Topological Qubits from Valence Bond Solids

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Topological qubits based on $SU(N)$-symmetric valence-bond solid models are constructed. A logical topological qubit is the ground subspace with twofold degeneracy, which is due to the spontaneous breaking of a global parity symmetry. A logical Z rotation by an angle $2\pi/N$, for any integer $N > 2$, is provided by a global twist operation, which is of a topological nature and protected by the energy gap. A general concatenation scheme with standard quantum error-correction codes is also proposed, which can lead to better codes. Generic error-correction properties of symmetry-protected topological order are also demonstrated.

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In recent years there has been significant interplay between quantum computing and topological states of quantum matter [1]. Topological qubits have been proposed in various systems [2–9], and excitations, e.g., anyons, support topologically protected gates by braiding [10]. The valence-bond solids [11,12], which are prototype models for matrix (and tensor) product states [13] and allow symmetry-protected topological (SPT) order [14–17], have also been exploited as resources for quantum computing [18–21].

Fault-tolerant quantum computing and error-correction codes benefit from, and also mostly rely on, the stabilizer formalism and Pauli Hamiltonians [22]. However, there are limitations [23–30]: e.g., the set of transversal qubit Z rotations is restricted to be of angles $2\pi/2^n$, $n \in \mathbb{N}$ [27–29]. At the same time, more general coding theory has been developed and nonstabilizer codes have been found [31–33]. In this work, we go beyond the stabilizer framework and construct topological qubits and gates from a class of $SU(N)$ valence-bond solids [34–38] with 1D SPT order. We find, for any integer $N > 2$, there exists a valence-bond solid qubit (VBSQ) such that the logical gate set $\{\bar{X}, e^{i(2\pi/N)\bar{Z}}\}$ is transversal. Larger sets of transversal gates can help reduce the circuit cost [27–29,39], making fault-tolerant quantum computing more efficient.

In this Letter, a VBSQ is based on the degeneracy due to the spontaneous breaking of a global parity symmetry, while the logical space is also protected by a global $SU(N)$ symmetry. We find both the broken and unbroken symmetries provide transversal logical gates, namely, the logical $\bar{X}$ (bit flip) is the generator of the global parity symmetry, while the logical $e^{i(2\pi/N)\bar{Z}}$ is provided by a global *twist* operation (Fig. 1). An appealing feature we show is that the twist is topologically robust and extracts the SPT order of ground states [14–17,21]. The gate $e^{i(2\pi/N)\bar{Z}}$ is implemented in a transversal and topologically stable fashion, and it is furthermore outside the stabilizer formalism. The existence of such gate implementations is a main finding of our work.

As a consequence of their SPT order and distinct from topological stabilizer codes [2,23,40], VBSQs have a code distance that grows linearly with the length of the code for bit flips (and a little more), but is constant for generic phase flips, which is due to the exponentially decaying correlation functions. Furthermore, VBSQs, viewed as a class of SPT codes, and standard codes can be concatenated, improving error resilience. That is, VBSQs provide error protection at the hardware level, afforded by an energy gap, and error syndrome in addition. By concatenation, bit flips are corrected at the hardware level, while phase flips are

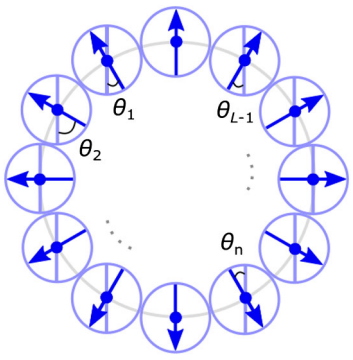

FIG. 1. Schematic diagram of a twist on a $SU(N)$ valence-bond solid qubit, which executes the logical operation $e^{i(2\pi/N)\bar{Z}}$. In general, a twist can be implemented by a product of rotations around a fixed direction on every site, with the rotation angles θ_n increasing smoothly from 0 to $2\pi\Omega$ along the system, for $n = 1, 2, \ldots, L$, L the size of the system, and winding number Ω. The angle Φ of the logical rotation is proportional to the winding number, $\Phi = 2\pi\Omega/N$.

corrected at the software level. The situation encountered thus resembles classical hard drives, which have a layer of physical error correction provided by the bulk magnetization of spins, and a layer of software error correction on top.

