

Viscoelastic Pipe Flow is Linearly Unstable

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Newtonian pipe flow is known to be linearly stable at all Reynolds numbers. We report, for the first time, a linear instability of pressure-driven pipe flow of a viscoelastic fluid, obeying the Oldroyd-B constitutive equation commonly used to model dilute polymer solutions. The instability is shown to exist at Reynolds numbers significantly lower than those at which transition to turbulence is typically observed for Newtonian pipe flow. Our results qualitatively explain experimental observations of transition to turbulence in pipe flow of dilute polymer solutions at flow rates where Newtonian turbulence is absent. The instability discussed here should form the first stage in a hitherto unexplored dynamical pathway to turbulence in polymer solutions. An analogous instability exists for plane Poiseuille flow.

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Since the discovery by Toms that the addition of small amounts of a high molecular weight polymer to a Newtonian fluid significantly reduces the pressure drop in turbulent pipe flow [1,2], turbulent flows of dilute polymer solutions have been widely studied for both their fundamental and industrial importance [1–6]. Understanding the transition to turbulence in shearing flows of viscoelastic fluids, including dilute polymer solutions, is thus crucial [7,8]. A central question underlying this field of study is if the laminar state is stable to infinitesimal amplitude perturbations [7–9].

Newtonian pipe flow is known to be linearly stable at all Reynolds numbers (Re) [10–14]. By carefully minimizing external perturbations, laminar flow has been maintained in experiments up to $Re \sim 100000$ [15]; in contrast, when forced with finite amplitude disturbances, transition occurs around a Re of 2000 [16–20]. Theoretically, this subcritical scenario is explained by the appearance, above a threshold Re , of nontrivial three-dimensional solutions of the Navier-Stokes equations (termed exact coherent states) which are disconnected from the laminar state [14,19,21]. Rectilinear shearing flows, including pipe flow, of dilute polymer solutions are also believed to be linearly stable at all Weissenberg numbers (Wi) in the inertialess limit ($Re = 0$) [9,22–24], Wi here being the ratio of the polymer relaxation time to the flow time scale. A nonlinear mechanism has been proposed for transition to (elastic) turbulence in such flows, where an initial finite amplitude perturbation induces curved streamlines, which then become unstable to a hoop-stress-driven elastic instability that operates at linear order in canonical curvilinear geometries [8,25–29]. Theoretical work explaining transition, and turbulent drag reduction, at finite Re and Wi has focused on the modification of the Newtonian scenario, by mapping the domain of existence of the exact coherent states in the Re - Wi plane [7,30–32]. That these finite

amplitude solutions do not exist above a critical Wi , for fixed Re , is indicative of a distinct transition mechanism at larger Wi [7,33]. A separate line of work has focused on the linear transient growth of disturbances from a stable laminar state [34–36]. The theoretical viewpoint is thus rooted in the (assumed) linear stability of the laminar state everywhere in the Re - Wi plane [7–9].

There have, however, been scattered observations that point to a linear instability in pipe flow experiments involving dilute polymer solutions. In a series of experiments in the 1960s and 1970s, transition to turbulence was observed in dilute polymer solutions, at Reynolds numbers much lower than the Newtonian threshold by several groups, the phenomenon being dubbed “early turbulence” [37–46]. Later, Draad *et al.* [47] observed an order of magnitude reduction in the natural (unforced) transition Re for a polymer solution. More recently, Samanta *et al.* [48] studied transition in polyacrylamide solutions, in smaller diameter pipes, thereby accessing higher Weissenberg numbers. In a 4 mm diameter pipe, the transition process for concentrations lesser than 200 ppm was analogous to the Newtonian one with forced and natural transitions occurring at disparate Reynolds numbers. In sharp contrast, for the 500 ppm solution, the transition occurred at $Re \sim 800$ independent of the perturbation amplitude. Further, spatially localized structures (puffs), characteristic of the bistability associated with the Newtonian subcritical transition [49–51], were absent. Subsequently, this new transitional pathway, connecting the laminar state to a novel elastoinertial turbulent state, has been demonstrated over a much wider parameter range [52].

