Novel Soft-Pion Theorem for Long-Range Nuclear Parity Violation

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The parity-odd effect in the standard model weak neutral current reveals itself in the long-range parity-violating nuclear potential generated by the pion exchanges in the $\Delta I = 1$ channel with the parity-odd pion-nucleon coupling constant h_π^1. Despite decades of experimental and theoretical efforts, the size of this coupling constant is still not well understood. In this Letter, we derive a soft-pion theorem relating h_π^1 and the neutron-proton mass splitting induced by an artificial parity-even counterpart of the $\Delta I = 1$ weak Lagrangian and demonstrate that the theorem still holds exact at the next-to-leading order in the chiral perturbation theory. A considerable amount of simplification is expected in the study of h_π^1 by using either lattice or other QCD models following its reduction from a parity-odd proton-neutron-pion matrix element to a simpler spectroscopic quantity. The theorem paves the way to much more precise calculations of h_π^1, and thus a quantitative test of the strangeness-conserving neutral current interaction of the standard model is foreseen.

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The study of the standard model (SM) parity (P) violation in nonleptonic processes is extremely difficult due to the overwhelming background of the strong interaction governed by quantum chromodynamics (QCD). Nevertheless, it is essential in order to better understand the general properties of the hadronic weak interaction and shed light on many unresolved puzzles such as the unexpectedly large violation of Hara's theorem in the hyperon weak radiative decays and the failure to simultaneously fit the S- and P-wave amplitudes of the hyperon decays. The P violation in the strangeness-conserving, $\Delta I = 1$ nucleon-nucleon interaction is a perfect ground to study the properties of the neutral weak current in hadronic systems which is otherwise poorly constrained, as it cannot be probed in the usual strangeness-changing weak processes due to the absence of a tree-level flavor-changing neutral current [1]. Ongoing P-violation experiments with an unprecedented level of precision (see the discussion below) call for a new round of theoretical study of the hadronic P violation so that the experimental results can be utilized to their largest extent in testing our current understanding of the SM hadronic weak interaction.

On the theory side, Desplanques, Donoghue, and Holstein (DDH) [2] formulated both the P-conserving (due to the strong interaction) and the P-violating (due to the weak interaction including all $\Delta I = 0, 1, 2$ channels) nucleon-nucleon interaction in terms of the single exchange of the lowest-lying light mesons, i.e., π, ρ, and ω. This description forms the basis of many experimental analyses. More recently, the description of the P-violating nucleon-nucleon forces and the associated currents has been based on effective field theory (EFT) frameworks such as the pionless EFT [3,4] or the chiral EFT [5–7]. Experimental progress has been made as well, although mainly in the last decades of the previous century, as exemplified by measurements of the P-violating longitudinal analyzing powers (LAPs) [8–12], the gamma-ray asymmetries [13–17], and the gamma-ray circular polarization [18–23]. For recent reviews, we refer to Refs. [1,7,24].

A long-standing problem in the field of hadronic P violation is the theoretical determination of the P-odd hadronic coupling constants. Since the underlying weak operators and their Wilson coefficients are rather well known, the outstanding problem is to compute the associated hadronic matrix elements directly with nonperturbative methods or to fit them to data. In particular, the P-violating, $\Delta I = 1$ pion-nucleon coupling h_π^1 has attracted much attention, as it is formally the single leading-order (LO) operator in the chiral EFT framework [25]. As such, it is expected to dominate the long-range part of the P-violating nucleon-nucleon potential and the resulting P-odd phenomenology in various processes. The above conclusion, however, depends crucially on the actual size of h_π^1, and, if it turns out smaller than originally expected, other

P-violating hadronic interactions can become dominant. Such interactions are described by the P-odd couplings between nucleons and heavier mesons (like the ρ and ω) in the DDH approach or as the P-odd nucleon-nucleon contact [5,26] and derivative pion-nucleon [27] interactions in the EFT language.

The simplest order-of-magnitude estimate of the size of h_π^1 is based on the naive dimensional analysis which gives $h_\pi^1 \sim \mathcal{O}(G_F F_\pi \Lambda_\chi) \sim 10^{-6}$ in terms of the Fermi constant G_F, the pion decay constant F_π, and the chiral-symmetry-breaking scale Λ_χ. This estimate does not capture the potential suppression due to powers of $\sin\theta_W$ or large N_c arguments [1,28]. In the DDH paper, a "reasonable range" for h_π^1 is given as $(0–11) \times 10^{-7}$ together with a "best value" around 4.6×10^{-7} based on a quark model and SU(6) flavor-spin symmetry. Other phenomenological studies of h_π^1 include the use of quark models [29–31], Skyrme models [32–34], and QCD sum rules [35,36]. In general, both the quark model and QCD sum rules predict an order of 10^{-7} for h_π^1; meanwhile, early chiral Skyrmion approaches [32,33] predict $\mathcal{O}(10^{-8})$, but subsequent work [34] gives 10^{-7}, all rather small values that are difficult to probe experimentally.

