Causal Modeling the Delayed-Choice Experiment

Rafael Chaves, Gabriela Barreto Lemos, and Jacques Pienaar

International Institute of Physics, Universidade Federal do Rio Grande do Norte,
Campus Universitario, Lagoa Nova, Natal, Rio Grande do Norte 59078-970, Brazil

Wave-particle duality has become one of the flagships of quantum mechanics. This counterintuitive concept is highlighted in a delayed-choice experiment, where the experimental setup that reveals either the particle or wave nature of a quantum system is decided after the system has entered the apparatus. Here we consider delayed-choice experiments from the perspective of device-independent causal models and show their equivalence to a prepare-and-measure scenario. Within this framework, we consider Wheeler's original proposal and its variant using a quantum control and show that a simple classical causal model is capable of reproducing the quantum mechanical predictions. Nonetheless, among other results, we show that, in a slight variant of Wheeler's gedanken experiment, a photon in an interferometer can indeed generate statistics incompatible with any nonretrocausal hidden variable model, whose dimensionality is the same as that of the quantum system it is supposed to mimic. Our proposal tolerates arbitrary losses and inefficiencies, making it specially suited to loophole-free experimental implementations.

Wave-particle duality is at the heart of the most renowned debates in quantum theory. Although light and matter produce individual counts on a detector, they also exhibit interference in certain experimental arrangements. In Wheeler's delayed-choice experiment (WDCE) (Fig. 1) [1,2], the experimenter chooses whether or not to remove the beam splitter BS_2 after a photon has entered a Mach-Zehnder interferometer (at BS_1), thereby observing no interference (particlelike behavior) or interference (wavelike behavior) accordingly [3]. By excluding any causal link between the experimental setup and a hidden variable that predefines the photons behavior (retrocausality), delayed-choice experiments are usually about defining and testing wave-particle objectivity models, in which a quantum system is intrinsically either a wave or a particle (see [5] and references therein).

Within this mindset, Ionicioiu and Terno suggested a particular wave-particle objective model [6] (hereafter called the IT model), which they ruled out using a quantum delayed-choice experiment (QDCE), realized in [7], in which the BS_2 in WDCE is replaced by a quantum control that can be in a superposition of being present or absent until after the photon is detected. This reasoning relies upon a device-dependent argument that the beam splitter was truly in a quantum superposition, since the statistics alone could not distinguish a superposition from an incoherent mixture. This motivated entanglement-assisted QDCE [8–10], in particular, the experiments in [11,12], which rely on the violation of a Bell inequality to rule out the IT model in a device-independent (DI) manner, i.e., from the measurement statistics alone without prior assumptions about the quantum nature of the control. However, the IT

model of wave-particle objectivity makes rather strong assumptions that, as we show below, are trivially inconsistent with the assumption of no retrocausality and the quantum predictions. Could wave-particle objectivity be tested more generally, using causal assumptions rather than specifically tuned models? Furthermore, can we devise a DI proof of the nonclassical behavior of a delayed-choice experiment without the need for entanglement (or the violation of a Bell inequality)?

In this Letter, we employ tools from the field of causal inference [13] to give a positive answer to these questions. We propose that WDCE can be described by a DI "prepare-and-measure" scenario [14]. In this framework, the only relevant constraint on the classical model being tested is its
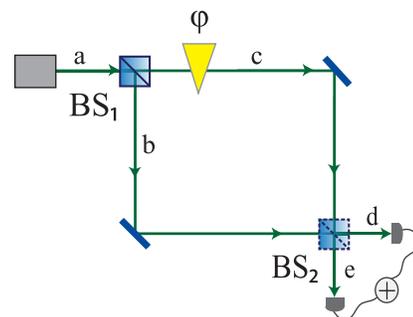

FIG. 1. In Wheeler's delayed-choice gedanken experiment, the choice of removing or not the beam splitter BS_2 in a Mach-Zehnder interferometer is made after the photon has entered the interferometer (at BS_1). With BS_2 present, the photon counting rate at either detector is a function of ϕ; when absent, the counting rate is independent of ϕ.

dimension, which leads us to suggest replacing wave-particle objectivity by the assumption that the hidden variable (HV) has the same dimension as the quantum system under test. We demonstrate that—contrary to intuition—a two-dimensional classical variable can explain the outcomes of WDCE (and of the QDCE). We then propose a delayed-choice experiment where, instead of removing BS_2, the experimenter can chose to slightly displace it, imparting a relative phase shift on the beam. Using previously derived dimension witnesses inequalities [14,15], we can exclude any model where the classical variable has at most two dimensions (assuming, e.g., the values wave and particle). Since we only detect one mode (we do not consider coincidences), our proposal tests the nonclassicality of this system without needing to introduce entanglement. Our proposal has the additional advantage of being robust to arbitrarily small (but larger than zero) detection efficiencies and to loss inside the interferometer, making it specially suited for loophole-free experimental implementations. Finally, we quantify how much retrocausal influence would be needed to explain the observations.

