Coulomb-Gas Electrostatics Controls Large Fluctuations of the Kardar-Parisi-Zhang Equation

Ivan Corwin,[1] Promit Ghosal,[2] Alexandre Krajenbrink,[3] Pierre Le Doussal,[3] and Li-Cheng Tsai[1]
[1]*Columbia University, Department of Mathematics 2990 Broadway, New York, New York 10027, USA*
[2]*Columbia University, Department of Statistics 1255 Amsterdam, New York, New York 10027, USA*
[3]*CNRS—Laboratoire de Physique Théorique de l'Ecole Normale Supérieure, 24 rue Lhomond, 75231 Paris Cedex, France*

(Received 27 March 2018; published 8 August 2018)

We establish a large deviation principle for the Kardar-Parisi-Zhang (KPZ) equation, providing precise control over the left tail of the height distribution for narrow wedge initial condition. Our analysis exploits an exact connection between the KPZ one-point distribution and the Airy point process—an infinite particle Coulomb gas that arises at the spectral edge in random matrix theory. We develop the large deviation principle for the Airy point process and use it to compute, in a straightforward and assumption-free manner, the KPZ large deviation rate function in terms of an electrostatic problem (whose solution we evaluate). This method also applies to the half-space KPZ equation, showing that its rate function is half of the full-space rate function. In addition to these long-time estimates, we provide rigorous proof of finite-time tail bounds on the KPZ distribution, which demonstrate a crossover between exponential decay with exponent 3 (in the shallow left tail) to exponent $5/2$ (in the deep left tail). The full-space KPZ rate function agrees with the one computed in Sasorov *et al.* [J. Stat. Mech. (2017) 063203] via a WKB approximation analysis of a nonlocal, nonlinear integrodifferential equation generalizing Painlevé II which Amir *et al.* [Commun. Pure Appl. Math. **64**, 466 (2011)] related to the KPZ one-point distribution.

DOI: 10.1103/PhysRevLett.121.060201

Since its birth in 1986, the Kardar-Parisi-Zhang (KPZ) equation [1] has been applied to describe growth of interfaces [2], transport in one-dimension (1D) and Burgers turbulence [3], directed polymers [4], chemical reaction fronts [5], bacterial growth [6], slow combustion [7], coffee stains [8], conductance fluctuations in Anderson localization [9], polar active fluids [10], Bose Einstein superfluids [11], and quantum entanglement growth [12].

Whereas some stochastic models (e.g., exclusion processes [13], random permutations [14], random walks in random media [15]) are directly related (via mappings to "height functions") to the universality class for the 1D KPZ equation; others—namely, random matrix theory (RMT)—rely on hidden connections to KPZ, which are only seen from exact solutions to both KPZ and RMT models [16]. In this Letter, we describe such a relationship between the KPZ equation and the Airy point process—an infinite particle Coulomb gas [17] that arises at the spectral edge in random matrix theory—and exploit variational techniques of electrostatics to precisely quantify the large fluctuations for the KPZ equation.

The 1D KPZ equation describes the stochastic growth of an interface of height $h(t,x)$ at $x \in \mathbb{R}$ and time $t > 0$

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \xi(t,x), \qquad (1)$$

in convenient units, starting from an initial condition $h(t=0,x)$. Here $\xi(x,t)$ is a centered Gaussian white noise with $\overline{\xi(t,x)\xi(t',x')} = 2\delta(x-x')\delta(t-t')$ and $\overline{\cdots}$ denotes expectations with respect to this noise. Typically, the fluctuations of the height field scale, at large time, like $t^{1/3}$. Recent progress has yielded exact solutions for the probability density function (PDF) of the height at a given space point at arbitrary time when starting from special initial conditions (e.g., droplet, flat, stationary) [18–20]. Focusing here and below on the droplet (also known as the narrow wedge) initial condition, $h(0,x) = -(|x|/\delta) - \ln(2\delta)$ for $\delta \ll 1$, the exact formula for the PDF is expressed in terms of a Fredholm determinant. Using this, the scaled and centered height $\mathcal{H}(t)/t^{1/3}$, where $\mathcal{H}(t) = h(t,0) + (t/12)$, was shown to converge in law as $t \to +\infty$ to the Tracy-Widom GUE distribution, which also describes the fluctuations of the largest eigenvalue, $\lambda_{\max}$, of a large random matrix from the Gaussian unitary ensemble (GUE).

Despite considerable interest, much less is known about large deviations and tails of the KPZ field or PDF $P(H,t) = (\partial/\partial H)\mathbb{P}(\mathcal{H}(t) \le H)$. For general nonequilibrium systems, large deviation rate functions play a role similar to the free energy or entropy in equilibrium systems (see Ref. [21] and references therein). Existing large deviation theories fail to apply in the KPZ growth setting. The macroscopic fluctuation theory [22] requires local thermodynamic equilibrium, not realized here. The weak noise theory (see, e.g., Ref. [23]) applies, but only at very short times. Understanding the large deviations for the KPZ equation poses an important conceptual challenge.

