Distributed Quantum Metrology with Linear Networks and Separable Inputs

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We derive a bound on the ability of a linear-optical network to estimate a linear combination of independent phase shifts by using an arbitrary nonclassical but *unentangled* input state, thereby elucidating the quantum resources required to obtain the Heisenberg limit with a multiport interferometer. Our bound reveals that while linear networks can generate highly entangled states, they cannot effectively combine quantum resources that are well distributed across multiple modes for the purposes of metrology: In this sense, linear networks endowed with well-distributed quantum resources behave classically. Conversely, our bound shows that linear networks can achieve the Heisenberg limit for distributed metrology when the input photons are concentrated in a small number of input modes, and we present an explicit scheme for doing so.

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By taking advantage of the quantum mechanical properties of micro- and mesoscopic systems, it is possible to increase the sensitivity of precision sensors beyond classical limitations [1–9]. Recently there has been increasing interest in understanding how such quantum metrological techniques can be used to enhance measurements that are spatially distributed [10–16], for applications such as phase imaging [10] and global-scale clock synchronization [17]. In this setting, the quantity of interest is often a linear combination of the results of a number of simultaneous measurements at different locations [18–20]. Examples of this problem, referred to as *distributed metrology*, are the inference of a field gradient or properties of the spatial fluctuations of a field. Even in single parameter estimation problems, a precise description of which nonclassical features of light are necessary for quantum enhancements is notoriously subtle [21–24]. In the multiparameter case, such an understanding has only very recently begun to emerge, with entanglement seemingly playing a crucial role. Moreover, there exist few examples of entangled states for achieving quantum-enhanced distributed metrology and those that do exist are fragile and difficult to create [18–20].

It is well known that linear-optical networks consisting purely of beam splitters and phase shifters can transform nonclassical but unentangled states into highly entangled states [25–31]. Given that linear networks are also quite easily implemented experimentally, much attention has been paid to applications of the entanglement they can generate, for example, towards demonstrations of quantum supremacy via boson sampling [32–36]. However, it is also clear that linear-optical networks cannot produce *arbitrary* quantum states at the output [37,38], given as input a particular quantum state. It is therefore natural to ask whether—for a particular application requiring a particular type of entanglement—linear-optical networks can or cannot provide a quantum advantage, and if so what types of input states they require.

Here we prove that entanglement is a necessary condition in quantum-enhanced distributed metrology, and we investigate whether or not entangled states generated by linear networks suffice for obtaining a quantum advantage, i.e., the Heisenberg limit [18–20]. Our results show that the answer to this important question depends crucially on how the initial (nonclassical but unentangled) resources are distributed amongst the network inputs. For well-distributed inputs, we show that the entangled states created via any linear network do not lead to the Heisenberg limit, unifying and generalizing several recent works [39–42]. More importantly, when the input photons are concentrated into a few modes, our bound (along with a specific example that verifies saturability) shows that linear networks can achieve the Heisenberg limit for distributed metrology, therefore identifying a large set of useful states for distributed quantum metrology in a linear network.

While our results have implications for distributed metrology in a rather general context, it is useful to consider the following concrete scenario depicted in Fig. 1(a). Imagine that there are d observers at different locations in space (which we will call "nodes"), and that each has the

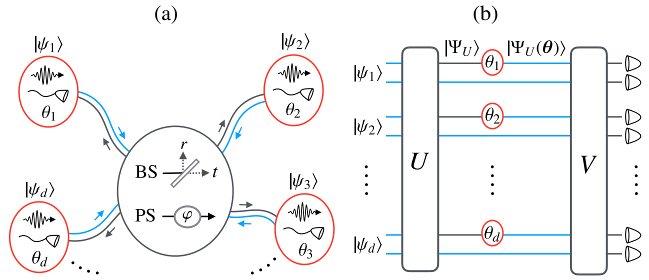

FIG. 1. (a) Spatial layout of a network of sensors designed to measure a linear combination of d spatially distributed phase shifts. Each node is equipped with a measurement device and a locally prepared nonclassical state, which can be sent to a central linear network prior to probing the local phase shifts and once again before making local measurements. (b) Formal representation of the situation described in (a): To make a fair comparison to the independent metrology of each phase, for which each node would require an additional reference mode in order to form an interferometer, each node is endowed with two input modes. These $2d$ modes are fed into a unitary U, and the phases are then probed, after which we allow further linear-optical processing (V) followed by local measurements.

