Exact Solutions for Nonlinear Development of a Kelvin-Helmholtz Instability for the Counterflow of Superfluid and Normal Components of Helium II

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Relative motion of the normal and superfluid components of helium II results in the quantum Kelvin-Helmholtz instability (KHI) at their common free surface. We found the integrability and exact growing solutions for the nonlinear stage of the development of that instability. Contrary to the usual KHI of the interface between two classical fluids, the dynamics of a helium II free surface allows reduction to the Laplace growth equation, which has an infinite number of exact solutions, including the generic formation of sharp cusps at the free surface in a finite time.

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The Kelvin-Helmholtz instability (KHI) is perhaps the most important hydrodynamic instability which commonly occurs either at the interface between two fluids moving with different velocities or in the presence of the tangential velocity jump or shear flow in the same fluid [1]. Recently, the KHI has attracted significant experimental and theoretical attention in superfluids. KHI was studied either for the interface between different phases of ^{3}He [2–6], which has many similarities with KHI in classical fluids, or for the relative motion of components of ^{4}He [7–11], which has no classical analog; thus, we refer to it as quantum KHI. We focus on the second case—i.e., on the quantum KHI of the free surface of ^{4}He in the superfluid phase (He-II state) in the presence of the counterflow of superfluid and normal fluid components [12,13]. The principal difference here from the KHI of classical fluids is that relative fluid motion in quantum KHI occurs not from different sides of the interface but from the same side of the He-II free surface, with fluid components coexisting in the same volume, which is purely a quantum effect. A counterflow is achieved in experiment by the action of a stationary heat flow within the liquid in the direction tangent to the free surface, as shown in Fig. 1.

Linear analysis of both classical KHI and quantum KHI results in the exponential growth of surface perturbations [1,12,13]. As these initial perturbations reach amplitudes comparable with their wavelength, nonlinear effects must be considered. Weak nonlinearity approximation takes into account the leading-order nonlinear correction over the small parameter, which is the typical slope of the surface. Weakly nonlinear equations for the development of the KHI of classical fluids result in a finite time singularity [14], which means that solutions become strongly nonlinear beyond the perturbation theory. Two-dimensional (2D) dynamics of the interface between two fluids in weak nonlinearity approximation can be reduced to the motion of complex singularities through the analytical continuation into the complex plane from the interface [15,16]. The approach of a singularity to the interface always means a formation of its geometric singularity. Other examples of the analysis of weakly nonlinear 2D dynamics through the motion of singularities include the interface between ideal fluid and light highly viscous fluid [17], and vortex sheets in ideal fluid [18]. Extending weakly nonlinear solutions into strongly nonlinear solutions is challenging and was mostly done for the particular case of free surface hydrodynamics (i.e., the density of the second fluid goes to zero) [19–24], including drops pinch-off [25,26]. Another exception is the ideal fluid pushed through a viscous fluid in a narrow gap between two parallel plates (Hele-Shaw flow), which can be approximately reduced to the Laplace growth equation (LGE), admitting an infinite set of exact solutions [27–32].

We use a key property of quantum KHI: that both fluid components share the same volume. It allows us to find the

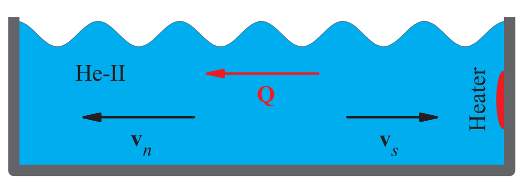

FIG. 1. A schematic of counterflow in superfluid ^{4}He. Heating results in the flux of heat $\mathbf{Q}$, which is carried by the normal fluid component with velocity $\mathbf{v}_n$, while the superfluid component moves in the opposite direction with the velocity $\mathbf{v}_s$. Both components coexist in the same volume of fluid and share the same free surface.

exact strongly nonlinear solutions, and moreover, the general integrability of growing solutions. This is achieved through the exact reduction of quantum KHI dynamics to the LGE for an arbitrary level of nonlinearity. These new solutions, in particular, describe the formation of cusps (dimples) on the He-II free surface in a finite time with both a surface curvature and the velocities of components of He-II diverging at singular points. We expect that these singularities will be possible to observe in He-II experiments, which is different from weaker singularities of the Moore's type (which were identified from approximate analysis in Refs. [15,16,18]) and predicts a smooth surface with jumps only in the second derivative. The LGE is integrable in the sense of the existence of an infinite number of integrals of motion and relation to the dispersionless limit of the integrable Toda hierarchy [33]. We suggest that the obtained reduction of quantum KHI to the LGE is important to the general problem of the integrability of surface dynamics [34]. It provides a very rare example of an integrable physical system.