Although a linear instability has occasionally been speculated upon [7,38], the consensus in the field assumes otherwise [8,27,29,48,53]. This assumption is sometimes stated as a fact, for instance in Refs. [27,29,53]. Even the

authors of Ref. [48], despite observing signatures of a linear instability, expect the transition to be governed by a non-linear process. Contrary to this widely held view, and for the first time, we demonstrate a linear instability in the Re-Wi plane. Our results offer the first theoretical explanation of observations of early turbulence spanning decades [37,38,48,52] and point to a novel pathway to turbulence in viscoelastic pipe flow which has thus far remained completely unexplored in earlier theoretical studies.

The governing system of equations for an incompressible viscoelastic fluid (in nondimensional form) is

$$\text{Re} \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{u} = -\nabla p + \frac{1-\beta}{\text{Wi}} \nabla \cdot \mathbf{A}_p + \beta \nabla^2 \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0, \quad (1)$$

where \mathbf{u} , p , and \mathbf{A}_p are the velocity field, pressure, and the elastic stress tensor, respectively. The relevant nondimensional parameters are $\beta = \mu_s / (\mu_p + \mu_s)$, $\text{Wi} = (U_s \tau / a)$, and $\text{Re} = \rho U_s a / (\mu_s + \mu_p)$ where μ_s and μ_p are the solvent and polymer contributions to the viscosity, τ the relaxation time of the polymer molecule, ρ the density of the fluid, a the pipe radius, and U_s (the centerline velocity) the imposed velocity scale. (The Deborah number (De) is sometimes used in the literature analogous to Wi defined here [54–56].) The elastic stress is assumed to be governed by the Oldroyd-B constitutive equation, corresponding to polymer molecules in the solution being modeled as non-interacting Hookean dumbbells. This gives $\mathbf{A}_p \propto \langle \mathbf{R}\mathbf{R} \rangle$, where \mathbf{R} is the dumbbell end-to-end vector and $\langle \dots \rangle$ denotes a configurational average. The affine deformation of \mathbf{R} , together with linear relaxation in a time τ , leads to the following equation for \mathbf{A}_p [57]:

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{A}_p - \mathbf{A}_p \cdot \nabla \mathbf{u} - (\nabla \mathbf{u})^\dagger \cdot \mathbf{A}_p = -\frac{\mathbf{A}_p - \mathbf{I}}{\text{Wi}}. \quad (2)$$

The Oldroyd-B model predicts a shear-rate independent viscosity and first normal stress coefficient in viscometric flows [57]. It has been shown to reproduce observations of linear instabilities in polymer solutions in various curvilinear [26] and extensional flows [58] as well as the inertialess nonlinear instability in rectilinear shearing flows [8,28], and is thus appropriate for a first effort. For $\beta = 0$, Eqs. (1) and (2) reduce to the Upper Convected Maxwell (UCM) model, with no solvent stress contribution.

The laminar pipe flow profile for an Oldroyd-B fluid is the same as the Newtonian one, $U = 1 - r^2$. An associated first normal stress difference, $N_1 = 8r^2 \text{Wi}^2$, arises owing to the polymer molecules being stretched and aligned with the flow. Assuming infinitesimal perturbations, $\mathbf{u} = U + \mathbf{u}'$, $\mathbf{A}_p = \mathbf{A} + \mathbf{a}'$, $p = p_0 + p'$, of the normal mode form, $f' = \hat{f}(r) e^{[ik(z-ct) + im\theta]}$ (where k and m are the axial and azimuthal wave numbers, respectively), and

linearizing about the aforementioned base-state, one obtains the following eigenvalue problem for pipe flow,