Quantitative control over the tails of the KPZ equation plays an important role in experimental and numerical works. Precise results can be used, e.g., as benchmarks for broadly applicable numerical Monte Carlo methods such as those used in Ref. [24]. In experimental work (such as reviewed in Ref. [25]), the tail behavior we are probing corresponds to excess growth. While unlikely at a single point, if the growing substrate is sufficiently long, disparate regions (spaced as time$^{2/3}$) will see roughly independent growth. Hence, by standard extreme-value theory, the maximal and minimal height of the entire substrate will be determined by the one-point tail behaviors. The KPZ equation also models semiconductor film growth [26]. In technological applications, the roughness of these films determines device performance. As many films are grown independently, large deviations dictate failure rates.

In population growth and mass transport models, the KPZ tails play contribute to multifractal intermittency [27]. The $H/t^{1/3} \gg 1$ tail is associated with excess mass growth which comes from locally favorable effects; in contrast, the $-H/t^{1/3} \gg 1$ tail is associated with mass die-out, which arises from collective effects of wide-spread unfavorable growth regions. Because of this collective effect, the left tail is intrinsically more difficult to analyze at large time. A similar situation arises in RMT for the tails of the PDF of $\lambda_{\max}$: while positive fluctuations arise from the largest eigenvalue $\lambda_{\max}$ simply detaching from the bulk of the spectrum, negative ones require a reorganization of the entire Wigner semicircle density of eigenvalues (the pushed Coulomb gas) [28]. This analogy leads to the prediction [29] that for $t \gg 1$ and large fluctuations $|H| \sim t$ the right tail ($H \gg 0$) scales as $-\ln P(H,t) \sim t$ while the left tail ($H \ll 0$) scales as $-\ln P(H,t) \sim t^2$.

For short times $t \ll 1$ the left tail of the PDF ($H \ll 0$) behaves as $P(H,t) \sim \exp[-(4/15\pi)|H|^{5/2}/t^{1/2}]$, as was shown analytically (via weak noise theory and exact solutions) [30,31] and numerically [24] (see also Refs. [23,32,33] for other initial conditions). Extracting this tail in the intermediate or large time limit is much harder. For $t \gg 1$, in the typical scaling region $H \sim t^{1/3}$, the left tail should behave like the Tracy-Widom GUE distribution, i.e., $P(H,t) \sim_{H\to-\infty} \exp(-\frac{1}{12}|H|^3/t)$. Until recently, nothing was known about how far this cubic exponent persists into the very far left tail region $|H| \sim t^{\alpha}$ with $\alpha > 1/3$, or whether it holds for intermediate times.

Given the similarities between the KPZ and RMT problems, it is natural to try to attack these tail questions using methods inspired by RMT. The left tail behavior for $\lambda_{\max}$ can be accessed by either (i) the Coulomb-gas and associated electrostatic variational problem for the GUE spectrum [28,34] (see also Ref. [35] for other large deviation applications of the Coulomb gas) or (ii) the relationship between gap probabilities and certain classical integrable systems [36] (which, in $N \to \infty$ edge limit, relate to the Painlevé II equation [37]). Reference [19] introduced a nonlocal, nonlinear integrodifferential equation which generalizes Painlevé II by including a "Fermi factor," and showed that its solution relates to the KPZ PDF. Studying this generalized equation via standard "integrable-integral operator" methods [38] involves an infinite-dimensional Riemann-Hilbert problem steepest descent analysis which is beyond current techniques. Employing a certain approximation ansatz [29] attempted to analyze this equation. While they successfully predicted the scaling form for the large deviation tail $P(H,t) \sim \exp[-t^2\Phi_-(H/t)]$ for $-H \sim t \gg 1$, the approximations were too reductive and Ref. [29] predicted $\Phi_-(z) = \frac{1}{12}|z|^3$, which turns out only to hold true for z near 0. Reference [39] revisited this analysis and employed a WKB approximation along with a "self-consistency" ansatz for the form of the solution to a Schrödinger equation in which the potential depends upon the solution. Given these assumptions, Ref. [39] extracted a formula

$$\Phi_-(z) = \frac{4}{15\pi^6}(1-\pi^2 z)^{5/2} - \frac{4}{15\pi^6} + \frac{2}{3\pi^4}z - \frac{1}{2\pi^2}z^2, \quad (2)$$

which predicts a crossover between $\Phi_-(z) \simeq_{z\to-\infty} (4/15\pi)|z|^{5/2}$ and $\simeq_{z\to-0} \frac{1}{12}|z|^3$. This, taken with the short-time estimates, suggests that the $|H|^{5/2}$ tail remains valid at all times (see also Ref. [24] and Refs. [23,30,32]) and that there is a crossover between the $\frac{1}{12}|H|^3/t$ and $(4/15\pi)|H|^{5/2}/t^{1/2}$ tail when $|H| \approx t$ (once $t \gg 1$).