The superfluid component of He-II necessary has quantized vortices if the counterflow velocity exceeds several millimeters per second, with their density growing with that velocity [35]. Here we consider the dynamics of He-II at a macroscopic scale where we can average over vortices. We neglect average vorticity from such averaging, as well as we ignore the vorticity of the normal component, similarly to Refs. [12,13], which refer to that approximation as a nondissipative two-fluid description (dissipative effects were taken into account, e.g., in Ref. [36]). In that approximation, the dynamics of both fluid components is a potential one—i.e., $\mathbf{v}_s = \nabla\Phi_s$ and $\mathbf{v}_n = \nabla\Phi_n$—where $\mathbf{v}_s$, $\mathbf{v}_n$ are the velocities of the superfluid and normal components, with Φ_s and Φ_n being the corresponding velocity potentials. We assume that both components are incompressible with densities $\rho_s \equiv \text{const.}$, $\rho_n \equiv \text{const.}$, and the total density $\rho \equiv \rho_s + \rho_n$. Incompressibility implies a Laplace equation for each component, $\nabla^2\Phi_{n,s} = 0$. We focus on 2D flow $\mathbf{r} \equiv (x, y)$, where x and y are horizontal and vertical coordinates, respectively. We assume that both fluids occupy the region $-\infty < y \le \eta(x,t)$, where $y = \eta(x,t)$ is the free surface elevation, with the unperturbed surface given by $\eta(x,t) \equiv 0$. The flow of both components deep inside He-II ($y \to -\infty$) as well as at $|x| \to \infty$ is assumed to be uniform following the x direction, which implies $\Phi_{n,s} \to V_{n,s}x$, where $V_{n,s}$ are the corresponding horizontal velocities. We use the reference frame of the center of mass such that $\rho_n V_n + \rho_s V_s = 0$ and introduce the relative velocity $V = V_s - V_n > 0$ between fluid components, meaning that $V_{n,s} = \mp \rho_{s,n} V/\rho$.

The dynamic boundary condition (BC) at the free surface ($y = \eta$) follows from the generalization of the Bernoulli equation into two fluid components (see, e.g., Chap. 140 of Ref. [1] and Refs. [12,13]):

$$\rho_n\left(\frac{\partial\Phi_n}{\partial t} + \frac{(\nabla\Phi_n)^2}{2}\right) + \rho_s\left(\frac{\partial\Phi_s}{\partial t} + \frac{(\nabla\Phi_s)^2}{2}\right)\bigg|_{y=\eta} = \Gamma - P_\alpha - P_g, \tag{1}$$

where $P_\alpha = -\alpha(\partial/\partial x)[\eta_x(1+\eta_x^2)^{-1/2}]$ is the pressure jump at the free surface due to the surface tension α (the pressure is zero outside the fluid assuming that there is a vacuum there), $\eta_x \equiv \partial\eta/\partial x$, $P_g = \rho g \eta$ is the gravity pressure (the contribution of the acceleration due to gravity g), and $\Gamma = \rho_n\rho_s V^2/(2\rho)$ is the Bernoulli constant which ensures that Eq. (1) is satisfied at $|x| \to \infty$.

The kinematic BCs at the free surface are given by

$$\eta_t(1+\eta_x^2)^{-1/2} = \partial_n\Phi_n|_{y=\eta} = \partial_n\Phi_s|_{y=\eta}, \tag{2}$$

where $\eta_t \equiv \partial\eta/\partial t$, and $\partial_n \equiv \mathbf{n}\cdot\nabla$ is the outward normal derivative to the free surface with $\mathbf{n} = (-\eta_x, 1)(1+\eta_x^2)^{-1/2}$. Equations (1) and (2) together with $\nabla^2\Phi_n = \nabla^2\Phi_s = 0$ and the BC at infinity form a closed set of equations of two-fluid hydrodynamics for the KHI problem.

