] The operation with Kraus operators $\sqrt{p_i} I \otimes U_i$, where I is the identity operator on $\mathcal{H}_{d_1}$, and U_i is a unitary operator on $\mathcal{H}_{d_2}$ such that $|ij\rangle_2 = U_i|1j\rangle_2$, transforms $|\phi_1\rangle\langle\phi_1|$ into ρ_A. A basis of $\mathcal{H}_{d_2}$ can always be formed with the states $|ij\rangle_2$ and, if necessary, other ones, denoted $|l\rangle_2$. The operation with Kraus operators $I \otimes \sum_{j=1}^{d_1} |1j\rangle_{22}\langle ij|$ and, if necessary, $I \otimes |l\rangle_{22}\langle l|$, changes ρ_A into $|\phi_1\rangle\langle\phi_1|$.

[49] J. Oppenheim, M. Horodecki, P. Horodecki, and R. Horodecki, Thermodynamical Approach to Quantifying Quantum Correlations, Phys. Rev. Lett. **89**, 180402 (2002).

[50] M. Piani, P. Horodecki, and R. Horodecki, No-Local-Broadcasting Theorem for Multipartite Quantum Correlations, Phys. Rev. Lett. **100**, 090502 (2008).

[51] R. Horodecki and M. Horodecki, Information-theoretic aspects of inseparability of mixed states, Phys. Rev. A **54**, 1838 (1996).

[52] G. Gour, M. P. Müller, V. Narasimhachar, R. W. Spekkens, and N. Y. Halpern, The resource theory of informational nonequilibrium in thermodynamics, Phys. Rep. **583**, 1 (2015).

[53] T. Baumgratz, M. Cramer, and M. B. Plenio, Quantifying Coherence, Phys. Rev. Lett. **113**, 140401 (2014).

[54] S. Camalet, Simple state preparation for contextuality tests with few observables, Phys. Rev. A **94**, 022106 (2016).