The delayed-choice experiment from a causal perspective.—How does quantum behavior manifest in the WDCE? On one hand, a single photon in an interferometer can, in principle, exhibit spatial mode entanglement. On the other hand, Wheeler's argument did not appeal to spatial separation, but to temporal causality: the state leaving the interferometer exists in the causal future of the state it possessed upon entry. This leads us to ask the question: could quantum behavior manifest itself even between the *temporal* parts of the WDCE, i.e., between the preparation of its causes and the measurement of its effects?

We begin by reviewing some basic concepts in causal modeling [13]. We use uppercase characters to denote random variables and lowercase to denote their possible values. Given a set of n experimental random variables $X_1, ..., X_n$, a hypothesis about the causal relationships between these variables can be represented by a directed acyclic graph (DAG) with the variables as nodes. Each arrow represents a direct causal influence of one variable upon another, in the following sense: if $\mathcal{P}(X_i)$ are the parents (direct causes) of X_i in the DAG, then there exists a local noise variable U_i (having no causes) and a deterministic function f_i such that $x_i = f_i(\mathcal{P}(x_i), u_i)$ [16]. Consequently, the joint probability distribution factorizes as a product, $p(\vec{x}) = \prod_{i=1}^{n} p(x_i|\mathcal{P}(x_i))$, relative to the

DAG. The factorization of probabilities is a DI constraint, as it depends only on the causal relationships described in the DAG and not on the particular choices of functions and noise variables $\{f_i, u_i\}$. This simplifies the task of causal hypothesis testing: if the observed probability does not factorize according to the structure of the hypothesized DAG, we can exclude any causal model based on that DAG from being a valid explanation for our observation.

We propose that the causal relations in WDCE (Fig. 1) can be represented by the DAG shown in Fig. 2(a). The variable X determines the phase shift ϕ_x between the interferometer arms. The variable Λ corresponds to the intrinsic state of the photon upon entering the interferometer, just after the phase ϕ_x. The values of Λ could correspond to possible quantum states or to values of some hidden variable encoding instructions for its future behavior (In Ref. [6], for example, this would correspond to particle or wave). For our purposes, only the dimensionality of Λ is relevant; its interpretation is unimportant. Since the experiment involves a single photon with two modes, which encodes only a single classical bit [17], Λ should likewise be binary. That is, we propose that the HV should not be able to encode more classical bits than the quantum system it is supposed to simulate. Assuming no retrocausal influences, the probability of Λ respects $p(\lambda|x, y) = p(\lambda|x)$, depending only on X (plus, possibly, local noise). In particular, it cannot depend on the delayed-choice Y, which determines the experimental arrangement $y = 1$ when BS_2 is present and $y = 0$ when it is removed [18]. Finally, the variables D and E represent the photon detectors and take the values $d, e \in \{1, 0\}$, depending whether the detector has clicked or not, respectively [19].

We will restrict our attention to the probability for the detector D to click [23]. The causal model in Fig. 2(a) implies that any observed distribution compatible with it should factorize as

$$p(d|x, y) = \sum_{\lambda} p(d|y, \lambda) p(\lambda|x). \qquad (1)$$

For comparison, let us compute the probabilities predicted by quantum mechanics in this setup. We treat the photon in the Mach-Zehnder interferometer as a two-level quantum system in the path degree of freedom. The initial state is a single photon in mode a. The state emerging from the interferometer is denoted $|\Psi(x, y)\rangle$. In the Fock basis $|\hat{n}_d \hat{n}_e\rangle$ at the output modes d, e, it has the form
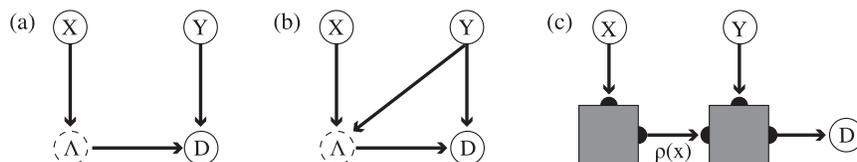

FIG. 2. (a) The DAG representing the causal model for WDCE. (b) A causal model allowing for retrocausal influence from Y to Λ. (c) The representation of the PAM scenario in terms of preparation and measurement devices (black boxes).