We introduce the average velocity $\mathbf{v} = (\rho_n\mathbf{v}_n + \rho_s\mathbf{v}_s)/\rho$ and the auxiliary potentials $\Phi = (\rho_n\Phi_n + \rho_s\Phi_s)/\rho$, $\phi = \sqrt{\rho_n\rho_s}(\Phi_n - \Phi_s)/\rho$, which are linear combinations of Φ_n and Φ_s, thus satisfying the Laplace equation together with $\nabla\Phi = \mathbf{v}$. BCs at either $y \to -\infty$ or $|x| \to \infty$ are reduced to

$$\Phi \to 0 \quad \text{and} \quad \phi \to -Vx\sqrt{\rho_n\rho_s}/\rho. \tag{3}$$

Equation (1) turns into

$$\frac{\partial\Phi}{\partial t} + \frac{(\nabla\Phi)^2}{2} + \frac{(\nabla\phi)^2}{2}\bigg|_{y=\eta} = \frac{c^2}{2} - \frac{P_\alpha + P_g}{\rho}, \tag{4}$$

where $c = \sqrt{2\Gamma/\rho}$ is the constant which has the dimension of velocity. Equation (2) is reduced to

$$\eta_t(1+\eta_x^2)^{-1/2} = \partial_n\Phi|_{y=\eta} \tag{5}$$

and

$$\partial_n\phi|_{y=\eta} = 0. \tag{6}$$

We replace ϕ with its harmonic conjugate ψ such that the Cauchy-Riemann equations $\phi_x = \psi_y$ and $\phi_y = -\psi_x$ are valid. The BC [Eq. (6)] for the Laplace equation

$$\nabla^2\psi = 0 \tag{7}$$

at the free surface reduces to the vanishing of tangential derivatives $\partial_\tau\psi|_{y=\eta} = 0$, because $\partial_\tau\psi|_{y=\eta} = -\partial_n\phi|_{y=\eta}$. Without the loss of generality, we set

$$\psi|_{y=\eta} = 0. \tag{8}$$

The BCs at either $y \to -\infty$ or $|x| \to \infty$ are reduced to

$$\psi \to -Vy\sqrt{\rho_n\rho_s}/\rho = -cy. \tag{9}$$

If we introduce the stream functions $\Psi_{n,s}$ for the components of He-II (they satisfy Cauchy-Riemann equations $\partial_x\Phi_{n,s} = \partial_y\Psi_{n,s}$ and $\partial_y\Phi_{n,s} = -\partial_x\Psi_{n,s}$), then $\psi = (\Psi_n - \Psi_s)\sqrt{\rho_n\rho_s}/\rho.\psi$ is fully determined by $\eta(x,t)$ from Eqs. (8) and (9) while being independent on Φ. The dynamic BC [Eq. (4)] in terms of Φ and ψ is given by

$$\frac{\partial\Phi}{\partial t} + \frac{(\nabla\Phi)^2}{2} + \frac{(\nabla\psi)^2}{2}\bigg|_{y=\eta} = \frac{c^2}{2} - \frac{P_\alpha + P_g}{\rho}. \tag{10}$$

Equations (5), (8), and (10), together with $\nabla^2\Phi = \nabla^2\psi = 0$ and BCs (3) and (9) at infinity, form a closed set of equations equivalent (through harmonic conjugation) to solving two-fluid He-II hydrodynamics for the KHI problem. It is remarkable that this set is equivalent (up to a trivial change of constants) to the problem of the 2D dynamics of a charged surface of ideal fluid in the limit where surface charges fully screen the electric field above the fluid free surface. This limit was realized experimentally for the He-II (with negligible ρ_n) free surface charged by electrons [37]. In that case, Φ has the meaning of the only (ideal) fluid component, and ψ represents (up to multiplication to the constant) the electrostatic potential in the ideal fluid. The term $\propto (\nabla\psi)^2$ in Eq. (10) corresponds to the electrostatic pressure.

References [38,39] found exact time-dependent solutions for this problem of the dynamics of the charged surface of superfluid He-II in the limit of zero surface tension and gravity, as well as for the limit of zero temperature (i.e., neglecting the normal component of He-II). We apply that approach for the full (nonlinear) KHI problem with finite temperature. We set $\alpha = g = 0$ on the right-hand side of Eq. (10). Below, we provide estimates of the applicability of such neglect of surface tension and gravity for two-component dynamics of He-II.