ability to generate a nonclassical state $|\psi_j\rangle$ and use this state to measure a local phase shift θ_j ($j = 1, \ldots, d$). We quantify the resources of each state by $n_j^{\mathrm{LO}} \equiv \sqrt{\langle\psi_j|\hat{n}_j^2|\psi_j\rangle}$ [43], which we assume to be constrained locally, i.e., to be less than some constant n that does not scale with d. Furthermore, suppose that rather than ascertaining all of the individual phases, we wish to estimate a particular linear combination $q = \sum_j w_j\theta_j$ for some set of weights $\boldsymbol{w} = (w_1, \ldots, w_d)$ (this linear combination could be, e.g., an average or the overlap of the phases θ_j with some spatial mode). To make a comparison with the metrology of a single quantity straightforward, we normalize the weights such that $\max_j|w_j| = 1/d$. With this normalization, choosing the w_j all equal to each other recovers the spatial average of the fields $q = (1/d)\sum_{j=1}^{d}\theta_j$. If the observers *do not* have a network for sharing their quantum states, the best strategy for estimating q is for each of them to make the best possible estimate of θ_j and then compute q by sharing their results via classical communication. We assume that each individual node has access to a reference mode with no phase shift and that the measurement of θ_j can be made at the Heisenberg limit $\Delta\theta_j \sim 1/n_j^{\mathrm{LO}}$. If all of the phases θ_j (or at least a number of them that scales with d) contribute meaningfully to q, a notion captured formally by the requirement $|\boldsymbol{w}|^2 \sim 1/d$, then we will say that the weights are *well distributed*. Standard error analysis shows that in the case of well-distributed weights, the estimation strategy just described produces an error in the estimate of q scaling at best as $\Delta q \sim 1/(n\sqrt{d})$. This scaling can be shown to persist for any unentangled input state with locally constrained resources, even if the observers share classical correlations [44]. On the other hand, a fully quantum protocol using an optimal entangled state of all the modes, and subject to the same resource constraints, can achieve a scaling of $\Delta q = 1/(nd)$ [18–20], thus providing a collective quantum enhancement proportional to $1/\sqrt{d}$. In the context of the above scenario, our central question is as follows: If the observers are allowed to share their initially unentangled states through a central linear network, can they beat the aforementioned classical limit in measuring q for well-distributed weights, in the sense of enhanced scaling with respect to d? If so, we wish to know what kinds of unentangled input states we must send into a linear network to obtain output states that can be used to obtain a measurement precision scaling at the Heisenberg limit.

Our conclusion is that the ability of a linear network to generate entangled states that are useful for quantum metrology is strongly bounded by specific properties of the input states. Our results are most simply stated by first considering the case in which the jth node possesses a state with exactly n_j photons, which is just the amount of local resources in that node. In this case, and denoting by $\boldsymbol{n}$ the vector of input photon numbers n_j, we show that the metrological precision that can be achieved by utilizing a linear network satisfies

$$\Delta q \geq d|\boldsymbol{w}|^2/(2|\boldsymbol{n}|). \tag{1}$$

Two important conclusions can be drawn from Eq. (1). First, for locally constrained resources, for example, if each node has exactly n photons, then we have $\Delta q \gtrsim 1/(n\sqrt{d})$. Considering this scaling (i.e., ignoring any prefactors) and comparing to the classical scheme above, we conclude that linear networks cannot generate useful entangled states for the purposes of combining metrological resources across modes, thereby establishing a strong operational sense in which linear networks are classical. Second, for globally constrained resources such that the same total number of photons is distributed across a finite number of input ports, for example, just two, the bound reduces to $\Delta q \gtrsim 1/(nd)$. Thus, if the bound is tight (at least in the sense of scaling), then Heisenberg scaling for Δq can be achieved so long as the total number of photons employed is placed in a small number of modes ("small" meaning not scaling with d). By constructing an explicit measurement scheme, we confirm that the bound, and thus the Heisenberg scaling, can be saturated when the total number of photons is divided evenly between just two input modes.