$$\mathcal{L} \hat{f} = c \hat{f},$$

such that $c = c_r + ic_i \equiv c(\text{Re}, \text{Wi}, k, m, \beta)$ where c_r is the wave speed and c_i the growth rate; $c_i > 0$ implies exponentially growing normal modes. We only consider axisymmetric perturbations ($m = 0$) in this Letter, since nonaxisymmetric disturbances were found to be stable over the parameter range considered. Two different numerical methods are used to solve the eigenvalue problem: a spectral collocation method in which the perturbation fields are expanded in terms of Chebyshev polynomials [59] and a shooting method that numerically integrates the governing equations and iterates over the eigenvalue c (with a Newton-Raphson procedure) in order to satisfy the boundary conditions [13]. We have verified our numerical schemes by reproducing earlier stability results for plane Poiseuille flow of an Oldroyd-B fluid [60,61] and for Newtonian pipe flow [13]. To avoid spurious modes, convergence was checked, for both eigenvalues and eigenfunctions, with respect to N (the number of Chebyshev polynomials in the spectral expansion) as well as against the shooting method. The only prior work on linear stability of viscoelastic pipe flow neglected the convected derivative in Eq. (2) and hence is of restricted validity [62].

The eigenvalue spectrum in Fig. 1, for $\text{Re} = 800$, $\text{Wi} = 65$, $\beta = 0.65$, $k = 1$, shows a single unstable mode, multiple damped discrete modes, and a pair of continuous spectra (these appear as balloons due to the finite discretization). The continuous spectrum eigenvalues are given by $c = U - i/(k\text{Wi})$ and $c = U - i/(\beta k\text{Wi})$ and correspond to singular modes whose decay rates are set by the polymeric stress relaxation [22,60,61,63–65]. The unstable mode is an axisymmetric center mode propagating at a speed close to the base-state maximum. Figure 2 shows the associated perturbation velocity and polymer force density ($\nabla \cdot \mathbf{a}'$) fields. The polymer force field is localized near the

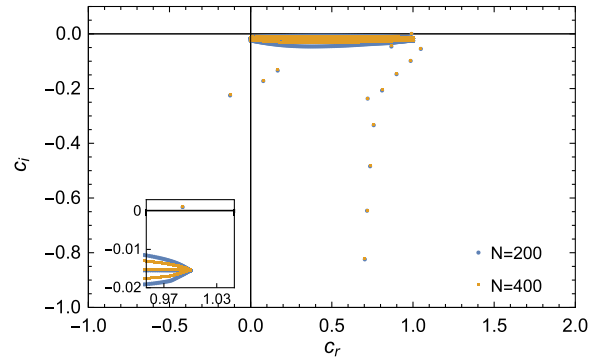


FIG. 1. Eigenspectrum for pipe flow of an Oldroyd-B fluid for $\text{Re} = 800$, $\text{Wi} = 65$, $\beta = 0.65$, and $k = 1$ (for $N = 200$ and 400); the inset zooms into the region around the unstable mode.

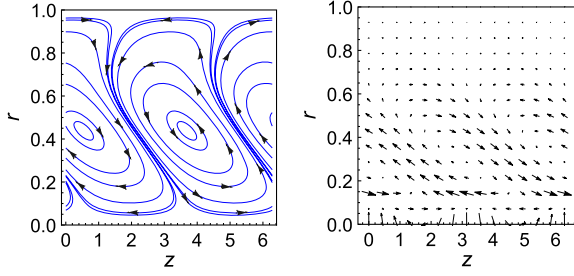


FIG. 2. Perturbation velocity (left) and polymer force (right) fields for the unstable mode for $Re = 800$, $Wi = 65$, $\beta = 0.6$, and $k = 1$.

centerline and reinforces the velocity field, leading to the instability. The radial structure of the polymer forcing is reminiscent of recent simulations of elastoinertial turbulence (for plane Poiseuille flow) wherein regions of high polymer stretch, localized in the gradient direction, were observed [48,53,66].