$|\Psi(x,0)\rangle = (1/\sqrt{2})(|01\rangle + e^{i\phi_x}|10\rangle)$ when BS_2 is absent and $|\Psi(x,1)\rangle = \cos(\phi_x/2)|01\rangle - i\sin(\phi_x/2)|10\rangle$ when BS_2 is present. The probabilities $p(d|x,y)$ for detector D to click are therefore given by $p(d|x,0) = 1/2$ and $p(1|x,1) = 1 - p(0|x,1) = \sin^2(\phi_x/2)$, respectively.

In light of this formulation, we now reexamine arguments put forward in Refs. [6,24] claiming that no hidden variable model of the form (1) could account for the quantum predictions of the QDCE. In that reference, it is assumed that wave-particle objectivity implies $\lambda = $ wave if and only if interference is observed, and $\lambda = $ particle otherwise [6,24]. In the present notation, this assumption would require Λ to be perfectly correlated with the variable D. However, quantum mechanics predicts correlations between D and the delayed-choice variable Y, and since D is binary (the detector can either click or not in each run), this leads to the conclusion that Λ is correlated with Y—but this can only be explained by a retrocausal influence of Y on Λ, which has been excluded by assumption. Our causal analysis shows that the assumptions in the IT model are trivially mutually inconsistent.

We now show that, absent any special assumptions about the relationship of Λ to the detector response D, we can reproduce the predictions of WDCE using a classical two-valued hidden variable. We introduce a local noise term U_D for the detector D, such that $p(d|y,\lambda) = \sum_{u_D} p(d|y,\lambda,u_D)p(u_D)$. Choosing

$$p(u_D = 0) = p(u_D = 1) = 1/2,$$

$$p(\lambda = 0|x) = 1 - p(\lambda = 1|x) = \cos^2\frac{\phi_x}{2},$$

$$p(d|y = 0, \lambda, u_D) = p(d|y = 0, u_D) = \delta_{d,u_D},$$

$$p(d|y = 1, \lambda, u_D) = p(d|y = 1, \lambda) = \delta_{d,\lambda}, \qquad (2)$$

one can verify that we recover the predictions of quantum theory for any choices of the phases ϕ_x. Therefore, WDCE can be explained by a classical causal model without the need of retrocausality. In the Supplemental Material [25], we show how this classical model can be extended to the QDCE [6].

The delayed-choice experiment as a prepare-and-measure scenario.—Given that there is a causal model (hence, a hidden variable model) explaining WDCE, it is natural to ask whether small modifications to the experiment would allow us to rule it out. We will continue to assume no retrocausality and that Λ is two dimensional. First, we draw a correspondence to the device-independent prepare-and-measure (PAM) scenario [14], shown schematically in Fig. 2(c). In the PAM scenario, an initial black box prepares different physical systems (upon pressing a button labeled by x) that are then sent to a second black box, where the systems are measured (upon pressing a button labeled by y) and produce an outcome labeled by d. The essential feature of the PAM scenario is that

quantum systems can produce statistical distributions that can only be reproduced by classical systems of higher dimensionality. In particular, there is a quadratic gap between classical and quantum dimensions, as one can devise situations where the statistics produced by a $(\sqrt{k} + 1)$-dimensional quantum system can only be reproduced by a classical system of at least $k + 1$ dimensions [15]. Moreover, the PAM scenario implies constraints on the probabilities [27] that are equivalent to the causal constraints of the delayed-choice experiment [Eq. (1)]. Therefore, general results pertaining to the PAM scenario can be readily adapted to analyze the experiment. For instance, for $2k$ preparations (choices of X) and k possible measurements (choices of Y), the $(k \times k)$ matrix [15]

$$W_k(i,j) = p(2j,i) - p(2j+1,i), \qquad (3)$$

with $p(i,j) = p(d = 0|x = i, y = j)$ $(0 \le i,j \le k - 1)$ satisfies $|\mathrm{Det}(W_k)| = 0$ for any classical system of dimension $\le k$. In WDCE, we have $k = 2$, since there are two possible experimental arrangements and we aim to test a classical model of dimension 2. The matrix of interest is given by

$$W_2 = \begin{pmatrix} p(0,0) - p(1,0) & p(2,0) - p(3,0) \\ p(0,1) - p(1,1) & p(2,1) - p(3,1) \end{pmatrix}, \qquad (4)$$

for which it can easily be verified that $|\mathrm{Det}(W_2)| = 0$ for the statistics predicted by quantum theory. Thus, the experiment cannot rule out a classical explanation such as the hidden variable model we described earlier. Strikingly, as we show next, by slightly modifying Wheeler's scenario, we can generate statistics that violate this dimensional witness and thus prove in a device-independent manner the incompatibility of a nonretrocausal hidden variable model with the generated data.