Our goal is to reduce Eqs. (5), (8), and (10), together with $\nabla^2\Phi = \nabla^2\psi = 0$ and BCs (3) and (9), to the solution of LGE. Differentiation of Eq. (8) over t and x results in

$$\eta_t = -\psi_t/\psi_y|_{y=\eta}, \qquad \eta_x = -\psi_x/\psi_y|_{y=\eta},$$

respectively. Using these expressions in kinematic BC (5) rewritten in the equivalent form $\eta_t = \Phi_y - \eta_x\Phi_x|_{y=\eta}$ allows us to obtain that

$$\psi_t + \nabla\psi\cdot\nabla\Phi|_{y=\eta} = 0. \tag{11}$$

The sum and difference of Eqs. (10) and (11) (with $P_\alpha = P_g = 0$) result in

$$F_t^{(\pm)} \mp cF_y^{(\pm)} + (\nabla F^{(\pm)})^2|_{y=\eta} = 0, \tag{12}$$

where we introduce the harmonics potentials

$$F^{(\pm)} = (\Phi \pm \psi \pm cy)/2, \tag{13}$$

which satisfy the Laplace equations.

$$\nabla^2 F^{(\pm)} = 0, \quad F^{(\pm)} \to 0 \text{ for } y \to -\infty \text{ or } |x| \to \infty. \tag{14}$$

According to Eqs. (8) and (13), the motion of the free surface is determined by the implicit expression

$$c\eta = F^{(+)} - F^{(-)}|_{y=\eta}. \tag{15}$$

Returning to physical $\Phi_{n,s}$ and $\Psi_{n,s}$, we find that

$$2\rho F^{(\pm)} = \rho_n\Phi_n + \rho_s\Phi_s \pm \sqrt{\rho_n\rho_s}(\Psi_n - \Psi_s + Vy). \tag{16}$$

Equation (14) together with Eqs. (12) and (15) are equivalent to the KHI problem. It is crucial that the nonlinear Eq. (12) decouple into separate equations for $F^{(+)}$ and $F^{(-)}$. We note that such decoupling does not occur for the classical KHI problem (the interface between two fluids) where the velocity potentials and stream functions of each of two fluids are defined in physically distinct regions ($y < \eta$ and $y > \eta$), thus making impossible a superposition of the type in Eq. (16). (Decoupling is, however, possible by other methods in small-angle approximations with leading quadratic nonlinearity in perturbation series for classical KHI [16].)

The full set of Eqs. (12), (14), and (15) is still generally coupled through Eq. (15). But an additional assumption (reduction) that either

$$F^{(+)} = 0 \quad \text{or} \quad F^{(-)} = 0 \tag{17}$$

ensures the closed equations which have a wide family of exact nontrivial solutions described below. That assumption remains valid as time evolves. It follows from Eq. (13) that Eq. (17) ensures the relations between $\Phi_{n,s}$ and $\Psi_{n,s}$ as

$$\mp\sqrt{\rho_n\rho_s}(\Psi_n - \Psi_s + Vy) = \rho_n\Phi_n + \rho_s\Phi_s.$$

We look at the physical meaning of our reductions (17), based on the particular limit of small-amplitude surface waves. We neglect the nonlinear term in Eq. (12), resulting in the linear system which we solve in the form of plane waves:

$$F^{(\pm)} = a^{(\pm)}\exp(ikx + ky - i\omega^{(\pm)}t),$$
$$\eta = b^{(+)}\exp(ikx - i\omega^{(+)}t) + b^{(-)}\exp(ikx - i\omega^{(-)}t), \tag{18}$$

where $a^{(\pm)}$ and $b^{(\pm)}$ are small constants, $\omega^{(\pm)}$ are frequencies, and k is the wave number. The first equation in Eq. (18) ensures the exact solution of Eq. (14) with decaying BCs at $y \to -\infty$. The substitution of Eq. (18) into Eq. (15) and the linearization of Eq. (12) results in the relations

$$\omega^{(\pm)} = \pm ick, \qquad cb^{(\pm)} = \pm a^{(\pm)}, \tag{19}$$

which are two branches of the dispersion relation of KHI with $g = \alpha = 0$ [1,12,13]. Superscripts "+" and "−" correspond to exponentially growing and decaying perturbations of the flat free surface, respectively. Equation (17) chooses one of these two branches. Thus, Eqs. (12), (14), and (15), together with (17) represent the fully nonlinear stage of such separation into two branches.

