# Random Search with Resetting: A Unified Renewal Approach

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We provide a unified renewal approach to the problem of random search for several targets under resetting. This framework does not rely on specific properties of the search process and resetting procedure, allows for simpler derivation of known results, and leads to new ones. Concentrating on minimizing the mean hitting time, we show that resetting at a constant pace is the best possible option if resetting helps at all, and derive the equation for the optimal resetting pace. No resetting may be a better strategy if without resetting the probability of not finding a target decays with time to zero exponentially or faster. We also calculate splitting probabilities between the targets, and define the limits in which these can be manipulated by changing the resetting procedure. We moreover show that the number of moments of the hitting time distribution under resetting is not less than the sum of the numbers of moments of the resetting time distribution and the hitting time distribution without resetting.

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*Introduction.*—The first passage processes are ubiquitous in nature and lie in the heart of physics of complex systems, where rare events dominate the long-time dynamics [1–4]. Similar processes dominate searching for randomly located targets, which is a central problem in biological applications on all scales, from the smallest in gene regulation [5] to the largest in movement ecology [6–8]. Other prominent examples come from the realm of operations research [9,10]. Different examples and applications are discussed in Refs. [11,12] and Ref. [13] and in references cited therein.

Recently, it was found that in a simple Brownian search the search efficiency (that is, the inverse time to find a target) can be essentially increased if the searcher returns to its starting point and recommences the search [14]. This finding gave rise to a whole series of works, in which the increase of search efficiency in the presence of resetting has been studied for a wide range of search processes with different waiting time distributions between the resetting events. Particular attention was paid to the existence of phase transitions in the space of parameters, which minimize the mean first passage time [15–18]. Thus, Brownian search processes under different resetting distributions were investigated in Refs. [19–24]. The optimal Brownian search for a team of independent searchers looking for a single target was considered in Refs. [21,25], while the effects of partial absorption and bounded domain on the diffusion with resetting have been addressed in Ref. [26] and Ref. [16], respectively. The first passage processes under resetting have been also analyzed for continuous time random walk [13,27], Lévy flights [15,28], and Sisyphus random walk [29].

A wide diversity of the random processes and restart mechanisms considered have brought into life the attempts to find the universal features of the first passage processes subject to stochastic restart. General expressions for the mean first passage time and distribution of first passage (hitting) times under restart have been obtained, and a conjecture that the constant pace resetting is the optimal search strategy in the general case has been made [11,12]. This general framework was recently used to show that a stochastic restart could also optimize splitting probabilities [30].

The present Letter was inspired by the unified treatment suggested in Ref. [12]. We here develop a renewal approach to the first passage processes with resetting. The renewal nature of the process allows for making several general statements on its properties which will be discussed below. The approach allows for simpler derivations and proofs of known results, and leads to new ones. We concentrate on the case of the single searcher, but discuss in detail the case of several targets, i.e., the properties of splitting probabilities. Three examples of derivations of some results obtained previously are given in the Supplemental Material [31], and more could be presented easily.

*Problem position.*—The search process corresponds to the motion of a searcher starting at the origin; the aim of the search is to hit one of $n \geq 1$ targets, none of them at the origin. A single realization of the search process will be called a run. A successful run is completed by hitting a

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target, which terminates the whole process; an unsuccessful (idle) run is terminated after some time $t$ (which might be fixed or random), the searcher is reset to the origin, and a new run starts. The whole procedure is repeated until the last run is successful.

The random variables characterizing a run and a resetting procedure are considered independent, and each new run is independent from all previous idle ones. The whole search process until completing the last run is therefore a renewal process.

*Notation.*—Let $\psi(t)$ be the probability density function (PDF) of resetting intervals, and $\Psi(t) = 1 - \int_0^t \psi(t')dt'$ the probability that no reset took place up to time $t$, i.e., the survival probability in resetting. The resetting probability density is assumed to be a proper one: $\Psi(t \to \infty) = 0$.