In our proposed experiment, the interferometer is always closed. Instead of removing BS_2, we gently displace it such that different measurement choices y now correspond to a new phase shift σ_y applied just before BS_2 (Fig. 3). From an experimental perspective, absorbing σ_y into the preparation phase shift ϕ_x has no effect on the photon counting rates. However, to ensure that no retrocausality implies the independence of Y and Λ, it is essential that Y lies in the causal future of Λ and hence of X. Therefore, the choice of phase σ_y must be delayed until long after the preparation phase ϕ_x, which is applied shortly after the photon passes BS_1.

By considering this extra phase σ_y and allowing for losses inside the interferometer, the statistics is given by $p(x,y) = \frac{1}{4}(T_a^2 + T_b^2) + (T_a T_b/2)\cos(\phi_x - \sigma_y)$, where $0 < T_a \le 1$ and $0 < T_b \le 1$ are the real transmittance coefficients of each arm. In this case,
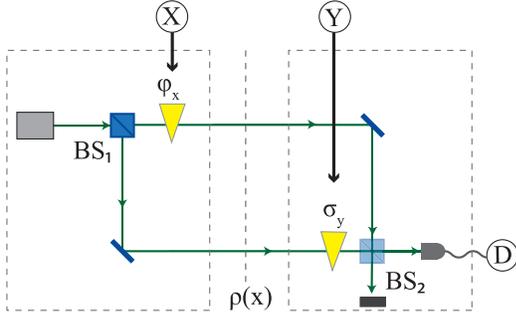
FIG. 3.   Our proposed modification of WDCE, which can be used to discard two-dimensional HV models. The gray dashed lines allude to Fig. 2(c). In addition to a phase shift ϕ_x in the preparation stage, another phase shift σ_y is applied after the single photon has entered the interferometer, for instance, by slightly moving BS_2.

$$\text{Det}(W_2) = \frac{T_a^2 T_b^2}{2} \sin(\sigma_0 - \sigma_1) \sin\left(\frac{\phi_2}{2} - \frac{\phi_3}{2}\right)$$

$$\times \left[-\cos\left(\phi_0 - \frac{\phi_2}{2} - \frac{\phi_2}{3}\right)\right.$$

$$\left. + \cos\left(\phi_1 - \frac{\phi_2}{2} - \frac{\phi_3}{2}\right)\right]. \tag{5}$$

Choosing $\phi_0 = 0$, $\phi_1 = \pi$, $\phi_2 = -\pi/2$, and $\phi_3 = \pi/2$, we obtain that $|\text{Det}(W_2)| = T_a^2 T_b^2 \sin(\sigma_0 - \sigma_1)$. If we make $\sigma_0 = \pi/2$ and $\sigma_1 = 0$, we obtain $|\text{Det}(W_2)| = T_a^2 T_b^2$, that is, the dimension witness is violated for any transmittance strictly larger than zero. This test is also resilient to detection inefficiencies. If our detector has efficiency η, then $p^\eta(b=0|x,y) = (1-\eta) + \eta p^{\eta=1}(b=0|x,y)$. Inserting this into W_2, we see that $|\text{Det}(W_2^\eta)| = \eta|\text{Det}(W_2^{\eta=1})|$. Even though the violation is less at lower efficiencies η, it can, in principle, be observed for any positive efficiency.

So far, we have also implicitly assumed that all noise terms are independent, and in particular, that the hidden variable Λ is independent of any noise term that might also influence the output of the interferometer. However, such a dependence would not be forbidden by causality, since we could imagine that the output of the interferometer depends on a noise variable Γ that exists prior to the preparation of ϕ_x and Λ and therefore might affect Λ as well. One might wonder whether it is possible to rule out hidden variable models that allow this dependence. Surprisingly, it can be done. To achieve this, we employ the dimension witness (DW) inequality [14]

$$I_{\text{DW}} = \langle D_{00}\rangle + \langle D_{01}\rangle + \langle D_{10}\rangle - \langle D_{11}\rangle - \langle D_{20}\rangle \le 3, \tag{6}$$

where $\langle D_{xy}\rangle = p(d=0|x,y) - p(d=1|x,y)$. This inequality involves three preparations and two measurements, and its violation witnesses incompatibility with any HV model of dimension 2, even in the presence of correlations

between the preparation and measurement devices, i.e., even when both Λ and the outcome of detector D have access to shared information. Using the same setup as above, we obtain $\langle D\rangle_{xy} = T_a T_b \cos(\phi_x - \sigma_y)$. For $T_a = T_b = 1$, and choosing $\sigma_0 = \pi/2$, $\sigma_1 = 0$, $\phi_0 = \pi/4$, $\phi_1 = 3\pi/4$, and $\phi_2 = -\pi/2$, we obtain the optimum quantum violation given by $I_Q = 1 + 2\sqrt{2}$ [28].