On one hand, such small values are in good agreement with the absence of a P-odd signal in the γ emission from ^{18}F, which gives an upper bound of $h_\pi^1 \leq 1.3 \times 10^{-7}$ [15,23,37]. On the other hand, the size of the Cs anapole moment [38,39] indicates a larger h_π^1, and the same holds for the measurement of the LAP in proton-alpha scattering [12,40]. The interpretation of the latter experiments, however, suffers from significant theoretical uncertainties due to the complicated systems involved in the experiments. A more promising approach seems to rely only on experiments involving a few nucleons, where the nuclear theory is under better control. In this light, Refs. [6,41] tried to extract h_π^1 from measurements of the proton-proton LAP. Unfortunately, h_π^1 contributes only to the proton-proton scattering at the subleading order, leading to a large uncertainty on the extraction $h_\pi^1 = (1.1 \pm 2.0) \times 10^{-6}$. More promising is the extraction of h_π^1 from the upcoming measurement of the gamma-ray asymmetry in the neutron capture on the proton by the NPDGamma Collaboration [42,43].

In view of these experimental efforts to extract h_π^1, there is the need to calculate its value reliably using, e.g., lattice QCD, in order to quantitatively test the strangeness-conserving neutral current aspect of the SM. So far, the only direct lattice QCD calculation of h_π^1 was attempted in Ref. [44] by studying a three-point correlation function. The result was incomplete partially due to its inability to extract signals from the so-called quark-loop diagram, which suffered from a too small signal-to-noise ratio. The existence of an explicit pion in the final state also brought about extra technical complications. For example, an extra total-derivative operator with an unknown coefficient must be introduced for the insertion of energy into the weak vertex.

In this Letter, we propose a new starting point for the theoretical investigation of h_π^1 by deriving a soft-pion theorem. This theorem relates h_π^1 to the neutron-proton mass splitting induced by an artificial P-even counterpart of the $\Delta I = 1$ weak Lagrangian. This approach is parallel to one of the techniques in the study of P-and-time-reversal-odd pion-nucleon coupling $\bar{g}_\pi^i$, which also attempts to relate $\bar{g}_\pi^i$ to the nucleon mass shifts generated by the underlying P-even operators. Such relations were first derived using the current algebra [45,46] and later refined under the framework of the chiral perturbation theory (ChPT) [47–52]. In its application to h_π^1, we find that the simple matching relation derived in the LO ChPT is exactly preserved by all corrections at the next-to-leading order (NLO), i.e., $\mathcal{O}(M_\pi^2/\Lambda_\chi^2)$ with M_π the pion mass and $\Lambda_\chi \approx 1$ GeV the chiral-symmetry-breaking scale, including both the one-loop and low-energy constant (LEC) contributions, which is a unique feature not shared by $\bar{g}_\pi^i$. Hence, the accuracy of such a simple matching relation is expected to be better than one percent. Considerable advantages are expected by studying the neutron-proton mass splitting instead of h_π^1 itself using either lattice QCD or other nonperturbative approaches.

We start by reviewing the underlying physics of the flavor-conserving nuclear P violation in the SM. Well below the electroweak scale, the W or Z bosons can be integrated out in exchange of four-quark operators in the form of current-current products. In the limit of vanishing Cabibbo angle θ_C, the charged current will not contribute to the $\Delta I = 1$ parity violation, and in reality it is suppressed by $\sin^2\theta_C \simeq 0.05$. Hence, the flavor-conserving $\Delta I = 1$ parity-violating nuclear processes serve as a unique probe to test the otherwise poorly constrained neutral current interaction of the SM. If one considers only the three lightest quarks, the effects of P violation in the $\Delta S = 0$, $\Delta I = 1$ channel are carried by seven independent P-odd four-quark operators which take the following form [53] (there is another operator θ_4 defined in Ref. [25], but it is not independent from $\{\theta_1, \theta_2, \theta_3\}$):

$$\mathcal{L}_{\mathrm{PV}}^w = -\frac{G_F}{\sqrt{2}}\frac{\sin^2\theta_W}{3}\sum_i (C_i^{(1)}\theta_i + S_i^{(1)}\theta_i^{(s)}), \tag{1}$$