Multiparameter quantum Cramér-Rao bound.—A formal analysis of the scenario depicted in Fig. 1(a) is summarized in Fig. 1(b). Note that we have now explicitly introduced one reference mode per phase θ_j, with index $d + j$. We assume that the input state is a product state $|\Psi\rangle = |\psi_1\rangle\ldots|\psi_d\rangle$ and that each $|\psi_j\rangle$ is itself a product state between the two input modes associated with phase θ_j,

$|\psi_j\rangle = |\varphi_j\rangle|\varphi_{d+j}\rangle$, so that $|\Psi\rangle = \otimes_{j=1}^{2d} |\varphi_j\rangle$. The state that the linear network produces is then denoted by $|\Psi_U\rangle = U|\Psi\rangle$. Defining $\hat{H} = \sum_j \theta_j \hat{n}_j$, interrogation of the phases θ_j then maps $|\Psi_U\rangle \to \exp(-i\hat{H})|\Psi_U\rangle \equiv |\Psi_U(\boldsymbol{\theta})\rangle$. We aim to infer the quantity $q = \sum_j w_j \theta_j$ by making local measurements on $|\Psi_U(\boldsymbol{\theta})\rangle$, preceded (if desired) by an additional linear-optical unitary V. The primary tool in our analysis is the multiparameter quantum Cramér-Rao bound, which states that a set of unbiased estimators Θ_j for the parameters θ_j satisfy [47]

$$\mathrm{cov}(\boldsymbol{\Theta}) \geq \mathcal{F}^{-1}. \tag{2}$$

Here the covariance matrix is defined by its matrix elements as $\mathrm{cov}(\boldsymbol{\Theta})_{jk} \equiv E[(\Theta_j - \theta_j)(\Theta_k - \theta_k)]$, where $E[X]$ is the expected value of the quantity X, and in this context the quantum Fisher information matrix $\mathcal{F}$ is defined by its matrix elements as

$$\mathcal{F}_{jk} \equiv 4(\langle\Psi_U|\hat{n}_j\hat{n}_k|\Psi_U\rangle - \langle\Psi_U|\hat{n}_j|\Psi_U\rangle\langle\Psi_U|\hat{n}_k|\Psi_U\rangle). \tag{3}$$

Using $Q = \sum_j w_j \Theta_j$ as an unbiased estimator of q, the uncertainty $\Delta^2 q \equiv E[(Q-q)^2] = \sum_{j,k} w_j \mathrm{cov}(\boldsymbol{\Theta})_{jk} w_k$ is bounded by Eq. (2) as [18–20,49]

$$\Delta^2 q \geq \sum_{j,k} w_j (\mathcal{F}^{-1})_{jk} w_k. \tag{4}$$

Note that $\mathcal{F}$ is a real, symmetric, positive *semidefinite* matrix, and it need not be invertible in general. However, in the case that $\mathcal{F}$ is not invertible, the estimation procedure will only succeed if $\boldsymbol{w}$ has vanishing projection onto the kernel of $\mathcal{F}$. In that case, $\mathcal{F}^{-1}$ should be interpreted as the inverse of $\mathcal{F}$ *after* projection onto the subspace spanned by eigenvectors with nonzero eigenvalues [20]. Thus we hereafter assume that $\mathcal{F}$ has been projected in this manner and therefore is positive definite (as opposed to positive semidefinite) and invertible. The bound in Eq. (4) is tight in the sense that it is guaranteed to be saturable for some choice of a measurement protocol, because the generators $\hat{n}_j$ commute with each other [16]. However, to obtain our result, it will be useful to further bound the rhs of Eq. (4) by something more easily computable for a general unitary U. To this end, we use the Cauchy-Schwarz inequality to write

$$\sum_{j,k} w_j (\mathcal{F}^{-1})_{jk} w_k \sum_{l,m} w_l \mathcal{F}_{lm} w_m \geq |\boldsymbol{w}|^4. \tag{5}$$

Defining $\mathcal{F}_{\boldsymbol{w}} \equiv \sum_{j,k} w_j \mathcal{F}_{jk} w_k$, we then obtain the bound

$$\Delta^2 q \geq \frac{|\boldsymbol{w}|^4}{\mathcal{F}_{\boldsymbol{w}}}. \tag{6}$$