In the limit $Re, Wi \rightarrow \infty$ with $Wi/Re^{1/2}$ (and k) fixed, the unstable eigenfunctions become increasingly localized in a boundary layer of $\mathcal{O}(Re^{-1/4})$ around the centerline (Fig. 3). Viscous diffusion balances inertia in this boundary layer, analogous to a Newtonian center mode [10], and for the perturbation polymeric stress to stay comparable requires $Wi \sim \mathcal{O}(Re^{1/2})$. The instability thus requires a balance of inertia, viscous, and elastic polymer stresses close to the centerline. The centerline localization is in contrast to the original Newtonian and the elastically modified Tollmien-Schlichting instability for plane Poiseuille flow, where the eigenfunction is localized near the channel walls for large Re [13,60,61].

For a given polymer solution, the elasticity number $E = (Wi/Re)$ and β are fixed, and independent of the imposed flow velocity. Hence, in Fig. 4, we characterize the instability in terms of a critical Reynolds number, Re_c , as a function of E and β . At a given E and β , Re_c is found by minimizing the threshold Re over all k . Both branches of the neutral curve in the $Re-k$ plane (see Fig. 4, inset) show the expected long wavelength scaling, $Re \sim \mathcal{O}(1/k)$, for

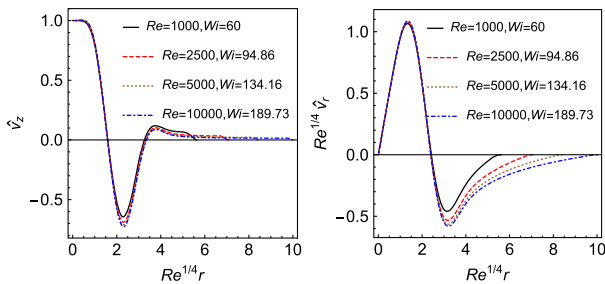


FIG. 3. The unstable center-mode eigenfunctions for the axial velocity (left) and radial velocity (right) in scaled boundary-layer coordinates in the limit $Re \rightarrow \infty$ and $Wi \rightarrow \infty$ for a fixed $Wi/Re^{1/2}$ ($k = 1$ and $\beta = 0.5$).

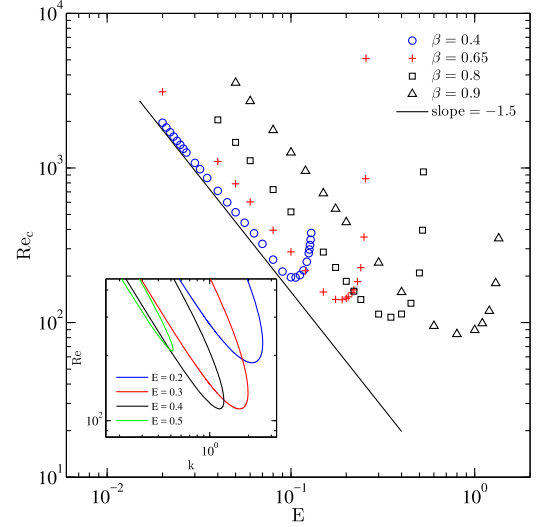


FIG. 4. The critical Reynolds number, Re_c , as a function of E for different viscosity ratios, β . The inset shows neutral curves in the $Re-k$ plane for $\beta = 0.8$ for different E .

$k \rightarrow 0$. Further, Fig. 4 shows that, at a fixed β , the left branch of the $Re_c - E$ curve is such that $Re_c \propto E^{-3/2}$ with $k_c \propto E^{-1/2}$ (not shown). The right branch is almost vertical and hence for a given β there appears to be an E_{crit} such that the instability does not exist for any Re for $E > E_{crit}$, while for $E < E_{crit}$, the laminar state is always unstable at large enough Re . As β is increased, the minima in the $Re_c - E$ curves shift to higher E and lower Re_c and the unstable region in the $Re_c - E$ plane increases in extent.