where

$$\begin{aligned}
\theta_1 &= \bar{q}_a\gamma^\mu q_a \bar{q}_b\gamma_\mu\gamma_5\tau_3 q_b, & \theta_2 &= \bar{q}_a\gamma^\mu q_b \bar{q}_b\gamma_\mu\gamma_5\tau_3 q_a,\\
\theta_3 &= \bar{q}_a\gamma^\mu\gamma_5 q_a \bar{q}_b\gamma_\mu\tau_3 q_b, & &\\
\theta_1^{(s)} &= \bar{s}_a\gamma^\mu s_a \bar{q}_b\gamma_\mu\gamma_5\tau_3 q_b, & \theta_2^{(s)} &= \bar{s}_a\gamma^\mu s_b \bar{q}_b\gamma_\mu\gamma_5\tau_3 q_a,\\
\theta_3^{(s)} &= \bar{s}_a\gamma^\mu\gamma_5 s_a \bar{q}_b\gamma_\mu\tau_3 q_b, & \theta_4^{(s)} &= \bar{s}_a\gamma^\mu\gamma_5 s_b \bar{q}_b\gamma_\mu\tau_3 q_a.
\end{aligned} \tag{2}$$

Here $q = (u, d)^T$ is the quark isospin doublet field, s is the strange quark field, $\{a, b\}$ are color indices, and θ_W is the weak mixing angle. The mixing of these operators under one-loop perturbative QCD corrections introduces a scale dependence to the Wilson coefficients $\{C_i^{(1)}, S_i^{(1)}\}$ [25,53,54]. The uncertainties of the Wilson coefficients are relatively under control: For instance, the higher-order corrections to the LO QCD running are generally of the order of 10%–20% [53].

The $\Delta I = 1$ P-violating coupling constants in either the DDH formalism or the EFT description are then just the QCD matrix elements of the Lagrangian (1) at the scale $\mu \approx \Lambda_\chi$ with respect to appropriate external hadronic states. The aim of this Letter is to relate these matrix elements to another set of P-even matrix elements with fewer external states. The easiest way to understand this formalism is to realize that the partially conserved axial current relation relates matrix elements with and without a soft external pion:

$$\lim_{p_\pi \to 0} \langle N' \pi^i | \mathcal{L}^w_{\mathrm{PV}} | N \rangle \approx \frac{i}{F_\pi} \langle N' | [\mathcal{L}^w_{\mathrm{PV}}, \hat{Q}^i_A] | N \rangle, \tag{3}$$

where $\hat{Q}^i_A$ is the axial charge; the right-hand side can be further reduced to a flavor-diagonal matrix element through the Wigner-Eckart theorem. This observation inspires us to construct a P-even chiral partner of $\mathcal{L}^w_{\mathrm{PV}}$ as follows:

$$\mathcal{L}^w_{\mathrm{PC}} = -\frac{G_F}{\sqrt{2}} \frac{\sin^2\theta_W}{3} \sum_i (C_i^{(1)} \theta'_i + S_i^{(1)} \theta_i^{(s)\prime}), \tag{4}$$

where

$$\begin{aligned}
&\theta'_1 = \bar{q}_a \gamma^\mu q_a \bar{q}_b \gamma_\mu \tau_3 q_b, \qquad \theta'_2 = \bar{q}_a \gamma^\mu q_b \bar{q}_b \gamma_\mu \tau_3 q_a, \\
&\theta'_3 = \bar{q}_a \gamma^\mu \gamma_5 q_a \bar{q}_b \gamma_\mu \gamma_5 \tau_3 q_b, \\
&\theta_1^{(s)\prime} = \bar{s}_a \gamma^\mu s_a \bar{q}_b \gamma_\mu \tau_3 q_b, \qquad \theta_2^{(s)\prime} = \bar{s}_a \gamma^\mu s_b \bar{q}_b \gamma_\mu \tau_3 q_a, \\
&\theta_3^{(s)\prime} = \bar{s}_a \gamma^\mu \gamma_5 s_a \bar{q}_b \gamma_\mu \gamma_5 \tau_3 q_b, \qquad \theta_4^{(s)\prime} = \bar{s}_a \gamma^\mu \gamma_5 s_b \bar{q}_b \gamma_\mu \gamma_5 \tau_3 q_a.
\end{aligned} \tag{5}$$

Regardless of the DDH formalism or the (pionful) EFT description, the long-range P-violating nuclear potential always features pion exchanges. If we write $N = (p, n)^T$ as the nucleon isospin doublet, then there are four kinds of $NN\pi$ couplings one could write down in terms of the isospin decomposition: $\bar{N} \vec{\tau} \cdot \vec{\pi} N$, $\pi^0 \bar{N} N$, $\bar{N}(\vec{\tau} \times \vec{\pi})^3 N$, and $\bar{N}(\vec{\tau} \cdot \vec{\pi} - 3\pi^0 \tau_3) N$, where the first term has $\Delta I = 0$, the second and third terms have $\Delta I = 1$, and the last term has $\Delta I = 2$. Since Barton's theorem [55] excludes the possibility of exchanging neutral pseudoscalars in the CP-conserving limit, the only available structure is $\bar{N}(\vec{\tau} \times \vec{\pi})^3 N$, which has $\Delta I = 1$ and is therefore dominated by the neutral current contribution. The LO P-odd pion-nucleon coupling term can thus be written as