Note that if we have at most n photons per mode after applying the unitary U, we can write $\mathcal{F}_{\boldsymbol{w}} \leq n^2 \sum_{j,k} |w_j||w_k| \leq n^2$, which for well-distributed weights ($|\boldsymbol{w}|^2 \sim 1/d$) gives $\Delta^2 q \gtrsim 1/(nd)^2$. Since nd is the maximum total number of photons, this coincides with the usual Heisenberg limit for measuring a single phase shift. However, whether or not $\mathcal{F}_{\boldsymbol{w}} \propto n^2$ can actually be achieved depends on the details of the states $|\Psi_U\rangle$ that a linear network can produce. As we demonstrate below, this in turn depends on the types of nonclassical states $|\psi_j\rangle$ that we have access to at the inputs.

Fisher information in linear-optical networks.—The next step in bounding Δq is to obtain a bound on $\mathcal{F}_{\boldsymbol{w}}$ in terms of the $2d$ input states. If we denote the annihilation operators for the $2d$ input (output) modes by a_j (b_j), then the action of the network is described by the relation $b_j^\dagger = \sum_k U_{jk} a_k^\dagger$, where U_{jk} are the elements of a $2d \times 2d$ unitary matrix. Since the phase shift θ_j is applied to the output mode b_j, the quantum Fisher information matrix can be computed by inserting the operators $\hat{n}_j = \hat{b}_j^\dagger \hat{b}_j$ into Eq. (3). Rewriting all operators in terms of the input mode operators and taking the expectation value in the initial product state $|\Psi\rangle = \otimes_{j=1}^{2d} |\varphi_j\rangle$, we obtain

$$\mathcal{F}_{jk} = 4 \sum_{l,m,r,s} U_{jl} U_{jm}^* U_{kr} U_{ks}^* (\langle \hat{a}_l^\dagger \hat{a}_m \hat{a}_r^\dagger \hat{a}_s\rangle - \langle \hat{a}_l^\dagger \hat{a}_m\rangle\langle \hat{a}_r^\dagger \hat{a}_s\rangle). \tag{7}$$

Later we will derive a bound on $\mathcal{F}_{\boldsymbol{w}}$ that holds for arbitrary separable input states, but it is useful to first consider the simpler situation in which all modes are initialized in Fock states, with the jth mode having photon number n_j. In this case $\mathcal{F}_{\boldsymbol{w}}$ reduces to

$$\mathcal{F}_{\boldsymbol{w}} = 4 \sum_{j,k} \sum_{r \neq s} n_s (n_r + 1)(U_{js} w_j U_{jr}^*)(U_{kr} w_k U_{ks}^*). \tag{8}$$

The restriction $r \neq s$ can be removed if we replace the equality with "$\leq$," because the additional term given by $r = s$ is non-negative. Defining Hermitian matrices $\mathcal{S}$ and $\mathcal{N}$ such that $\mathcal{S}_{rs} = \sum_j U_{js} w_j U_{jr}^*$ and $\mathcal{N}_{rs} = \delta_{rs} n_r$, Eq. (8) then takes the following compact form:

$$\mathcal{F}_{\boldsymbol{w}} \leq 4\mathrm{Tr}[\mathcal{N}\mathcal{S}(\mathcal{N}+1)\mathcal{S}]. \tag{9}$$

Standard trace inequalities [50,51] can now be used to write $\mathcal{F}_{\boldsymbol{w}} \leq 4\sum_j \mathrm{eigs}(\mathcal{N})_j [\mathrm{eigs}(\mathcal{N})_j + 1] \mathrm{eigs}(\mathcal{S})_j^2$, where $\mathrm{eigs}(M)$ is a list of the eigenvalues of the matrix M, sorted by absolute value. The eigenvalues of $\mathcal{N}$ are clearly n_j, and because $\mathcal{S}$ is a unitary transformation of the matrix $\mathrm{diag}(\boldsymbol{w})$, it has eigenvalues w_j. Therefore, remembering that $\max_j |w_j| = 1/d$, we have