Let $\phi_i(t)$ be the hitting (first passage) time PDF for the target $i$ in a single run: $\phi_i(t)dt$ is the probability that the target $i$ was hit in the time interval between $t$ and $t + dt$, and no target was ever hit before. The total hitting density will be denoted as $\phi(t) = \sum_{i=1}^n \phi_i(t)$. Let, moreover, $F_i(t) = \int_0^t \phi_i(t')dt'$ be the corresponding cumulative function, $F(t) = \sum_i F_i(t)$ be the total hitting probability in a single run, and $\Phi(t) = 1 - F(t) = 1 - \sum_{i=1}^n \int_0^t \phi_i(t')dt'$ be the total survival probability in a single run. The hitting probability may be nonproper: $\lim_{t\to\infty}\Phi(t)$ may be nonzero.

If the mean hitting time $T$ in a single run exists, this is given by $T = \int_0^\infty F(t)dt$. The nonproper situations, and the proper one with diverging first moment will be considered as cases $T \to \infty$.

*General approach.*—Let us calculate the PDF $p_i(t)$ of hitting the target $i$ under resetting. Up to time $t$ there may be 0,1,... idle runs, the last run is always complete. Therefore, $p_i(t)$ is given by the following expression: $p_i(t) = \Psi(t)\phi_i(t) + \int_0^t \psi(t')\Phi(t')\Psi(t - t')\phi_i(t - t')dt' + \int_0^t dt'\psi(t')\Phi(t') \int_0^{t-t'} dt''\psi(t'')\Phi(t'')\Psi(t - t' - t'')\phi_i(t - t' - t'') + \cdots$. The first term is the probability density of hitting the $i$th target in the first, complete, run, which was not reset until the hitting time. The second term describes the situation in which the first, idle run was terminated at $t'$, and the second run is complete, etc. Denoting $\kappa_i(t) = \Psi(t)\phi_i(t)$ and $R(t) = \psi(t)\Phi(t)$, and turning to the Laplace domain, we get

$$\tilde{p}_i(s) = \frac{\tilde{\kappa}_i(s)}{1 - \tilde{R}(s)}, \tag{1}$$

with $\tilde{\kappa}_i(s)$ being the Laplace transform of $\kappa_i(t)$, and $\tilde{R}(s)$ is the Laplace transform of $R(t)$. The total probability $\Pi_i = \int_0^\infty p_i(t')dt'$ of hitting the target $i$ (splitting probability) corresponds to $\tilde{p}_i(0)$ and is given by

$$\Pi_i = \frac{\tilde{\kappa}_i(0)}{1 - \tilde{R}(0)} = \frac{\int_0^\infty \psi(t')F_i(t')dt'}{\int_0^\infty \psi(t')F(t')dt'}. \tag{2}$$

To get the second form from the first one we note that $\tilde{\kappa}_i(0) = \int_0^\infty \Psi(t)\phi_i(t)dt$, and $\tilde{R}(0) = \int_0^\infty \psi(t)\Phi(t)dt$. Then we rewrite the expression for $\tilde{\kappa}_i(0)$ performing integration by parts: $\tilde{\kappa}_i(0) = \Psi(t)F_i(t)|_0^\infty + \int_0^\infty \psi(t')F_i(t')dt'$. The first term vanishes since $F_i(0) = 0$, because there is no target at the origin, and $\Psi(\infty) = 0$ because the distribution of the resetting times is a proper one. Note that the relation $\Phi(t) = 1 - \sum_i F_i(t)$ between $\Phi(t)$ and $F_i(t)$ guarantees the normalization: $\sum_i \tilde{\kappa}_i(0) = 1 - \tilde{R}(0)$, except for the cases when the probability of hitting any target in a run is exactly zero: $\tilde{\kappa}_i(0) = 0$. This is equivalent to the orthogonality of $\Psi(t)$ and $\phi_i(t)$ for all $i$ on $[0, \infty)$, and is, e.g., the case when the possible durations of the resetting intervals are smaller than the minimal time necessary to hit any target. Therefore, if hitting in a single run is possible at all, hitting under resetting takes place with probability 1.