We demonstrate our findings with the following main parts. The code space is first defined, and then transversal logical gates are studied. Detailed analysis of the code properties then follows, and, finally, a concatenation framework is laid out including a scheme of entangling gates. The model we study is the translation-invariant $SU(N)$ valence-bond solid

$$H_N := \sum_{n=1}^{L} (J_1 h_n + J_2 h_n^2 + J_0 \mathbb{1}_n) := \sum_{n=1}^{L} H_n \tag{1}$$

on a ring of L sites with periodic boundary conditions (see Fig. 1) and

$$h_n := \sum_{\alpha=1}^{N^2-1} T_n^\alpha \otimes T_{n+1}^\alpha, \tag{2}$$

with $\{T^\alpha\}$ the generators of $SU(N)$ in the adjoint irrep for each site n [34–38]. For each $N > 2$, and at $J_1 = (3N/2)J_2 = (3/N)J_0$ (without loss of generality let $J_1 = 1$), the system has two degenerate ground states, which span the code space of a VBSQ. The two ground states, denoted as $|\mathbf{L}\rangle$ and $|\mathbf{R}\rangle$, are the spatial reflection of each other, namely, they break a parity symmetry, denoted as $\mathbb{Z}_2^p$, but preserve $SU(N)$ symmetry. In the valence-bond picture (see Fig. 2), the parity refers to the interchange of a fundamental irrep and its conjugate. In terms of matrix product state we have

$$|\mathbf{L}\rangle = \vartheta \sum_{i_1=1}^{N^2-1} \cdots \sum_{i_L=1}^{N^2-1} \mathrm{tr}(A^{i_1} A^{i_2} \cdots A^{i_L}) |i_1, \ldots, i_L\rangle, \tag{3}$$

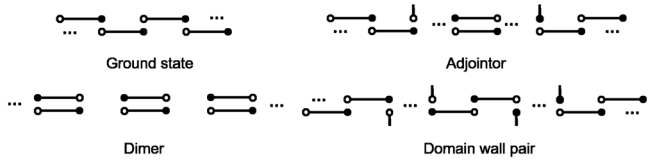

FIG. 2. Ground states and approximate excitations in the valence-bond picture. The on-site adjoint irrep is constructed from the projection from the product of a fundamental irrep (filled dot) and its conjugate (empty dot). Solid lines represent singlet (bond). (Upper left) A ground state that breaks $\mathbb{Z}_2^p$, and the other one is obtained by flipping each bond. (Upper right) A single adjointor excitation, whose size is confined by a linear potential, which is due to the dimerized region in between that has higher energy. (Lower left) A dimerized state, which preserves parity (reflection about a link) but breaks translation symmetry. (Lower right) A domain excitation with a pair of domain walls due to the periodic boundary condition, which roughly has the same energy as two adjointors.

and $\{A^{i_n}\}$ is independent of the site label n, which is then dropped, and for each site $A^i = \sqrt{2/N} t^i$, $t^i = \lambda^i/2$ for the generalized Gell-Mann matrices $\{\lambda^i\}$, while for $|\mathbf{R}\rangle$ the operators on each site are $A^i = \sqrt{2/N}(t^i)^*$. The normalization constant is $\vartheta := (N^2/N^2 - 1)^{L/2}$ such that the ground states are normalized and orthogonal

$$\langle \mathbf{L}|\mathbf{L}\rangle = \langle \mathbf{R}|\mathbf{R}\rangle = 1, \quad \langle \mathbf{L}|\mathbf{R}\rangle = \left(\frac{1}{N-1}\right)^L \to 0. \tag{4}$$

The model is frustration free; hence its gap at zero temperature is robust against local perturbation [41]. Some of its approximate excitations can also be obtained (see Fig. 2 and the Supplemental Material [42], Sec. I).