The purpose of this Letter is to demonstrate how the Coulomb gas can be utilized in a straightforward and assumption-free manner to (i) establish, using the large deviations for the Airy point process, an electrostatic variational formula for $\Phi_-(z)$ whose solution (which we derive) agrees with Eq. (2), and (ii) demonstrate the first precise tail bounds (13) that are valid for all intermediate and long times and which capture the crossover between the $\frac{1}{12}|H|^3/t$ tail for $|H| \ll t$ and the $(4/15\pi)|H|^{5/2}/t^{1/2}$ tail for $|H| \gg t$. Our work provides a description of the intermediate and late time left large deviations for the KPZ equation where the connection to RMT and the role of the collective effects is explicit: each fixed value of $z = H/t$ corresponds to an optimal eigenvalue density (see Fig. 1). Finally, we extend our study to the half-line KPZ equation in the critical case, which relates to the Gaussian orthogonal ensemble (GOE), leading (via our RMT approach) to the rate function $\Phi_-^{\text{half-space}}(z) = \frac{1}{2}\Phi_-^{\text{full-space}}(z)$.

Our starting point is a remarkable identity [41,42], obtained from the exact solution of the droplet initial condition [18,19], which directly connects KPZ and RMT (as well as fermions in a harmonic well at temperature of order $t^{-1/3}$ [43]): for $\varphi_{t,s}(a) = \log(1 + e^{t^{\frac{1}{3}}(a+s)})$,

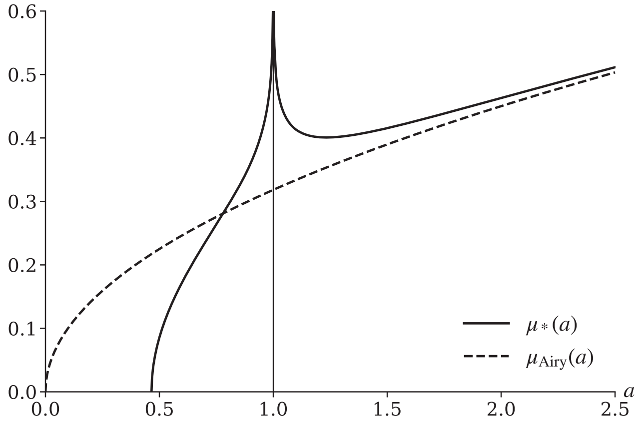

FIG. 1. Optimal density $\mu_*(a)$ at $z = -1$ compared to $\mu_{\mathrm{Airy}}(a)$. The density μ_* has a log singularity at $a = -z$ (cf. Ref. [40] for another Coulomb gas problem with similar behavior).

$$\overline{\exp(-e^{\mathcal{H}(t)+st^{1/3}})} = \mathbb{E}_{\mathrm{Airy}}\left[\exp\left(-\sum_{i=1}^{\infty}\varphi_{t,s}(\mathbf{a}_i)\right)\right]. \quad (3)$$

The left-hand side (l.h.s.) is an expectation over the KPZ white noise giving access to $P(H,t)$ while the right-hand side (r.h.s.) is the expectation of a Fermi factor over the Airy point process (Airy PP) generating the set $\{\mathbf{a}_i\} \in \mathbb{R}$. The Airy PP describes the largest few eigenvalues of a large GUE matrix. It is a "determinantal" measure on infinite point configurations $\mathbf{a} = (\mathbf{a}_1 > \mathbf{a}_2 > \cdots)$ on $\mathbb{R}$ which means that for all $k \geq 1$, the kth correlation function $\rho_k(x_1,\ldots,x_k)$ (which equals the probability density for the event that $\{x_i \in \mathbf{a}, \text{ for all } 1 \leq i \leq k\}$) takes the form $\rho_k(x_1,\ldots,x_k) = \det[K(x_i,x_j)]_{1\leq i,j\leq k}$ for some fixed "correlation kernel" $K\colon\mathbb{R}^2 \to \mathbb{C}$. The Airy PP correlation kernel is $K_{\mathbf{Ai}}(x,y) = \int_0^\infty \mathbf{Ai}(x+r)\mathbf{Ai}(y+r)dr$. In particular, the mean density is $\rho(a) = \rho_1(a) = K_{\mathbf{Ai}}(a,a) \simeq_{a\to-\infty} \pi^{-1}\sqrt{|a|}$. This agrees with the square-root behavior of the Wigner semicircle at the edge. Remarkably, the $|H|^{5/2}$ tail emerges quite simply from this $\sqrt{|a|}$ density as we show from the first term in the cumulant expansion of the right-hand side of Eq. (3); see Eq. (6). After observing this, we describe the Airy PP large deviation principle (LDP) derived via Coulomb gas, and use it to compute the full crossover rate function $\Phi_-(z)$. Finally, we provide the bounds (13) which describes intermediate time behavior of the tail.

Cumulant expansion.—As $st^{\frac{1}{3}} \to \infty$ the l.h.s. of Eq. (3) approaches $\mathbb{P}[\mathcal{H}(t) \leq -st^{\frac{1}{3}}] = \mathbb{P}[\mathcal{H}(t) \leq zt]$ with $z = -st^{-2/3}$. The r.h.s. of Eq. (3) is evaluated via cumulants as

$$\log[\text{r.h.s.}(3)] = \sum_{n=1}^{\infty}\frac{\kappa_n}{n!}, \quad (4)$$

where κ_n is the nth cumulant of the Airy PP whose general form is known [44], e.g., for $n = 1, 2$,

$$\kappa_1 = -\mathrm{Tr}(\varphi_{t,s}K_{\mathbf{Ai}}) = -\int_{-\infty}^{+\infty} da\varphi_{t,s}(a)\rho(a). \quad (5)$$