*Moments of the hitting time.*—Let us now discuss the existence of the moments of hitting time, concentrating on the single target case. The PDF of hitting a target in the Laplace domain reads $\tilde{p}(s) = \tilde{\kappa}(s)/[1 - \tilde{R}(s)]$. Now we can discuss under what conditions do the mean hitting time $\tau$, the mean squared hitting time, etc., exist. The number $N$ of the moments of $\tilde{p}(s)$ corresponds to the number of the last regular term in the Taylor expansion of $\tilde{p}(s) = a_0 + a_1 s + a_2 s^2 + \cdots + a_N s^N + o(s^N)$, where the remainder term is the first term singular in $s = 0$. This number can be found by counting regular terms in the expansion of the numerator and denominator of the equation for $p(s)$, and noting that the function $1/\tilde{f}(s)$ has the same number of regular terms in its expansion as the function $\tilde{f}(s)$, provided $\tilde{f}(0) \neq 0$. The details of such counting procedure are given in Ref. [31]. The result is that in nonproper cases, $\Phi(\infty) \neq 0$, the number of moments of $p(t)$ is at least equal to the number of moments of the resetting time distribution. For proper cases the number of moments of hitting time is not smaller than the sum of the numbers of moments of hitting time PDF $\phi(t)$ in a single run and of a reset time PDF $\psi(t)$.

*Mean hitting time and optimal resetting.*—Let $P(t) = 1 - \int_0^t p(t')dt'$ be the probability of no hitting up to time $t$. In the Laplace domain one has

$$\tilde{P}(s) = \frac{1}{s} - \frac{1}{s}\frac{\tilde{\kappa}(s)}{1 - \tilde{R}(s)} = \frac{\tilde{X}(s)}{1 - \tilde{R}(s)}, \tag{3}$$

with $\tilde{X}(s) = s^{-1}[1 - \tilde{R}(s) - \tilde{\kappa}(s)]$. In the time domain we have $X(t) = 1 - \int_0^t R(t')dt' - \int_0^t \kappa(t')dt' = 1 - \int_0^t \psi(t')\Phi(t')dt' - \int_0^t \Psi(t')\phi(t')dt'$. Now one uses the fact that $\psi(t) = -(d/dt)\Psi(t)$ and $\phi(t) = -(d/dt)\Phi(t)$, performs integration by parts in the second integral, and notes that $\Psi(0)\Phi(0) = 1$ to obtain $X(t) = \Psi(t)\Phi(t)$. The hitting time density is

$$p(t) = -\frac{d}{dt}P(t), \tag{4}$$

and therefore in the Laplace domain $p(s) = 1 - s\tilde{P}(s)$, where $\tilde{P}(s)$ is given by Eq. (3).

Let us assume that the mean hitting time does exist, e.g., that at least one of the PDFs $\psi(t)$ and $\phi(t)$ possesses the first moment. Then it is given by $\tau = \int_0^\infty p(t)tdt = \int_0^\infty P(t)dt$. In the Laplace domain we have $\tau = \lim_{s\to 0}\tilde{P}(s) = \tilde{X}(0)/[1-\tilde{R}(0)]$. The numerator of this expression is $\tilde{X}(0) = \int_0^\infty \Psi(t)\Phi(t)dt = \int_0^\infty \int_t^\infty \psi(t')\Phi(t)dt'dt = \int_0^\infty dt'\psi(t')\int_0^{t'}\Phi(t)dt$. Now we introduce a new function $\Lambda(t) = \int_0^t \Phi(t')dt'$ and write $\tilde{X}(0) = \int_0^\infty \psi(t)\Lambda(t)dt$. The expression in the denominator can be rewritten as $1-\tilde{R}(0) = \int_0^\infty dt\psi(t)[1-\Phi(t)]dt = \int_0^\infty \psi(t)F(t)dt$. Therefore, $\tau = [\int_0^\infty \psi(t)\Lambda(t)dt]/[\int_0^\infty \psi(t)F(t)dt]$.