The generic initial conditions include both unstable and stable parts [Eq. (18)], with the unstable part dominating as time evolves. Also, it was shown in Refs. [39,40] that small perturbations of $F^{(-)}$ on the background of large $F^{(+)}$ decay to zero. Thus, the choice of the reduction $F^{(-)} = 0$ (which is assumed below) in Eq. (17) is the natural one to address the nonlinear stage of KHI. Then Eq. (13) implies that $F^{(+)} = \Phi = \psi + cy$; i.e., Φ is determined by ψ. The boundary value problem (BVP) [Eqs. (7)–(9)] solves for ψ at each t. The motion of the free surface is determined by Eq. (5) as

$$(\eta_t - c)(1 + \eta_x^2)^{-1/2} = \partial_n \psi|_{y=\eta}. \tag{20}$$

To solve the BVP in Eqs. (7)–(9), we consider the conformal map $z = z(w, t)$ [41] from the lower complex half-plane $-\infty < v \leq 0$, $-\infty < u < +\infty$ of the complex variable $w = u + iv$ into the area $-\infty < y \leq \eta(x, t)$ occupied by the fluid in the physical plane $z = x + iy$ with the real line $v = 0$ mapped onto the fluid free surface. Then the free surface is given in the parametric form $y = Y(u, t) \equiv \mathrm{Im} z(u, t)$ and $x = X(u, t) \equiv \mathrm{Re} z(u, t)$. Solutions of both the BVP [Eqs. (7)–(9)] and the harmonically conjugated BVP $\nabla^2 \phi = 0$ [Eqs. (3) and (6)] in (u, v) variables are given by $\phi + i\psi = -c(u + iv)$. This means that the conformal variables u and v have a simple physical meaning: $u = -\phi/c$ and $v = -\psi/c$, corresponding (up to multiplication to the constant $-1/c$) to the harmonically conjugated potentials ϕ and ψ.

We consider w as an independent variable, while $z(w, t)$ is the unknown function. Equation (20) is given by $Y_t X_u - Y_u X_t = cX_u - c$, which can be rewritten as

$$\mathrm{Im}(\bar{G}_t G_u) = c, \tag{21}$$

where $G(u, t) = z(u, t) - ict$. Equation (21) has the exact form of LGE, which has an infinite number of exact solutions, often involving logarithms (see, e.g., Refs. [29–32]). We look at a periodic solution [31] with the wave number k:

$$z = w - ikA^2(t)/2 - iA(t)\exp[-ikw], \tag{22}$$

where $A(t)$ is the amplitude of the free surface perturbation satisfying a nonlinear ordinary differential equation $dA/dt = ckA(1 - k^2A^2)^{-1}$, which develops a finite-time singularity in dA/dt at the time $t = t_c$ with $A(t_c) = 1/k$. As t approaches t_c, a leading-order solution is given by $A = 1/k - \sqrt{c\tau/k} + O(\tau)$, where $\tau = t_c - t$. Singularities of the conformal map (22) are determined by a condition $z_w = 0$ implying that they approach the real line $v = 0$ from above with the increase of t. That line is reached at $\tau = 0$ and $u = 2\pi n/k$, $n = 0, \pm 1, \pm 2, \ldots$. In particular, choosing $n = 0$, expanding at $u = 0$, and assuming $\tau \to 0$, we obtain that

$$X = u\sqrt{ck\tau} + k^2u^3/6 + O(u\tau + u^3\tau^{1/2}),$$
$$Y = -3/2k + 2\sqrt{c\tau/k} + ku^2/2 + O(\tau + u^2\tau^{1/2}). \tag{23}$$

Figure 2 shows an example of such a solution at different t. It follows from Eq. (23) that a cusp pointing downward (a dimple) $y + 3/2k \propto |x|^{2/3}$ is formed at the free surface at $t = t_c$ (i.e., $\tau = 0$), with the vertical velocity diverging as $\tau^{-1/2}$ at the tip of the cusp [29,31].