and $\kappa_2 = \mathrm{Tr}(\varphi_{t,s}^2 K_{\mathbf{Ai}}) - \mathrm{Tr}(\varphi_{t,s}K_{\mathbf{Ai}}\varphi_{t,s}K_{\mathbf{Ai}})$, where $(\psi K)(x,y) = \psi(x)K(x,y)$, $\mathrm{Tr}K = \int_{\mathbb{R}} daK(a,a)$. In the limit $z \to -\infty$, it is sufficient to keep only the first cumulant (the $n = 1$ term) in Eq. (4), which, using the above asymptotics $\rho(a) \simeq_{a\to-\infty} \pi^{-1}\sqrt{|a|}$, is estimated as (we use the notation $(\cdot)_+ = \max(\cdot, 0)$ below)

$$\kappa_1 \simeq -t^{1/3}\int_{-\infty}^{+\infty} da(a+s)_+\rho(a)$$
$$\simeq -t^{1/3}\frac{4}{15\pi}s^{5/2} = -t^2\frac{4}{15\pi}|z|^{5/2}. \quad (6)$$

This simple argument gives the leading behavior as $z \to -\infty$ of the left large deviation rate function, $\Phi_-(z) \simeq (4/15\pi)|z|^{5/2}$, hence, the desired $|H|^{5/2}$ tail. Explicit calculation (see Ref. [45]) of the next higher cumulants,

$$\kappa_2 \simeq t^{2/3}\frac{s^2}{\pi^2} = t^2\frac{z^2}{\pi^2}, \quad \kappa_3 \simeq -t\frac{4|s|^{3/2}}{\pi^3} = -t^2\frac{4|z|^{3/2}}{\pi^3}, \quad (7)$$

shows their subdominance both (i) for $-z \gg 1$ with $t \gg 1$ and $z = H/t$ fixed and (ii) t fixed and large $s = -H/t^{1/3}$ and reproduces the large $|z|$ expansion of Eq. (2).

Coulomb-gas and large deviation rate function.—Using Eq. (3), $\Phi_-(z)$ can be computed as (write $\mathbb{E}$ for $\mathbb{E}_{\mathrm{Airy}}$)

$$\Phi_-(z) = \lim_{t\to\infty}\frac{1}{t^2}\log\mathbb{E}\left[\exp\left(-\sum_{i=1}^{\infty}\varphi_{t,-zt^{2/3}}(\mathbf{a}_i)\right)\right].$$

For large t, we have $\varphi_{t,-zt^{2/3}}(t^{2/3}a) \approx t(a-z)_+$. Let $\mu_t(a)da = t^{-1}\sum_{i\geq 1}\delta_{-t^{-2/3}\mathbf{a}_i}(a)da$ denote the *scaled*, space-reversed Airy PP empirical measure. Then we have

$$\Phi_-(z) = \lim_{t\to\infty}\frac{1}{t^2}\log\mathbb{E}\left[\exp\left(-t^2\int_{\mathbb{R}} da\mu_t(a)(-z-a)_+\right)\right]. \quad (8)$$

Like the GUE, the Airy PP should enjoy an LDP so that for a suitable class of functions μ, $\mathbb{P}(\mu_t \approx \mu) \approx \exp[-t^2 I_{\mathrm{Airy}}(\mu)]$. To our knowledge, this rate function is not in the literature, and we describe it below and in Ref. [46]. Given this, the r.h.s. of Eq. (8) can be evaluated via a variational problem, $\Phi_-(z) = \min_\mu \Sigma(\mu)$, with cost function

$$\Sigma(\mu) = \int_{\mathbb{R}} da\mu(a)(-z-a)_+ + I_{\mathrm{Airy}}(\mu). \quad (9)$$

To derive the LDP for the Airy PP we will appeal to the fact that the Airy PP arises as an edge limit of the GUE. The GUE spectrum is a 1D Coulomb gas with logarithmic interaction which immediately leads to an electrostatic variational formulation for the GUE LDP [17,34] (with the Wigner semicircle representing the minimizer of this electrostatic energy). Our approach is to rewrite the GUE LDP in such a manner that it admits an edge scaling limit to yield the Airy PP LDP.

Recall from Ref. [34] that the empirical measure $\Lambda_N(\lambda)d\lambda = (1/N)\sum_{i=1}^N \delta_{\boldsymbol{\lambda}_i}(\lambda)d\lambda$ associated to the eigenvalues $\{\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_N\}$ of the GUE (normalized to have typical support $[-2,2]$—see Ref. [46] for a precise definition) enjoys an LDP so that, for a generic density Λ with unit mass, $\mathbb{P}(\Lambda_N \approx \Lambda) \approx \exp[-N^2 I_2(\Lambda)]$. The rate function $I_\beta(\Lambda)$ is the difference of the electrostatic energy of a Coulomb-gas of charge β (with $\beta = 2$ for GUE, and $\beta = 1$ for GOE) with density Λ, as compared to that of the Wigner semi-circle density $\Lambda_{\mathrm{sc}}(\lambda) = (1/2\pi)\sqrt{4-\lambda^2}\mathbf{1}_{\{|\lambda|<2\}}$. I_β can be rewritten (see Ref. [46] for details) as