Now we define the VBSQs for each $N > 2$. In the large-L limit, the operator $P_{\mathcal{C}} := |\mathbf{L}\rangle\langle\mathbf{L}| + |\mathbf{R}\rangle\langle\mathbf{R}|$ is the projection on the code space $\mathcal{C}$ since the correction $1/(N-1)^L$ is exponentially suppressed. Logical $\bar{X}$ operation is the generator of the parity symmetry $\mathbb{Z}_2^p$. In a certain on-site basis (see Supplemental Material [42], Sec. I. B), it can be expressed as a permutation operator Π on each local site

$$\bar{X}|\mathbf{L}\rangle := \Pi \otimes \Pi \cdots \Pi |\mathbf{L}\rangle = |\mathbf{R}\rangle, \tag{5}$$

and

$$\Pi = \mathrm{diag}(\sigma_x, \sigma_x, \ldots, \sigma_x, \mathbb{1}), \tag{6}$$

for qubit Pauli operator $\sigma_x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Notably, $\bar{X}$ is stable against the single adjointor excitation since there is no way for it to make a logical $\bar{X}$ without introducing a second one. The situation is different if there are domains. The size of a domain is not confined as there is no binding potential between the two domain walls, with one soliton (two empty dots) and one antisoliton (two filled dots) (see Fig. 2), although the size of domain walls is instead confined. This means domain wall excitations can lead to logical bit flip error; however, the probability to induce $\bar{X}$ will be exponentially suppressed as the system size increases. The system size should not be too big in order to achieve a proper coding redundancy, while also not too small to satisfy (4).

The $SU(N)$ symmetry provides the logical $\bar{Z}$ rotations by a twist operator

$$U_{\mathrm{TW}} := \bigotimes_n e^{i(2\pi/L) n \mathcal{O}_n} := \bigotimes_n U_n, \tag{7}$$

and Hermitian operator $\mathcal{O}_n$ on each site n is in Cartan subalgebra; i.e., it is diagonal, and $e^{i2\pi\mathcal{O}_n} = \mathbb{1}$ (Supplemental Material [42], Sec. II). In the valence-bond picture, the twist is equivalent to a unitary operator

$$V = \mathrm{diag}(e^{i\ell}, 1, \ldots, 1), \qquad \ell := \frac{2\pi}{L}, \tag{8}$$

acting on each bond, while $e^{i\ell}$ can be at any of the N different positions (flavors) on the diagonal of V. For any flavor we find

$$\langle \mathbf{L}|U_{\mathrm{TW}}|\mathbf{L}\rangle = e^{i2\pi/N}, \qquad \langle \mathbf{R}|U_{\mathrm{TW}}|\mathbf{R}\rangle = e^{-i2\pi/N}, \tag{9}$$

and $\langle \mathbf{L}|U_{\mathrm{TW}}|\mathbf{R}\rangle = 0$. This provides the logical operator $e^{i(2\pi/N)\bar{Z}}$. Actually, we see that the phases $e^{\pm i2\pi/N}$ are the SPT index of the two ground states [17,21,55].

The twist is a weak perturbation on the system. The action of the twist is uniform on each bond, while it is not the same on each physical site. The disturbance of the twist to the local interaction terms h_n is of $O(1/L^2)$ and negligible since $[h_n, \mathcal{O}_{n+1} + \mathcal{O}_n] = 0$, and $\langle\psi|[h_n, \mathcal{O}_{n+1} - \mathcal{O}_n]|\psi\rangle \in O(1/L)$, $\forall|\psi\rangle \in \mathcal{C}$. This means that the twist operator approximates the symmetry of the system up to the $O(1/L)$ correction, which vanishes in the large-L limit.