$$\mathcal{F}_{\boldsymbol{w}} \leq 4|\boldsymbol{n}|^2/d^2. \tag{10}$$

Plugging this bound on the quantum Fisher information into Eq. (6) and taking the square root of both sides, we obtain Eq. (1). As mentioned earlier, the most important consequences of this bound are the following: (i) For locally constrained resources—that is, when $n_j \leq n$ such that n that does not scale with d—we have $\Delta q \gtrsim 1/(n\sqrt{d})$; thus, the bound proves that a linear-optical network endowed with locally constrained resources cannot improve upon the scaling of a classical scheme in which the estimates of all d phases are made independently. (ii) For globally constrained resources such that the total number ($N = nd$) of photons are all placed in a finite number of input ports, and assuming the bound can still be saturated, then Heisenberg scaling $\Delta q \sim 1/(nd)$ can be achieved. With considerably more effort, the bound in Eq. (1) can be generalized to the case of arbitrary separable input states. However, before presenting this generalization, we first give an explicit protocol that saturates the above bound in case (ii), thereby obtaining Heisenberg scaling for distributed metrology in a linear-optical network by concentrating the resources in a few modes [52].

Explicit protocol for concentrated-resource states.—We now show that Heisenberg scaling $\Delta q \sim 1/(nd)$ can be achieved if the total number of photons $N = nd$ is split evenly between just two input modes; i.e., $|\Psi\rangle = |N/2\rangle \otimes |N/2\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle$. The scheme can be viewed as a generalization of the "twin-Fock-state" proposal in Ref. [2] for the task of distributed quantum metrology. The basic idea is to find a unitary transformation U that distributes the input twin Fock state between the various modes in a way that explicitly encodes the weights w_j. Upon evolution through the network in Fig. 1(b) and subsequent measurement of an operator $\hat{O}$, the sensitivity of estimating q can be obtained through the standard error propagation,

$$\Delta q = \frac{\sqrt{\langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2}}{|\partial \langle \hat{O} \rangle / \partial q|}. \tag{11}$$

Here, the expectation values are taken with respect to the state $\hat{V} \exp(-i\hat{H})\hat{U}|\Psi\rangle$ at the output of the network. Choosing $V = U^\dagger$ [53] and measuring the observable $\hat{O} = |\Psi\rangle\langle\Psi|$, which can be accomplished with photon-number-resolving detectors at the $2d$ output ports, we have $\langle O \rangle = \langle O^2 \rangle = |\langle \Psi_U | e^{-i\hat{H}} | \Psi_U \rangle|^2$, which for small q becomes

$$\langle \hat{O} \rangle \approx 1 - (\langle \Psi_U | \hat{H}^2 | \Psi_U \rangle - \langle \Psi_U | \hat{H} | \Psi_U \rangle^2). \tag{12}$$

By choosing $U_{i,1} = U_{i+d,1} = \sqrt{|w_i|/2}$, and $U_{i,2} = -U_{i+d,2} = w_i/\sqrt{2|w_i|}$ for $i = 1, 2, \ldots, d$, we can encode each phase with its corresponding weight, obtaining

$$\langle \hat{O} \rangle \approx 1 - \frac{N(N+2)}{8} q^2. \tag{13}$$

Plugging this result into Eq. (11), we obtain an uncertainty in estimating q of

$$\Delta q = \frac{2}{\sqrt{2N(N+2)}}, \tag{14}$$

which exhibits Heisenberg scaling. Though we have used a $2d$ unitary matrix for estimating d phase shifts with d reference ports, it is in fact possible to simplify the scheme to include only one reference port while maintaining Heisenberg scaling (compare Fig. 2 for an example involving two phases). In implementing the scheme, we require Fock-state inputs [54], a linear-optical network [55,56], and photon-number-resolving detectors [57].

A general bound for arbitrary separable states.—Equation (1) was derived assuming Fock-state inputs; here we show that a similar bound can be derived for arbitrary separable input states [Eq. (17) below]. Though this more general bound depends on different specific properties of the input states, it shares with Eq. (1) the important characteristic that for fixed *local* resources, the minimal value of Δq obeys the classical scaling $\sim 1/\sqrt{d}$.