$$I_\beta(\Lambda) = \frac{\beta}{2}J(\Lambda) + \frac{\beta}{2}\int_{\mathbb{R}} d\lambda V(\lambda)\Lambda(\lambda), \tag{10}$$

with a Coulomb interaction term $J(\Lambda) = -\int_{\mathbb{R}^2} \log|\lambda_1 - \lambda_2| \prod_{i=1}^2 d\lambda_i[\Lambda(\lambda_i) - \Lambda_{\mathrm{sc}}(\lambda_i)]$ (note that $\Lambda - \Lambda_{\mathrm{sc}}$ is a signed density with integral over $\mathbb{R}$ equal to 0) and potential term $V(\lambda) = \int_0^{|\lambda|} d\lambda'[(\lambda'^2 - 4)_+]^{1/2}$. The (space-reversed) Airy PP arises as a scaling limit of the GUE spectrum near its lower edge $\lambda = -2$. To deduce the Airy PP LDP from that of the GUE, we introduce the scaling $\lambda = -2 + t^{2/3}N^{-2/3}a$. As $N \to \infty$, $N d\lambda \Lambda_N(\lambda) \simeq t da \mu_t(a)$, which when inserted into Eq. (10) gives $N^2 I_\beta(\Lambda) \simeq t^2 I_{\mathrm{Airy}}(\mu)$, with

$$I_{\mathrm{Airy}}(\mu) = J_{\mathrm{Airy}}(\mu) + U(\mu).$$

Here, $J_{\mathrm{Airy}}(\mu) = -\int \log|a_1 - a_2| \prod_{i=1}^2 da_i[\mu(a_i) - \mu_{\mathrm{Airy}}(a_i)]$ is defined for densities μ satisfying mass conservation $\int da[\mu(a) - \mu_{\mathrm{Airy}}(a)] = 0$, where $\mu_{\mathrm{Airy}}(a) = (1/\pi)\sqrt{a}\mathbf{1}_{\{a>0\}}$, and $U(\mu) = \frac{4}{3}\int_{-\infty}^0 da|a|^{\frac{3}{2}}\mu(a)$.

Instead of searching directly for the minimum of Σ in Eq. (9), we first consider a simpler cost function

$$\Sigma_J(\mu) = \int_{\mathbb{R}} da(-z-a)_+\mu(a) + J_{\mathrm{Airy}}(\mu),$$

that drops the term $U(\mu)$. The minimizer μ_* of Σ_J is the unique measure (see Ref. [46] for details) such that

$$(-z-a)_+ - 2\int_{\mathbb{R}} da' \log|a-a'|[\mu_*(a') - \mu_{\mathrm{Airy}}(a')] \geq c, \tag{11}$$

for some constant c with strict equality on the support of μ_*. Differentiating the l.h.s. of Eq. (11) in a yields

$$-\mathbf{1}_{\{a<-z\}} - 2\int_{\mathbb{R}} da' \frac{\mu_*(a') - \mu_{\mathrm{Airy}}(a')}{a-a'}. \tag{12}$$

Consider a generic interval $[u,\infty)$ and let

$$\mu_{*,u}(a) = \left[\frac{1}{\pi}\sqrt{a-u} + \frac{1}{2\pi^2}\log\left|\frac{\sqrt{a-u}+\sqrt{v}}{\sqrt{a-u}-\sqrt{v}}\right| + \frac{1}{\pi}\left(\frac{u}{2} - \frac{\sqrt{v}}{\pi}\right)\frac{1}{\sqrt{a-u}}\right]\mathbf{1}_{\{a>u\}},$$

where $v = -z-u$. Reference [46] verifies that substituting this density $\mu_{*,u}(a)$ for $\mu_*(a)$ implies that Eq. (12)$= 0$ on $[u,\infty)$. Furthermore, Ref. [46] shows that $u = u_0 = (2/\pi^2)(\sqrt{1-\pi^2 z} - 1)$ is the unique choice of u for which one also has Eq. (12)≥ 0 on $(-\infty, u_0)$ and $= 0$ on $[u_0,\infty)$. This means that $\mu_*(a) = \mu_{*,u_0}(a)$ satisfies Eq. (11) and hence is the unique minimizer of Σ_J. Evaluating yields (see Fig. 1)

$$\mu_*(a) = \left(\frac{1}{\pi}\sqrt{a-u_0} + \frac{1}{2\pi^2}\log\left|\frac{\sqrt{a-u_0}+\frac{\pi}{2}u_0}{\sqrt{a-u_0}-\frac{\pi}{2}u_0}\right|\right)\mathbf{1}_{\{a>u_0\}}.$$

The associated minimum of Σ_J is

$$\min_\mu \Sigma_J(\mu) = \frac{4}{15\pi^6}(1-\pi^2 z)^{\frac{5}{2}} - \frac{4}{15\pi^6} + \frac{2}{3\pi^4}z - \frac{1}{2\pi^2}z^2,$$

which coincides precisely with $\Phi_-(z)$ in Eq. (2).