Introducing another new function

$$G(t) = \frac{\Lambda(t)}{F(t)} = \frac{t - \int_0^t F(t')dt'}{F(t)}, \tag{5}$$

we may write

$$\tau = \frac{\int_0^\infty \psi(t)F(t)G(t)dt}{\int_0^\infty \psi(t)F(t)dt}. \tag{6}$$

Note that $\psi(t)F(t)$ is non-negative. By virtue of the mean value theorem we now have $\tau = [G(t^*)\int_0^\infty \psi(t')F(t')dt']/[\int_0^\infty \psi(t')F(t')dt'] = G(t^*)$, where $0 \le t^* \le \infty$. Therefore the important result follows:

$$\tau \ge \min_t G(t).$$

The bound obtained is optimal, i.e., can be attained for a specific resetting procedure. Let $t_r$ be the time at which the global minimum of $G(t)$ is reached, and let us assume that $t_r < \infty$. Taking $\psi(t) = \delta(t - t_r)$ we get $\tau = G(t_r)$: If the global minimum of $G(t)$ exists, and is attained at time $t = t_r$, the optimal resetting procedure is a resetting at a constant pace with resetting interval $t_r$. The fact that the constant pace resetting is the optimal strategy was already observed in Ref. [12].

*Looking for a needle in a haystack.*—The nonproper situations with low probability $F(t)$ of hit in a single run allow for a very simple approximation. In such situations the integral in the numerator in Eq. (5) can be neglected compared to $t$ at all times, and thus $G(t) \simeq t/F(t)$. Therefore the optimal resetting pace $t_r$ for which $G'(t_r) = 0$ is given by the solution of the equation

$$t\frac{d}{dt}\ln F(t) = 1. \tag{7}$$

To stress the quality of this approximation we consider an example allowing for the exact solution.

Let us discuss a diffusive search of a sphere starting from the outside [22]. Let $\phi(t)$ be PDF of the first passage time of a Brownian motion with diffusion coefficient $D$ to a sphere of radius $r$ in a 3d space placed at a distance $x > r$ from the origin. The result for $\phi(t)$ is given by theorem 5 of Ref. [34]: Its Laplace transform $\tilde{\phi}(s)$ follows by taking $n = 0$ and $h = 1/2$ in the corresponding formula. The inverse transform gives the nonproper Lévy-Smirnov-like distribution $\phi(t) = \{r(x-r)/[\sqrt{2\pi D}xt^{3/2}]\}\exp[-(x-r)^2/2Dt]$ normalized onto $r/x < 1$, from which the expressions for $F(t)$ and $\Lambda(t)$ follow. The explicit calculations reproducing the results of Ref. [22] are given in Ref. [31]. Knowing these functions one obtains the equation for $G(t)$ [Eq. (5)], solves numerically the equation $G'(t_r) = 0$ to obtain the optimal resetting pace as a function of $x$ and the corresponding $\tau = G(t_r)$. In what follows $x$ is measured in units of $r$, and $t$ in units of $r^2/D$, so that both get to be dimensionless. The results of the exact solution are shown as solid lines in Fig. 1 for $r = 1$ and $D = 1$.

The approximation given by Eq. (7) can be reformulated in the scaling form in a variable $\xi = (x-r)^2/2Dt$: $-(d/d\xi)\ln F(\xi) = \xi^{-1}$, and solved numerically giving the solution $\xi_r \approx 0.708\,77$, from which

$$t_r \approx 0.705\,44\frac{(x-r)^2}{D}, \qquad \tau \approx 3.017\,20\frac{x(x-r)^2}{rD} \tag{8}$$