For an arbitrary separable input state ρ, the correct way to calculate the Fisher information depends on whether external phase references are assumed to be available in the measurement protocol [58]. Here we make no such assumption, and therefore the Fisher information should be computed with respect to a phase averaged state,

$$\varrho = \frac{1}{2\pi} \int_0^{2\pi} d\theta \exp(i\theta\hat{N}) \rho \exp(-i\theta\hat{N}), \tag{15}$$

where $\hat{N} = \sum_j \hat{n}_j$ and θ is a global phase. However, because the Fisher information is convex and we need only obtain an upper bound, it is sufficient to calculate an

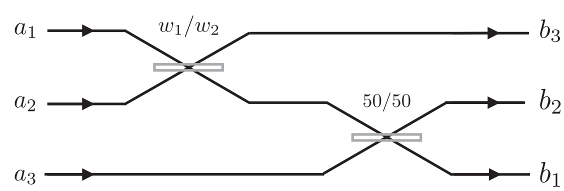

FIG. 2. Illustration of a simple linear network (consisting of only beam splitters) that implements the unitary U described for the twin-Fock-state approach to measuring q with Heisenberg scaling. Here we can measure $q = w_1\theta_1 + w_2\theta_2$ for $w_1 > 0$ and $w_2 > 0$ with precision $1/\sqrt{2n(n+1)}$, using only a single reference port. The input state is $|n, n, 0\rangle$, and the ratio above each beam splitter is the transmission/reflection rate. The "classical" estimation strategy for measuring q yields the precision $1/\sqrt{n(n+2)}$, a factor of $\sqrt{2}$ larger for large n, using the same total number of photons.

upper bound on $\mathcal{F}$ for an arbitrary pure product state $|\Psi\rangle$, from which we can infer an upper bound on $\mathcal{F}$ for the separable density matrix ϱ. Deriving a bound for an arbitrary pure state is still rather complex, and we defer a detailed analysis to the Supplemental Material [44]. Here, we simply quote the final result,

$$\mathcal{F}_{\boldsymbol{w}} \leq \frac{A}{d} + B|\boldsymbol{w}|^2. \tag{16}$$

We note that this bound is not tight (that is, it cannot necessarily be achieved). In Eq. (16), A and B depend only on moments of the input states (specifically, $\langle \hat{a}_j \rangle$, $\langle \hat{a}_j^\dagger \hat{a}_j \rangle$, $\langle \hat{a}_j \hat{a}_j \rangle$, $\langle \hat{a}_j^\dagger \hat{a}_j^\dagger \hat{a}_j \rangle$, and $\langle \hat{a}_j^\dagger \hat{a}_j^\dagger \hat{a}_j \hat{a}_j \rangle$). Their exact form can be found in the Supplemental Material [44], but for our purposes it suffices only to know that they obey the bound $A + B < C^2 \max_j m_j$, with $m_j \equiv \langle (a_j^\dagger a_j)^2 \rangle$ and $C = 20$. Recalling that for well-distributed weights we have $|\boldsymbol{w}|^2 \sim 1/d$, plugging Eq. (16) into Eq. (6), and defining $\langle n^2 \rangle_{\max} \equiv \max_j m_j$, we obtain

$$\Delta q \geq \frac{1}{C\sqrt{d \langle n^2 \rangle_{\max}}}. \tag{17}$$

If the resources are constrained locally rather than globally, such that $\langle n^2 \rangle_{\max}$ is independent of d, it follows that quantum metrology using linear-optical networks and separable input states cannot improve upon the classical scaling $\sim 1/\sqrt{d}$.

In addition to elucidating the resource requirements for quantum metrology with linear networks, the above results are also interesting from the point of view of quantifying nonclassicality [59]. Linear networks have the ability to reversibly transform nonclassical but unentangled states into entangled ones, thus providing a route to quantifying nonclassicality using measures of entanglement [28–30]. Our results suggest that it may be useful to refine this approach by using more stringent operational measures of entanglement such as the ability of the entangled state to realize quantum-enhanced metrology. These considerations suggest that it may be interesting to investigate how a quantitative measure of nonclassicality might be obtained via quantum metrology and more generally to explore the classes of states that are preserved under linear networks.