Returning to Σ from Eq. (9), we note that $U(\mu) \geq 0$ implies $(\min\Sigma) \geq (\min\Sigma_J)$. Since $\mu_*(a)$ vanishes for $a<0$ (since $u_0>0$), we have $U(\mu_*) = 0$ and, hence, $\Sigma(\mu_*) = \Sigma_J(\mu)$. Thus, the minimizer and minimum for Σ_J in fact also applies to Σ. Since $\Phi_-(z) = \min_\mu \Sigma(\mu)$, this confirms the formula in Eq. (2) and the calculation of Ref. [39].

Tail bounds for intermediate times.—While the KPZ LDP holds for $t\to\infty$, the crossover behavior between exponents 3 and 5/2 remains valid at all intermediate times. Precisely: For any $\varepsilon, \delta \in]0, \frac{1}{3}[$ and $t_0 > 0$ then there exists constants $S = S(\varepsilon,\delta,t_0)$, $K_1 = K_1(\varepsilon,\delta,t_0) > 0$, and $K_2 = K_2(t_0) > 0$ such that for all $s \geq S$ and $t \geq t_0$,

$$\begin{aligned}\mathbb{P}(H \leq -st^{\frac{1}{3}}) &\leq e^{-\frac{4(1-\varepsilon)}{15\pi}t^{\frac{1}{3}}s^{\frac{5}{2}}} + e^{-K_1 s^{3-\delta} - \varepsilon t^{\frac{1}{3}} s} + e^{-\frac{1-\varepsilon}{12}s^3},\\ \mathbb{P}(H \leq -st^{\frac{1}{3}}) &\geq e^{-\frac{4(1+\varepsilon)}{15\pi}t^{\frac{1}{3}}s^{5/2}} + e^{-K_2 s^3}.\end{aligned} \tag{13}$$

For $t^{2/3} \gg s \gg 1$, the second and third terms in the first line of Eq. (13) dwarf the first term and represent cubic decay (in the exponential) in s. In particular, as t gets large, only the third term survives and we recover (up to an ε correction) the predicted $\frac{1}{12}s^3$ decay. On the other hand, for $s \gg t^{2/3}$ the first term in the second line of Eq. (13) dwarfs the others and recovers the predicted $(4/15\pi)s^{5/2}$ decay for all t. The second line of Eq. (13) contains corresponding lower bounds—though notice that for t large and $t^{2/3} \gg s \gg 1$, our bounds do not recover the $\frac{1}{12}$ constant for the lower bound on the cubic decay. This result recovers the large and small z behavior of the $t \to \infty$ rate function $\Phi_{-}(z)$. Prior to Eq. (13), the only finite time bounds were in Ref. [47], which provided a Gaussian upper bound on the decay (hence, the wrong exponent). Moreover, those bounds are not adapted to large t center and scaling—becoming ineffective as t grows.

Equations (13) follow from two considerations. The typical locations of the $\mathbf{a}_i$ are governed by $\rho(a)$. Plugging these typical values into Eq. (3) yields the $5/2$ exponential term. However, the $\mathbf{a}_i$ are random and may deviate from their typical locations. For instance, $\mathbf{a}_1 \leq -s$ with probability $\approx \exp(-\frac{1}{12}s^3)$. Such deviations lead to the cubic exponential terms. In order to provide matching upper and lower tail bounds, we precisely control the LDP for the counting function of the Airy PP in large intervals. This can be done via asymptotics of the Ablowitz-Segur solution to Painlevé II [48,49], which relates to the exponential moment generating function for this counting process, as well as by using of the relation of the AAP to the stochastic Airy operator [50]. The main ideas and steps of this derivation are provided in Ref. [46] (and further technical details and complete rigorous proofs are in Ref. [51]).

Extensions and summary.—The approach developed in this Letter is applicable to certain variants of the KPZ equation which enjoy identities similar to Eq. (3)—namely, half-space KPZ [52], the stochastic six vertex model and ASEP [42,53]. Briefly we consider the half-space KPZ equation, i.e., Eq. (1) restricted to $x \in \mathbb{R}^+$, with Neumann b.c. $\partial_x h(t,x)|_{x=0} = A$, for the value $A = -1/2$ corresponding to the so-called critical case. In that case and for droplet initial condition Ref. [52] proved that

$$\overline{\exp\left(-\frac{1}{4}e^{\mathcal{H}(t)+st^{1/3}}\right)} = \mathbb{E}_{\mathrm{GOE}}\left(\prod_{i=1}^{+\infty}\frac{1}{\sqrt{1+e^{t^{1/3}(\mathbf{a}_i+s)}}}\right),$$

where the r.h.s. expectation is over the $\beta = 1$ version of the Airy PP (which describes the top few eigenvalues at the spectral edge for the GOE instead of GUE—see also Ref. [46]). Employing the Airy PP Coulomb-gas approach from this Letter, we find that due to the square root in the right-hand side above (which introduces a factor of $1/2$ in exponential form), and the value of $\beta = 1$ (instead of $\beta = 2$), the half-space KPZ rate function $\Phi_{-}^{\mathrm{half-space}}(z) = \frac{1}{2}\Phi_{-}(z)$, where $\Phi_{-}(z)$ is the full-space function in Eq. (2).