Figure 5 shows the expected absence of the instability in the Newtonian limit and its surprising absence in the UCM

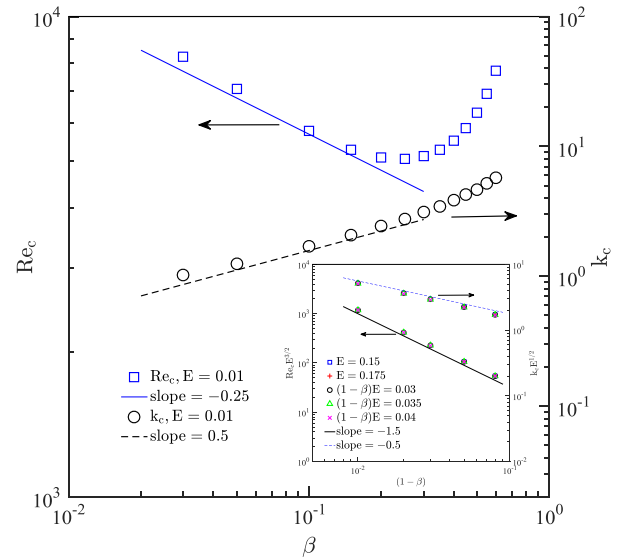


FIG. 5. The critical Reynolds number, Re_c , as a function of the viscosity ratio β for $E = 0.01$. The inset shows the scaling behavior in the dual limit $E \rightarrow 0$, $\beta \rightarrow 1$.

limit. In the Newtonian limit ($\beta \rightarrow 1$), $\text{Re}_c \propto (1 - \beta)^{-3/2}$ and $k_c \propto (1 - \beta)^{-1/2}$. The β and E scalings above may be combined in the dual limit $E(1 - \beta) \ll 1$ and $\beta \rightarrow 1$, so that $\text{Re}_c \propto [E(1 - \beta)]^{-3/2}$ and $k_c \propto [E(1 - \beta)]^{-1/2}$ (Fig. 5 inset). Thus, the instability survives provided $E \propto (1 - \beta)^{-1}$, which ensures that the perturbation polymer stress remains of order unity. For a given Re this implies that the minimum Wi for which the instability exists diverges in the Newtonian limit. In all approaches to the Newtonian limit, the axial wavelength (k_c^{-1}) of the center mode becomes comparable to the centerline boundary layer thickness [$\mathcal{O}(\epsilon^{1/2})$, ϵ being the relevant small parameter]. For $k_c \gg 1$, $\text{Re}_c \propto k_c^3$ ensures a balance between inertia and viscous stresses and, along with $k_c \propto \epsilon^{-1/2}$, predicts $\text{Re}_c \propto \epsilon^{-3/2}$, consistent with the observed scalings.

In contrast to Fig. 5 which shows an asymptotic scaling for Re_c , for $\beta \rightarrow 1$, an analysis of the spectrum for fixed Re , E , and k , shows that the instability doesn't persist until the Newtonian limit. Instead, the unstable center mode becomes stable at a finite $(1 - \beta)$, eventually falling off into the continuous spectrum with the polymer force field becoming singular. In the UCM limit ($\beta \rightarrow 0$), Re_c is shown to diverge as $\beta^{-1/4}$ in Fig. 5, for $E = 0.01$, but the associated critical wave number (k_c) decreases as $\beta^{1/2}$ for small β . The absence of the instability for $\beta = 0$ reinforces the idea that all three physical effects (inertia, the viscous solvent stress, and the elastic polymeric stress) are essential for the center-mode instability. This is in contrast to the expectation that the solvent stress generally plays a stabilizing role in elastic instabilities [26,61].