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[1] C. M. Caves, Phys. Rev. D **23**, 1693 (1981).
[2] M. J. Holland and K. Burnett, Phys. Rev. Lett. **71**, 1355 (1993).
[3] V. Giovannetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004).
[4] V. Giovannetti, S. Lloyd, and L. Maccone, Phys. Rev. Lett. **96**, 010401 (2006).
[5] R. A. Campos, C. C. Gerry, and A. Benmoussa, Phys. Rev. A **68**, 023810 (2003).
[6] L. Pezzé and A. Smerzi, Phys. Rev. Lett. **100**, 073601 (2008).
[7] J. J. Cooper, D. W. Hallwood, and J. A. Dunningham, Phys. Rev. A **81**, 043624 (2010).
[8] R. Birrittella, J. Mimih, and C. C. Gerry, Phys. Rev. A **86**, 063828 (2012).
[9] C. N. Gagatsos, D. Branford, and A. Datta, Phys. Rev. A **94**, 042342 (2016).
[10] P. C. Humphreys, M. Barbieri, A. Datta, and I. A. Walmsley, Phys. Rev. Lett. **111**, 070403 (2013).
[11] P. A. Knott, T. J. Proctor, A. J. Hayes, J. F. Ralph, P. Kok, and J. A. Dunningham, Phys. Rev. A **94**, 062312 (2016).
[12] M. Szczykulska, T. Baumgratz, and A. Datta, Adv. Phys. X **1**, 621 (2016).
[13] J. Liu, X.-M. Lu, Z. Sun, and X. Wang, J. Phys. A **49**, 115302 (2016).
[14] T. Baumgratz and A. Datta, Phys. Rev. Lett. **116**, 030801 (2016).
[15] M. A. Ciampini, N. Spagnolo, C. Vitelli, L. Pezzè, A. Smerzi, and F. Sciarrino, Sci. Rep. **6**, 28881 (2016).
[16] L. Pezzè, M. A. Ciampini, N. Spagnolo, P. C. Humphreys, A. Datta, I. A. Walmsley, M. Barbieri, F. Sciarrino, and A. Smerzi, Phys. Rev. Lett. **119**, 130504 (2017).
[17] P. Komar, E. M. Kessler, M. Bishof, L. Jiang, A. S. Sørensen, J. Ye, and M. D. Lukin, Nat. Phys. **10**, 582 (2014).
[18] Z. Eldredge, M. Foss-Feig, S. L. Rolston, and A. V. Gorshkov, Phys. Rev. A **97**, 042337 (2018).
[19] T. J. Proctor, P. A. Knott, and J. A. Dunningham, Phys. Rev. Lett. **120**, 080501 (2018).
[20] T. Proctor, P. Knott, and J. Dunningham, arXiv:1702.04271.
[21] R. Gaiba and M. G. Paris, Phys. Lett. A **373**, 934 (2009).
[22] J. Sahota and N. Quesada, Phys. Rev. A **91**, 013808 (2015).
[23] N. Friis, M. Skotiniotis, I. Fuentes, and W. Dür, Phys. Rev. A **92**, 022106 (2015).
[24] D. Braun, G. Adesso, F. Benatti, R. Floreanini, U. Marzolino, M. W. Mitchell, and S. Pirandola, arXiv:1701.05152.
[25] M. S. Kim, W. Son, V. Bužek, and P. L. Knight, Phys. Rev. A **65**, 032323 (2002).
[26] X. B. Wang, Phys. Rev. A **66**, 024303 (2002).
[27] M. M. Wolf, J. Eisert, and M. B. Plenio, Phys. Rev. Lett. **90**, 047904 (2003).
[28] J. K. Asbóth, J. Calsamiglia, and H. Ritsch, Phys. Rev. Lett. **94**, 173602 (2005).
[29] W. Vogel and J. Sperling, Phys. Rev. A **89**, 052302 (2014).
[30] N. Killoran, F. E. S. Steinhoff, and M. B. Plenio, Phys. Rev. Lett. **116**, 080402 (2016).
[31] W. Ge, M. E. Tasgin, and M. S. Zubairy, Phys. Rev. A **92**, 052328 (2015).
[32] S. Aaronson and A. Arkhipov, Theor. Comput. **9**, 143 (2013).

[33] M. Tillmann, B. Dakic, R. Heilmann, S. Nolte, A. Szameit, and P. Walther, Nat. Photonics **7**, 540 (2013).

[34] M. A. Broome, A. Fedrizzi, S. Rahimi-Keshari, J. Dove, S. Aaronson, T. C. Ralph, and A. G. White, Science **339**, 794 (2013).