Quantifying retrocausality.—Altogether, our proposed modification of WDCE can exclude in a DI manner all nonretrocausal classical models with dimension $d = 2$. Conversely, if one allows retrocausality [see Fig. 2(b)], any observed distribution $p(d|x,y)$ could be simulated classically. But how much retrocausality is actually needed to reproduce the quantum predictions? To answer this, we need to quantify the strength of the causal influence $Y \to \Lambda$ in the DAG. Without this causal arrow, and allowing for correlations between preparation and measurement devices (described by a HV Γ), $p(\lambda|x,\gamma,y) = p(\lambda|x,\gamma,y')$. This leads us to consider a measure of retrocausality given by

$$R_{Y\to\Lambda} = \sup_{\lambda,x,y,y'} \sum_\gamma p(\gamma)|p(\lambda|x,\gamma,y) - p(\lambda|x,\gamma,y')|, \tag{7}$$

the maximum shift in the probability of the HV Λ produced by changes in the measurement setting Y for a preparation X (averaged over Γ, since we do not have empirical access to it). As explained in [25], given an observed value of the dimension witness I_{DW}, the minimum value of $R_{Y\to\Lambda}$ required to explain it is

$$\min R_{Y\to\Lambda} = \max\left[\frac{I-3}{4}, 0\right], \tag{8}$$

thus showing that the maximal possible value of $I_Q = 1 + 2\sqrt{2}$ allows one to exclude any retrocausal model with $R_{Y\to\Lambda} \lesssim 0.207$.

Discussion.—The vague notion of wave-particle objectivity has been analyzed in different ways by different authors. For instance, Wheeler [1,2] associated wave and particle notions to the possibility (respectively, impossibility) of a photon being in a path superposition inside the interferometer. In turn, the IT model [6] associates wave and particle notions with the statistics obtained at the detectors, implying the label wave if the statistics depend on the phase shift ϕ and particle otherwise. This interpretation may be criticized for implying that $\lambda = $ wave cannot produce statistics independent of ϕ, whereas in an open interferometer this is precisely what we would expect from a classical wave. Furthermore, it implies a correlation between Λ and the detector D which, as we showed using causal modeling, makes it trivially incompatible with the quantum predictions and no retrocausality.

To avoid the difficulties arising from particular interpretations of wave-particle objectivity, one could consider different ways of describing delayed-choice experiments.

In Ref. [9], it was shown that a simple argument based on causal models could provide conceptual insights into entanglement-assisted delayed-choice experiments (see [25]). We have shown that causal models can also shed light on the simpler experiments proposed by Wheeler [1,2] and the quantum variants proposed in Refs. [6,7].

We argued that, regardless of any categorization as a wave or particle, *any* predefined classical state of the photon that is supposed to reproduce the results of WDCE or QDCE should have two values, corresponding to the dimension of the quantum system being probed. This holds for both the Wheeler and IT models' conception of wave-particle objectivity, and indeed both can be cast as two-dimensional classical hidden variable models. By taking the dimensionality to be the main relevant feature, we showed that these models could be excluded using DI methods, provided the experiment is modified such that BS_2 is always present. Since wavelike behavior (interference) is always present in our experiment, our approach does not rely on making any kind of wave or particle distinction: we merely probe how much information must be conveyed by a classical variable in order to produce the observed interference. In this respect, our proposal is conceptually distinct from previous approaches. Moreover, our results extend to any two-dimensional hidden variable, whether it be a wave, a particle, or something queer in between.

This gives us a new perspective on what is counterintuitive in delayed-choice experiments: the fact that any classical explanation requires a variable with more dimensions than its quantum counterpart or else requires retrocausal influences. Our results also show the benefits of viewing quantum phenomena from a causal perspective. For applications, since the PAM scenario plays an important role in recent work on quantum key distribution [29–31], our Letter suggests that interesting quantum information protocols could be performed with setups as simple as a Mach-Zehnder interferometer.

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