The instability is predicted to exist over a wide range of Re . The regime $\text{Re}_c \sim \mathcal{O}(100)$, $E \sim \mathcal{O}(1)$ is achievable in microfluidic devices [41,67]. For (E, β) values such that Re_c is $\mathcal{O}(2000)$ or greater, pertinent to macroscopic geometries, the subcritical Newtonian transition might mask the linear instability unless external perturbations are carefully minimized. A natural transition Re of around 8000 was reported by Draad *et al.* [47] for a 20 ppm solution of partially hydrolyzed polyacrylamide in demineralized water ($\beta \sim 0.1$ based on the zero shear viscosity), as opposed to the much higher transition Re of 60000 for Newtonian fluids for their experimental facility. Our calculations do yield an unstable mode at the corresponding Re and β , for $E = 0.01$, although the strong shear thinning exhibited by the solution prevents a quantitative comparison. Similar observations of a significantly lower natural transition Re have been reported for dilute solutions of polyethylene oxide [68]. The instability also qualitatively explains the observations of “early turbulence” in Refs. [37–39,41,42,45]. For the 500 ppm polyacrylamide solution used by Samanta *et al.* [48], $\beta = 0.65$. The Re_c for the instability at this β is well below that at which Newtonian turbulence sets in. This is in qualitative agreement with the experiment where transition was reported at $\text{Re} \sim 800$. However, the minimum E required for the

center-mode instability is around 0.05, which is an order of magnitude larger than the experimentally reported value of 0.004, based on a measurement of the relaxation time using a capillary break-up elongational rheometer (CaBER). This discrepancy may be attributed to the known difficulty in associating the time inferred from CaBER measurements to the relaxation rate relevant to the Oldroyd-B model [67,69–72]. The $(\text{Re}-\text{Wi}-\beta)$ dependent threshold of the center-mode instability calls for a reexamination of the expectation that early transition, even in the absence of finite amplitude perturbations, is governed by a critical Wi , regardless of Re [41,48]. We hope that the first theoretical evidence for the laminar state being unstable provided by this Letter will motivate the search for a decisive experimental demonstration.

Observations of pressure-driven flow through a channel of a polyacrylamide solution becoming turbulent at $\text{Re} \sim 350$, $\text{Wi} \sim 250$, and $\beta = 0.92$ were reported in Ref. [67]. We have verified that a center-mode instability, similar to the one described above for pipe flow, exists at these parameter values for plane Poiseuille flow of an Oldroyd-B fluid; the details will be reported elsewhere [73]. Plane Couette flow was, however, found to be stable at all Re and Wi values examined. Since the polymers are only weakly stretched near the centerline, the center-mode nature of the instability suggests the relative unimportance of finite extensibility of the polymer chains. Indeed, preliminary work using the FENE-P constitutive equation [66,74,75], confirms this expectation [73].

The instability described in this Letter should form the first step in a new pathway to turbulence, and the maximum drag reduction (MDR) asymptote, in dilute polymer solutions. The loss of bistability, characteristic of the Newtonian scenario [20,51], implies very different transitional dynamics. Complementing recent experimental [52,69] and numerical efforts [53,66], ours is the first theoretical work that points to a state of elastoinertial turbulence (EIT) with novel spatiotemporal dynamics, particularly at large Wi , underlying the MDR asymptote. At the linear instability threshold, elastoinertial traveling wave solutions, associated with the unstable center-mode eigenfunctions, would be created in a Hopf bifurcation from the laminar state [76,77]. These traveling wave solutions and associated phase space structures would thus be relevant for describing both EIT and MDR dynamics [52,53,66,69]. The implied contrast between the state space for viscoelastic pipe flow and the Newtonian one will have fundamental consequences for the dynamical systems interpretation of the maximum drag reduction state which, currently, crucially relies on a similarity between the two [7,30,32,53,78]. The general mechanism will also be applicable to inertial flows of other viscoelastic fluids such as wormlike micellar surfactant solutions which show drag reduction [69,79].

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