[35] J. B. Spring, B. J. Metcalf, P. C. Humphreys, W. S. Kolthammer, X.-M. Jin, M. Barbieri, A. Datta, N. Thomas-Peter, N. K. Langford, D. Kundys, J. C. Gates, B. J. Smith, P. G. R. Smith, and I. A. Walmsley, Science **339**, 798 (2013).

[36] A. Crespi, R. Osellame, R. Ramponi, D. J. Brod, E. F. Galvao, N. Spagnolo, C. Vitelli, E. Maiorino, P. Mataloni, and F. Sciarrino, Nat. Photonics **7**, 545 (2013).

[37] P. Kok and S. L. Braunstein, Phys. Rev. A **62**, 064301 (2000).

[38] P. Kok, W. J. Munro, K. Nemoto, T. C. Ralph, J. P. Dowling, and G. J. Milburn, Rev. Mod. Phys. **79**, 135 (2007).

[39] N. Spagnolo, L. Aparo, C. Vitelli, A. Crespi, R. Ramponi, R. Osellame, P. Mataloni, and F. Sciarrino, Sci. Rep. **2**, 862 (2012).

[40] K. R. Motes, J. P. Olson, E. J. Rabeaux, J. P. Dowling, S. J. Olson, and P. P. Rohde, Phys. Rev. Lett. **114**, 170802 (2015).

[41] Z.-E. Su, Y. Li, P. P. Rohde, H.-L. Huang, X.-L. Wang, L. Li, N.-L. Liu, J. P. Dowling, C.-Y. Lu, and J.-W. Pan, Phys. Rev. Lett. **119**, 080502 (2017).

[42] O. Zimmermann, Bachelor thesis, Cornell University, arXiv:1710.03805.

[43] H. F. Hofmann, Phys. Rev. A **79**, 033822 (2009).

[44] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.121.043604 for detailed derivations, which includes Refs. [45,46].

[45] D. Petz and C. Ghinea, Quantum Probab. Relat. Top. **1**, 261 (2011).

[46] P. J. D. Crowley, A. Datta, M. Barbieri, and I. A. Walmsley, Phys. Rev. A **89**, 023845 (2014).

[47] This inequality is shorthand for the asymptotic per-trial covariance. Strictly speaking, for ν trials of an estimation procedure, we have $\mathrm{cov}(\mathbf{\Theta}) \geq (\nu\mathcal{F})^{-1}$. This bound may also apply to a global-parameter estimation when the global unknown parameter becomes local after many repeated measurements [48].

[48] S. M. Kay, *Fundamentals of Statistical Signal Processing* (PTR Prentice Hall, Englewood Cliffs, NJ, 1993).

[49] M. G. A. Paris, Int. J. Quantum. Inform. **07**, 125 (2009).

[50] D. Petz, Banach Cent. Pub. **30**, 287 (1994).

[51] L. Mirsky, Monatshefte fur Mathematik **79**, 303 (1975).

[52] We note that if all of the photons are placed into a *single* port, then the bound in Eq. (10) can be tightened to $\mathcal{F}_w \leq 4N/d^2$, which leads to the standard quantum limit in estimating q. See the Supplemental Material [44] for the proof.

[53] J. P. Olson, K. R. Motes, P. M. Birchall, N. M. Studer, M. LaBorde, T. Moulder, P. P. Rohde, and J. P. Dowling, Phys. Rev. A **96**, 013810 (2017).

[54] M. Cooper, L. J. Wright, C. Söller, and B. J. Smith, Opt. Express **21**, 5309 (2013).

[55] M. Reck, A. Zeilinger, H. J. Bernstein, and P. Bertani, Phys. Rev. Lett. **73**, 58 (1994).

[56] M. Żukowski, A. Zeilinger, and M. A. Horne, Phys. Rev. A **55**, 2564 (1997).

[57] B. Kardynał, Z. Yuan, and A. Shields, Nat. Photonics **2**, 425 (2008).

[58] M. Jarzyna and R. Demkowicz-Dobrzański, Phys. Rev. A **85**, 011801(R) (2012).

[59] M. Hillery, Phys. Rev. A **35**, 725 (1987).