Furthermore, the twist operation is topological. First, the phase gate V [Eq. (8)] can be slightly disturbed such that the parameter ℓ_b for each bond b is different and, as long as $\sum_b \ell_b = 2\pi$ and the second order ℓ_b^2 is small, the twist angle remains the same. Second, the twist is also homotopic since the same twist angle can be achieved on any long-enough continuous segment. Third, if the set of ℓ_b sums to $2\pi f$ for a fractional number $f \in (0, 1)$, the system will be excited and there will be an energy penalty. This means the twist is protected by the gap. If f is slightly perturbed from 1 by δf, the twist angle remains the same with corrections of the order $(\delta f)^2$. Last but not least, the twist is also stable against the single adjointor excitation. When there are domains, however, the phase accumulated from one ground state may cancel that from the other, destroying the twist phase.

Next we consider error-correction properties of VBSQ. A correctable set of errors $\{E_i\}$ are defined such that the error-correction condition

$$P_{\mathcal{C}} E_i^\dagger E_j P_{\mathcal{C}} = C_{ij} P_{\mathcal{C}} \tag{10}$$

is satisfied, and $\{C_{ij} \in \mathbb{C}\}$ form a Hermitian matrix [56]. Usually, a code can detect more errors that it can correct. Different from error correction, a detectable set of errors $\{E_i\}$ are defined such that

$$P_{\mathcal{C}} E_i P_{\mathcal{C}} = e_i P_{\mathcal{C}} \tag{11}$$

for $e_i \in \mathbb{C}$. We study errors that are from independent local unitary errors. In the following we mainly study three sets of errors: errors from the set (12) below, $SU(N)$, and $SU(N^2 - 1)$, respectively. We find that the code distance is linear in L for the set (12), while shows nontrivial error correction and detection features for the other general errors.

First, we expect the VBSQ is robust against logical bit flip errors Π. This is indeed the case (Supplemental Material [42], Sec. III). Consider the noncommutative error set

$$\mathcal{E} := \{\Pi, \mathcal{P}^{jk}\} \tag{12}$$

with $\mathcal{P}^{jk}$ as the adjoint rep of the generalized Pauli operators $P^{jk} = X^j Z^k$ (Supplemental Material [42], Sec. III). Denote $E(n)$ as a product of n errors acting on n random positions of the system with each from the set (12), and define the effective operator as

$$\mathbf{U}(n) := \begin{pmatrix} \langle \mathbf{L}|E(n)|\mathbf{L}\rangle & \langle \mathbf{L}|E(n)|\mathbf{R}\rangle \\ \langle \mathbf{R}|E(n)|\mathbf{L}\rangle & \langle \mathbf{R}|E(n)|\mathbf{R}\rangle \end{pmatrix}. \tag{13}$$

We find $\mathbf{U}(n) = (-1/N^2 - 1)^n \mathbb{1}$ holds until a certain large enough n. In particular, $\mathbf{U}(1) = -(1/N^2 - 1)\mathbb{1}$, $\mathbf{U}(2) = (1/N^2 - 1)^2 \mathbb{1}$, which shows that any single error from the set (12) is correctable. To determine its distance from the ideal identity gate, we expand $\mathbf{U}(n)$ in terms of qubit Pauli matrices as $\mathbf{U}(n) = \frac{1}{2}(s_0 \mathbb{1} + \vec{s} \cdot \vec{\sigma})$. The numerical simulation in Fig. 3 for the $SU(3)$ case shows the value $\log_{10} \delta$ for $\delta := |\vec{s}|/|s_0|$ for different system sizes. We also find a similar behavior for the trace distance between $\mathbf{U}(n)$ and identity. We see that by an increase of 20 for the system size, the critical number of errors increases by 10. This shows that the critical number of errors is $L/2$, which means the code distance for the error set (12) is L, saturating the classical Singleton bound [22].