In conclusion, by relating the distribution of the height for the KPZ equation to an expectation over the Airy point process, we are able to employ the Coulomb-gas formalism and associated electrostatic problem large deviation principle (first for the GUE and, through a limit transition which we present, for the Airy point process) to identify the KPZ rate function. Solving the variational problem produces the formula in Eq. (2). This argument brings the role of random matrix theory in the study of KPZ to the forefront and provides a straightforward and assumption-free derivation of the KPZ rate function. Additionally, a similar approach should be applicable to other exactly solvable KPZ class models such as ASEP or the stochastic six vertex model which connect to discrete Coulomb gases. This approach also permits us to derive results valid for all intermediate times and opens the way to systematically calculate higher order corrections between the long time and finite time PDF, as is useful in experiments and numerics.

We thank G. Barraquand, A. Borodin, T. Bothner, E. Corwin, P. Deift, T. Halpin-Healy, S. N. Majumdar, B. Meerson, G. Schehr, H. Spohn, and B. Virag for helpful discussions and comments. The authors initiated this work during the 2017 session of PCMI, partly funded by the NSF Grant No. DMS:1441467. I. C. was partially funded by the NSF Grant No. DMS:1664650 and No. DMS:1811143, and the Packard Foundation through a Packard Fellowship for Science and Engineering. P. L. D. and A. K. acknowledge support from ANR Grant No. ANR-17-CE30-0027-01 RaMaTraF. L. C. T. was partially supported by a Junior Fellow award from the Simons Foundation, and by the NSF through DMS:1712575.

[1] M. Kardar, G. Parisi, and Y-C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).
[2] K. A. Takeuchi and M. Sano, Phys. Rev. Lett. **104**, 230601 (2010); K. A. Takeuchi, M. Sano, T. Sasamoto, and H. Spohn, Sci. Rep. **1**, 34 (2011); K. A. Takeuchi and M. Sano, J. Stat. Phys. **147**, 853 (2012).
[3] D. Forster, D. R. Nelson, and M. J. Stephen, Phys. Rev. A **16**, 732 (1977).
[4] D. A. Huse, C. L. Henley, and D. S. Fisher, Phys. Rev. Lett. **55**, 2924 (1985); M. Kardar and Y-C. Zhang, Phys. Rev. Lett. **58**, 2087 (1987); T. Halpin-Healy and Y-C. Zhang, Phys. Rep. **254**, 215 (1995).
[5] S. Atis, A. K. Dubey, D. Salin, L. Talon, P. Le Doussal, and K. J. Wiese, Phys. Rev. Lett. **114**, 234502 (2015).
[6] J. Wakita, H. Itoh, T. Matsuyama, and M. Matsushita, J. Phys. Soc. Jpn. **66**, 67 (1997); O. Hallatschek, P. Hersen, S. Ramanathan, and D. R. Nelson, Proc. Natl Acad. Sci. U.S.A. **104**, 19926 (2007).
[7] J. Maunuksela, M. Myllys, O.-P. Kähkönen, J. Timonen, N. Provatas, M. J. Alava, and T. Ala-Nissila, Phys. Rev. Lett. **79**, 1515 (1997).
[8] P. J. Yunker, M. A. Lohr, T. Still, A. Borodin, D. J. Durian, and A. G. Yodh, Phys. Rev. Lett. **110**, 035501 (2013).
[9] A. M. Somoza, M. Ortuno, and J. Prior, Phys. Rev. Lett. **99**, 116602 (2007); A. M. Somoza, P. Le Doussal, and M. Ortuno, Phys. Rev. B **91**, 155413 (2015).
[10] L. Chen, C. F. Lee, and J. Toner, Nat. Commun. **7**, 12215 (2016).
[11] L. M. Sieberer, G. Wachtel, E. Altman, and S. Diehl, Phys. Rev. B **94**, 104521 (2016); L. He, L. M. Sieberer, E. Altman, and S. Diehl, Phys. Rev. B **92**, 155307 (2015).