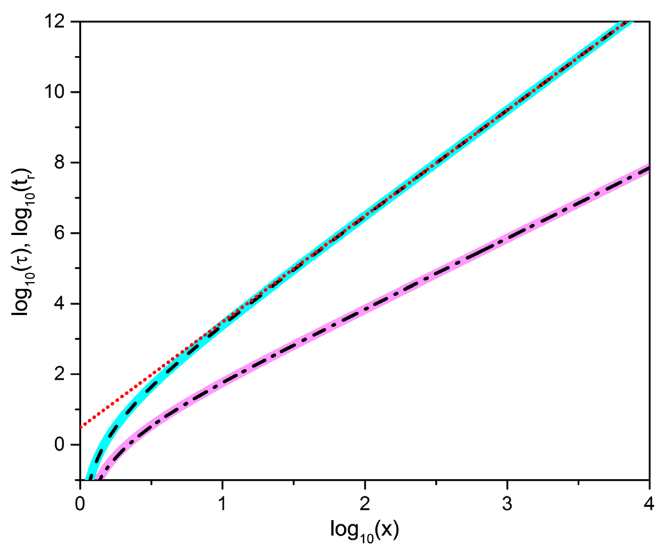


FIG. 1. The lower solid curve represents the numerical solution for the optimal resetting time $t_r$ as a function of the distance from starting point to the center of the sphere, as obtained by finding the minimum of the function $G(t)$ given by Eq. (5); see Ref. [31] for details. The corresponding dashed-dotted curve is the approximate solution, Eq. (8). The solid upper curve and the dashed upper one are the exact solution $\tau = G(t_r)$ and the approximation, Eq. (8), for the mean hitting time $\tau$. The asymptotics of these curves is shown as a dotted straight line having slope 3. Note the double logarithmic scales.

follow. These are the approximations shown in Fig. 1 by the dashed-dotted and by the dashed lines, respectively. We note that the approximation devised for the small total hitting probability in a single run works astonishingly well in the whole domain of parameters starting from $x/r$ as low as 1.5.

*When can resetting harm?*—In situations with diverging mean hitting time $T$ in a single run resetting with some $\psi(t)$ which does possess the mean leads to finite $\tau$. The resetting with $\psi(t)$ with diverging first moment [e.g., $\psi(t) \simeq t^{-1-\alpha}$ with $\alpha < 1$] leads to a faster decrease of $p(t)$ compared to $\Phi(t)$, see Eq. (19) of Ref. [31], i.e., increases search efficiency. Therefore the cases when resetting may harm are pertinent to finite $T$. Since the optimal resetting time corresponds to the position of the global minimum of $G(t)$, no resetting is the best option when this is attained at infinity. A necessary condition for this is that the limiting value $G(\infty)$ is approached from above.

Let us discuss, under which condition this takes place. The derivative of $G(t)$ can be put as $G'(t) = F^{-2}(t)[\Phi(t)F(t) - F'(t)\int_0^t \Phi(t')dt']$. Now we consider $t \gg T$ and note that $F(t) \to 1$ and $\int_0^t \Phi(t')dt' \to T$, respectively, and get $G'(t) \simeq \Phi(t) - TF'(t) = 1 - F(t) - TF'(t)$. This is negative at infinity if $F'(t) + T^{-1}F(t) > T^{-1}$. The borderline behavior corresponding to the equality in the previous relation is $F(t) = 1 - e^{-t/T}$, i.e., $\phi(t) \simeq T^{-1}\exp(-t/T)$. The faster decay corresponds to the case when resetting may harm. Therefore our next important conclusion is that the only cases when a resetting procedure may harm correspond to the single run hitting probability densities decaying faster than $\exp(-t/T)$. An example of the case when the resetting may harm is given by $\phi(t) = \delta(t - t_0)$, where the resetting at time $t < t_0$ makes the run idle. The exponential hitting PDF $\phi(t) = T^{-1}\exp(-t/T)$ is an interesting situation: here $G(t) = T =$ const, in which case the resetting does not influence the search efficiency.