General errors lead to more complicated behavior. For two local unitary errors $V, U \in SU(N)$ with a spatial distance r, $\mathcal{V}$ and $\mathcal{U}$ as their adjoint rep, we find

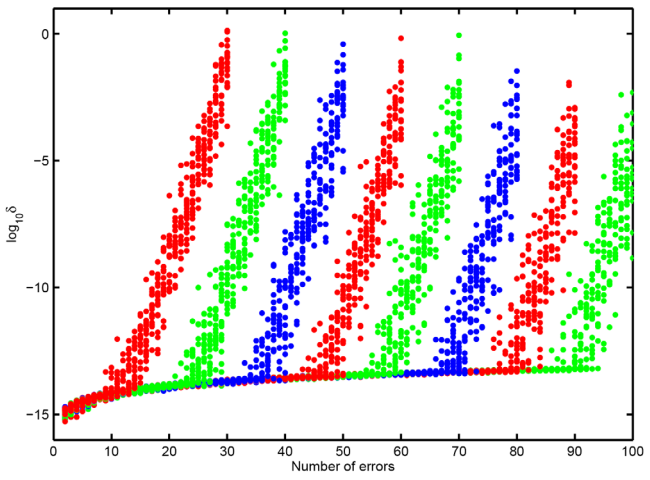

FIG. 3. Simulation of the deviation $\log_{10} \delta$ of $\mathbf{U}(n)$ from identity as the number of errors n from the discrete error set $\mathcal{E}$ (12) increases for different system sizes: (from left to right) from $L = 60$ to 200 by an increase of amount 20. For a fixed n, we chose 20 random errors at n random positions.

$$\langle \mathbf{L}|\mathcal{VU}|\mathbf{L}\rangle = \alpha + \beta, \qquad \langle \mathbf{R}|\mathcal{VU}|\mathbf{R}\rangle = \alpha + \beta^*, \tag{14}$$

and zero off-diagonal terms. Here the parameters $\alpha := [(N^2|u_0|^2 - 1)(N^2|v_0|^2 - 1)]/[(N^2 - 1)^2]$, $\beta := (-1/N^2 - 1)^{r+2}\beta_0$ for $\beta_0 \in \mathbb{C}$, and $V = v_0\mathbb{1} + \sum_i v_i P_i$, $U = u_0\mathbb{1} + \sum_i u_i P_i$, expanded in the Pauli basis $\{P_i\}$ of the virtual space. Note $\beta_0 = 0$ if $V = U$. We see that the imaginary part β decays exponentially with respect to their spatial distance r. The result above is consistent with the exponentially decaying correlation functions of the system, and it means that arbitrary unitary errors in the symmetry can cause a leakage together with a phase flip error, and they are not exactly correctable. However, the code will perform better for some cases, e.g., when the errors are dilute which could be met by lowering the temperature. This is confirmed by our simulations (Supplemental Material [42], Sec. III. B). Finally, we expect on-site random unitary errors from $SU(N^2 - 1)$ are not correctable since they destroy the SPT order of the system. Indeed this is the case, and fortunately, we find they are detectable (Supplemental Material [42], Sec. III. B).

Error syndrome can be determined by the energy of local terms H_n. For instance, after a Pauli error $\mathcal{P}$ on site n, the energy of local terms H_n and H_{n-1} each increases to $N^3(N^2 + 1)/3(N^2 - 1)^2$ (Supplemental Material [42], Sec. III. C). Recall that it is zero on ground states. The total energy penalty is about twice the energy of single adjointor excitation. This means a Pauli error requires higher energy to occur than an adjointor, and Pauli errors can be suppressed by avoiding two or more adjointor excitations. A general unitary error on site n can also be detected from the syndrome of H_{n-1} and H_n. Furthermore, if only one term shows the syndrome, this implies there is effectively only one adjointor, given a perfect energy check. This causes no problem for an error correction since a single adjointor cannot cause logical errors. By cooling the system to a low temperature, excitations will be suppressed and the Hamiltonian itself provides protection of the code.