[12] A. Nahum, J. Ruhman, S. Vijay, and J. Haah, Phys. Rev. X **7**, 031016 (2017).
[13] K. Johansson, Commun. Math. Phys. **209**, 437 (2000).
[14] J. Baik, P. Deift, and K. Johansson, J. Am. Math. Soc. **12**, 1119 (1999).
[15] G. Barraquand and I. Corwin, Probab. Theory Relat. Fields **167**, 1057 (2017); P. Le Doussal and T. Thiery, Phys. Rev. E **96**, 010102 (2017).
[16] A. Borodin and I. Corwin, Probab. Theory Relat. Fields **158**, 225 (2014).
[17] S. Serfaty, *Proceedings of the International Congress of Mathematicians 2018* (Rio De Janeiro, 2018).
[18] T. Sasamoto and H. Spohn, Phys. Rev. Lett. **104**, 230602 (2010); P. Calabrese, P. Le Doussal, and A. Rosso, Europhys. Lett. **90**, 20002 (2010); V. Dotsenko, Europhys. Lett. **90**, 20003 (2010).
[19] G. Amir, I. Corwin, and J. Quastel, Commun. Pure Appl. Math. **64**, 466 (2011).
[20] P. Calabrese and P. Le Doussal, Phys. Rev. Lett. **106**, 250603 (2011); P. Le Doussal and P. Calabrese, J. Stat. Mech. (2012) P06001; T. Imamura and T. Sasamoto, Phys. Rev. Lett. **108**, 190603 (2012); J. Stat. Phys. **150**, 908 (2013); A. Borodin, I. Corwin, P. L. Ferrari, and B. Veto, Math. Phys. Geo. Analy. **18**, 20 (2015).
[21] D. Nickelsen and H. Touchette, arXiv:1803.05708.
[22] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, Rev. Mod. Phys. **87**, 593 (2015).
[23] I. V. Kolokolov and S. E. Korshunov, Phys. Rev. E **80**, 031107 (2009); Phys. Rev. B **75**, 140201 (2007); **78**, 024206 (2008).
[24] A. K. Hartmann, P. Le Doussal, S. N. Majumdar, A. Rosso, and G. Schehr, Europhys. Lett. **121**, 67004 (2018).
[25] T. Halpin-Healy and K. A. Takeuchi, J. Stat. Phys. **160**, 794 (2015).
[26] I. Gupta and B. C. Mohanty, Sci. Rep. **6**, 33136 (2016); P. A. Orrillo, S. N. Santalla, R. Cuerno, L. Vazquez, S. B. Ribotta, L. M. Gassa, F. J. Mompean, R. C. Salvarezza, and M. E. Vela, Sci. Rep. **7**, 17997 (2017).
[27] R. A. Carmona and S. A. Molchanov, *Memoirs of the American Mathematical Society* (American Mathematical Society, 1994); A. Borodin and I. Corwin, Ann. Appl. Probab. **24**, 1172 (2014); D. Khoshnevisan, K. Kim, and Y. Xiao, Commun. Math. Phys. **360**, 307 (2018); J. Huang and D. Khoshnevisan, arXiv:1704.08334.
[28] D. S. Dean and S. N. Majumdar, Phys. Rev. Lett. **97**, 160201 (2006); Phys. Rev. E **77**, 041108 (2008).
[29] P. Le Doussal, S. N. Majumdar, and G. Schehr, Europhys. Lett. **113**, 60004 (2016).
[30] A. Kamenev, B. Meerson, and P. V. Sasorov, Phys. Rev. E **94**, 032108 (2016).
[31] P. Le Doussal, S. N. Majumdar, A. Rosso, and G. Schehr, Phys. Rev. Lett. **117**, 070403 (2016).
[32] B. Meerson, E. Katzav, and A. Vilenkin, Phys. Rev. Lett. **116**, 070601 (2016).
[33] M. Janas, A. Kamenev, and B. Meerson, Phys. Rev. E **94**, 032133 (2016); B. Meerson and J. Schmidt, J. Stat. Mech. (2017) 129901; A. Krajenbrink and P. Le Doussal, Phys. Rev. E **96**, 020102 (2017); N. Smith and B. Meerson, Phys. Rev. E **97**, 052110 (2018).
[34] G. B. Arous and A. Guionnet, Probab. Theory Relat. Fields **108**, 517 (1997).
[35] S. N. Majumdar and G. Schehr, J. Stat. Mech. (2014) P01012.
[36] M. Alder and P. van Moerbeke, Ann. Math. **153**, 149 (2001).
[37] C. A. Tracy and H. Widom, Commun. Math. Phys. **159**, 151 (1994).
[38] A. R. Its, A. G. Izergin, V. E. Korepin, and N. A. Slavonv, Int. J. Mod. Phys. B **04**, 1003 (1990).
[39] P. Sasorov, B. Meerson, and S. Prolhac, J. Stat. Mech. (2017) 063203.
[40] A. Grabsch, S. N. Majumdar, and C. Texier, J. Stat. Phys. **167**, 234 (2017).
[41] A. Borodin and V. Gorin, SIGMA **12**, 102 (2016).
[42] A. Borodin, J. Math. Phys. **59**, 023301 (2018).
[43] D. S. Dean, P. Le Doussal, S. N. Majumdar, and G. Schehr, Phys. Rev. Lett. **114**, 110402 (2015).
[44] A. Soshnikov, Ann. Probab. **30**, 171 (2002).
[45] A. Krajenbrink and P. Le Doussal, arXiv:1802.08618.
[46] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.121.060201 for details on the Airy point process large deviation principal, the solution to the associated electrostatic variational problem, and intermediate time tail results.
[47] C. Mueller, Stochastics, **37**, 225 (1991); C. Mueller and D. Nualart, Electron. J. Pro **13**, 2248 (2008); G. Moreno Flores, Ann. Probab. **42**, 1635 (2014).
[48] M. J. Ablowitz and H. Segur, Phys. Rev. Lett. **38**, 1103 (1977); *Solitons and the Inverse Scattering Transform* (Society for Industrial and Applied Mathematics, 1981).
[49] T. Bothner, Duke Math. J. **166**, 205 (2017).
[50] A. Edelman and B. D. Sutton, J. Stat. Phys. **127**, 1121 (2007); J. A. Ramírez, B. Rider, and B. Virág, J. Am. Math. Soc. **24**, 919 (2011).
[51] I. Corwin and P. Ghosal, arXiv:1802.03273.
[52] G. Barraquand, A. Borodin, I. Corwin, and M. Wheeler, arXiv:1704.04309.
[53] A. Borodin and G. Olshanski, Commun. Math. Phys. **353**, 853 (2017).