*Splitting probabilities.*—Let us return to our Eq. (2) for splitting probabilities. Parallel to what was done above, we introduce the functions $\gamma_i(t) = F_i(t)/F(t)$, rewrite the expression in the numerator via $\gamma_i$, and apply the mean value theorem so that $\Pi_i = \gamma_i(t^*)$ with $0 < t^* < \infty$. This immediately gives us the bounds on $\pi_i$:

$$\min_t \gamma_i(t) \le \Pi_i \le \max_t \gamma_i(t).$$

The bounds are optimal: the first one is reached applying the constant-pace resetting with reset time $t_{\min}$, the second one applying the constant-pace resetting with reset time $t = t_{\max}$, where $t_{\min}$ and $t_{\max}$ are the times at which $\gamma_i(t)$ attains it minimal and maximal values. Therefore the resetting gives the possibility to manipulate splitting probabilities.

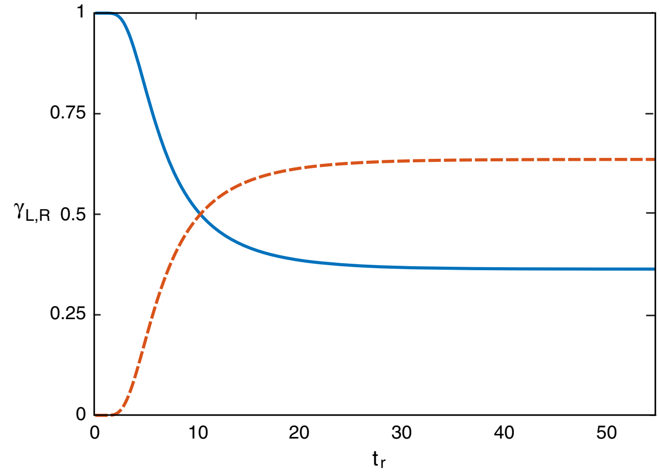


FIG. 2. The probability $\gamma_L$ to find first the left (upstream) boundary, full curve, blue, and the probability $\gamma_R$ to find first the right (downstream) boundary of the interval, dashed line, red, as functions of the resetting period $t_r$; see text for details.

As an example we consider the diffusive motion with drift velocity $v$ on an interval of length $L$. The resetting corresponds to return of the particle to its starting position $x_0$ within the interval. The probability density of the searcher's position within the interval is given by the Fokker-Planck equation $(\partial/\partial t)\rho(x,t) = -v(\partial/\partial x)\rho(x,t) + D(\partial^2/\partial x^2)\rho(x,t)$ with absorbing boundary conditions at $x = 0$ and $x = L$. The solution of this problem is known (see Ref. [35]): $\rho(x,t) = (2/L)\exp[-(v/2D)(x_0 - x) - (v^2 t/4D)] \times \sum_{n=1}^{\infty} \sin(n\pi x/L) \sin(n\pi x_0/L)\exp(-Dn^2\pi^2 t/L^2)$. The hitting time densities $\phi_L(t)$ and $\phi_R(t)$ of the left and of the right end of the interval are given by the diffusion fluxes $\phi_{L,R}(t) = \pm D(\partial/\partial x)\rho(x,t)|_{x=0,L}$, and the functions $\phi_{L,R}$ follow in a form of rapidly converging series. Now we can discuss the behavior of splitting probabilities $\gamma_{L,R}(t_r)$ under constant-pace resetting with resetting interval $t_r$. In Fig. 2 we plot these quantities for the case $x_0 = 1$, $v = 1$, $D = 1$ and $L = 10$. We see that for short resetting times, when the diffusion "wins" over the drift, the upstream (left) end of the interval is hit with the probability close to unity, while for longer resetting intervals the downstream end is hit with higher probability.

*Conclusions.*—The search problem under renewal resetting can be solved in full generality leading to astonishingly simple expressions in the Laplace domain. These can be used for the unified derivation of known and new results. Let us summarize some of them. The total number of moments of the hitting time is not less than the sum of the numbers of moments of a search probability in a single run and of the resetting time distribution. The optimal resetting procedure always corresponds to resetting at a constant pace. No resetting can be a better option only if the probability density of finding a target in a single run decays at longer times faster than $e^{-t/T}$, with $T$ being the mean hitting time in a single run. If several targets are present, resetting allows for manipulation of the probabilities of finding each of them.

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