As a final part of this work, we find the code performance of VBSQs can be improved by the concatenation with an outer code, e.g., a certain stabilizer code. A necessary ingredient of concatenation is state preparation and coupling of many VBSQs. We find there exists a measurement-based scheme [25] when $N = 4k$, with positive integers $k \in \mathbb{N}$, for which the logical $\bar{Z}$ gate can be realized by the twist (Supplemental Material [42], Sec. II). Measurements of a logical $\bar{Z}$ or $\bar{X}$ will prepare their eigenstates $|\mathbf{L}/\mathbf{R}\rangle$ or $|\mathbf{U}/\mathbf{D}\rangle = 1/\sqrt{2}(|\mathbf{L}\rangle \pm |\mathbf{R}\rangle)$, respectively. Using the five-qubit code [22], which has the parity property that the Hamming weight of its logical $|0\rangle$ ($|1\rangle$) is even (odd), a VBSQ at state $|\mathbf{L}\rangle$ can be prepared at state $1/\sqrt{2}(|\mathbf{L}\rangle + (-1)^x|\mathbf{R}\rangle)$ by the transversal coupling between the five-qubit code and each particle in VBSQ sequentially, and measuring the logical X of the five-qubit code (see Fig. 4) at the end, for x as the parity extracted from the measurement outcomes. This also works for other stabilizer codes that have the even-odd parity property, such as the Steane code and the Shor code. Error correction on the ancillary stabilizer code can be performed during the process to enhance fault tolerance. Logical operators $\bar{X}\bar{X}$, $\bar{Z}\bar{Z}$, etc., for two VBSQs can be measured in a similar fashion, which can then realize logical entangling gates such as the CNOT and controlled-phase gates.

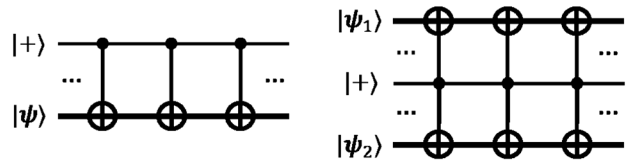

FIG. 4. The measurement of $\bar{X}$ (left) and $\bar{X}\bar{X}$ (right). Here each controlled symbol is for the operator Π, and each controlled Π represents the transversal controlled Π between a particle in VBSQ and the five-qubit code (at its logical state $|+\rangle$). The particles in VBSQ are acted upon sequentially.

Given the robustness against the $\bar{X}$ error, the repetition code is the most efficient choice to deal with a $\bar{Z}$ error on a VBSQ. For a series of VBSQ rings in parallel, enforcing the stabilizers $\bar{X}_i\bar{X}_{i+1}$ for all nearest-neighbor pairs of rings prepares the repetition code with codeword

$$|0/1_{\mathrm{L}}\rangle := \frac{1}{\sqrt{2}}(|\mathbf{UUU}\cdots\rangle \pm |\mathbf{DDD}\cdots\rangle), \tag{15}$$

and the logical operator $X_{\mathrm{L}} := \bar{X}$ for $\bar{X}$ on any ring and logical $Z_{\mathrm{L}} := \bar{Z}\bar{Z}\bar{Z}\cdots$. Notably, our code can correct more $\bar{Z}$ errors. When the parity check on the software level indicates a $\bar{Z}$ error, it can then be located with the help of error detection of individual rings. As a result, the minimal number of rings is two instead of three for the repetition code, and, in general, it can correct up to $\kappa - 1$, instead of $\frac{1}{2}(\kappa - 1)$, phase flip errors for κ rings. More generally, concatenation with stabilizer codes can be employed to safely avoid $\bar{X}$ error on a single VBSQ ring. The error detection of a single ring will also improve the code performance. For transversal gates, VBSQs for $N = 2^{k+1}$ concatenated with stabilizer codes allow transversal implementation of Z rotation of an angle $\pi/2^k$. For instance, there is a transversal phase gate for $N = 8$ and a T gate for $N = 16$, which are constrained by the structure of the Clifford hierarchy (also see Supplemental Material [42], Sec. IV).

In summary, we have proposed a construction of valence-bond solid qubits and concatenation schemes with stabilizer codes. We notice a distinction for even and odd N, while a scheme for odd N cases remains to be discovered. Our work demonstrates fundamental features of valence-bond solids and symmetry-protected topological order for quantum error correction and quantum memory, and can also be generalized to other valence-bond solids or crystals for various symmetries or spatial dimensions.

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