Near the singularity (the tip of the cusp), one has to take into account the surface tension and the finite viscosity of the normal component to regularize the singularity. Surface tension near the singularity is given by $P_\alpha \approx \alpha/r$, where r is the radius of curvature of the free surface. It follows from Eq. (23) that $r \approx c\tau$, which implies that $P_\alpha \approx \alpha/c\tau$. The dynamic pressure P_v, which determines the development of KHI in LGE reduction, is given by

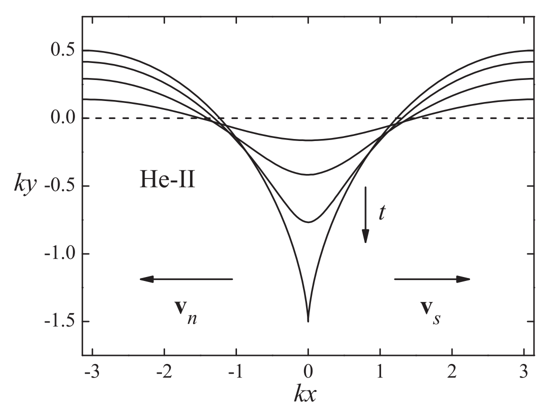

FIG. 2. Evolution of an initial periodic perturbation of the free surface $y(x)$ for Eq. (22) with $kA(0) \approx 0.15$. The surface shape is shown over one spatial period for the times $ckt = 0$, 0.8, 1.2, 1.4 until the cusp singularity is formed. The dashed line shows the unperturbed free surface, $y \equiv 0$.

$P_v = \rho v^2/2 \equiv \rho[(\nabla\Phi)^2) + (\nabla\psi)^2]/2$, where v is the typical velocity. Near singularity, $v \simeq \sqrt{2c/k\tau}$ and $P_v = \rho c/k\tau$. Thus, both P_v and $P_\alpha \propto \tau^{-1}$. The surface tension effect is small if the Weber number $\mathrm{We} = P_v/P_\alpha$, the ratio of dynamic and surface tension pressures, is well above 1. Using $\mathrm{We} \approx \rho c^2/\alpha k = \rho_n\rho_s V^2/(\rho\alpha k)$, and assuming We $\gtrsim 1$ for the applicability of the LGE regime, we obtain the condition for the wavelength: $\lambda = 2\pi/k \gtrsim 2\pi\rho\alpha/(\rho_n\rho_s V^2)$. He-II at temperature 1.5 K has $\rho_n = 0.016\ \mathrm{g/cm^3}$, $\rho_s = 0.129\mathrm{g/cm^3}$, and $\alpha = 0.332$ dyn/cm [42]. E.g., if $V = 15$ cm/s, then $\lambda \gtrsim 0.64$ cm.

The relative strength of inertial and viscous forces near the singularity is determined by the Reynolds number $\mathrm{Re} \equiv vr/\nu$, where ν is kinematic viscosity of He-II. Using the fact that $v \approx \sqrt{2c/k\tau}$ and $r \approx c\tau$ implies that $\mathrm{Re} \approx c\nu^{-1}\sqrt{2r/k}$; i.e., Re turns small for $r \to 0$, and viscosity has to be taken into account. A typical scale r_ν below which the flow of the normal component cannot be considered as a potential one is estimated by setting $\mathrm{Re} \approx 1$, which gives $r_\nu \approx k\nu^2/2c^2$. For the temperature 1.5 K, we use $\nu = 9.27 \times 10^{-5}\ \mathrm{cm^2/s}$ [42]. Then $r_\nu \approx 1.8 \times 10^{-10}$ cm, i.e., $r_\nu \ll \lambda$; thus, the viscous effect is much less than the surface tension. The influence of gravity, which is determined by the Froude number $\mathrm{Fr} = P_v/P_g$, is small near the singularity because the gravity pressure $P_g \simeq \rho g y$ is finite, while P_v diverges as τ^{-1}, implying the divergence of Fr.

We conclude that we have reduced fully nonlinear quantum KHI dynamics to LGE, which has an infinite set of exact solutions with the generic formation of cusps at the free surface in a finite time. The key is the exact transform from a two-fluid description into the effective single-fluid description of Eq. (10). It suggests a road map for the efficient use of conformal mapping to include gravity and capillarity in dynamics. Adding capillarity would ensure singularity regularization at small spatial scales. Conformal mapping can be used for electrohydrodynamic instability [37,40] and Faraday waves [43] of He-II. Viscosity can be taken into account through a conformal map in the Stokes flow regime of the normal component which would go beyond a weakly nonlinear result [17].

The free surface represents a vortex sheet which results in the additional generation of quantized vortices at the nonlinear stage of KHI. It is expected to push quantum turbulence states T1 towards T2/T3 states [35].

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