$$\mathcal{L}^w_{\mathrm{PV}} = -\frac{h^1_\pi}{\sqrt{2}} \bar{N} (\vec{\tau} \times \vec{\pi})^3 N + \cdots = i h^1_\pi (\bar{n} p \pi^- - \bar{p} n \pi^+) + \cdots, \tag{6}$$

where the pion-nucleon coupling constant h^1_π may be expressed in terms of the hadronic matrix element

$$h^1_\pi = -\frac{i}{2 m_N} \lim_{p_\pi \to 0} \langle n \pi^+ | \mathcal{L}^w_{\mathrm{PV}}(0) | p \rangle. \tag{7}$$

Here m_N is the averaged nucleon mass.

We shall now derive the promised soft-pion theorem using the ChPT. As far as this work is concerned, it is sufficient to restrict ourselves to the SU(2) version of the ChPT, since strangeness is conserved in the weak Lagrangian of our interest. We should stress that this does not mean that we are disregarding the effects of the operators with strange fields, i.e., $\{\theta_i^{(s)}\}$; they are just having the same isospin structure as the nonstrange operators $\{\theta_i\}$ and can be described by the same spurion involving only SU(2) indices, as we shall demonstrate later. The LO chiral Lagrangians for QCD in the pion and nucleon sector are given by

$$\begin{aligned}
&\mathcal{L}_\pi = \frac{F_0^2}{4} \mathrm{Tr}[\partial_\mu U \partial^\mu U^\dagger] + \frac{F_0^2 B_0}{2} \mathrm{Tr}[M_q U^\dagger + U M_q^\dagger], \\
&\mathcal{L}_N = \bar{N} i v \cdot \mathcal{D} N + g_A \bar{N} u_\mu S^\mu N,
\end{aligned} \tag{8}$$

respectively. Here we adopt the standard notations of the ChPT as in Ref. [56]: $U = \exp\{i \vec{\pi} \cdot \vec{\tau} / F_0\}$, $u = \sqrt{U}$, and $u_\mu = i(u^\dagger \partial_\mu u - u \partial_\mu u^\dagger)$, while $M_q = \mathrm{diag}(m_u, m_d)$ is the quark mass matrix that gives rise to the LO pion mass $M_\pi^2 = B_0(m_u + m_d)$. In the nucleon sector, we adopt the heavy baryon chiral perturbation theory formalism [57,58] so that the nucleon field N appears as a massless excitation with four-velocity v and the chiral covariant derivative $\mathcal{D}_\mu = \partial_\mu + (u^\dagger \partial_\mu u + u \partial_\mu u^\dagger)/2$. The finite quark mass effect could be implemented to the baryon Lagrangian at higher orders through the matrices $\chi_\pm = 2B_0(u^\dagger M_q u^\dagger \pm u M_q^\dagger u)$. Next, we turn to the discussion of the weak chiral Lagrangian. The effects of both $\mathcal{L}^w_{\mathrm{PV}}$ and $\mathcal{L}^w_{\mathrm{PC}}$ can be implemented into the chiral Lagrangian by means of the spurion method. To understand the procedure, we first combine the two Lagrangians to obtain

$$\mathcal{L}^w_{\mathrm{tot}} = \mathcal{L}^w_{\mathrm{PV}} + \mathcal{L}^w_{\mathrm{PC}} = -\frac{G_F}{\sqrt{2}} \frac{\sin^2\theta_W}{3} \sum_i (C_i^{(1)} \tilde{\theta}_i + S_i^{(1)} \tilde{\theta}_i^{(s)}), \tag{9}$$

where $\tilde{\theta}_i = \theta_i + \theta'_i$ and $\tilde{\theta}_i^{(s)} = \theta_i^{(s)} + \theta_i^{(s)\prime}$. One immediately observes that the operators $\{\tilde{\theta}_i, \tilde{\theta}_i^{(s)}\}$ break the SU(2) chiral symmetry via the matrix τ_3. Therefore, the effect of $\mathcal{L}^w_{\rm tot}$ can be implemented to the chiral Lagrangian through a Hermitian, traceless spurion, $X_R = u^\dagger \tau_3 u$. The LO weak Lagrangian in the nucleon sector that incorporates such a spurion is simply [25,59]

$$\begin{aligned}\mathcal{L}^w_{\rm tot,LO} &= \alpha \bar{N} X_R N \\ &= \alpha \bar{N}\tau^3 N - \frac{\sqrt{2}i}{F_0}\alpha(\bar{n}p\pi^- - \bar{p}n\pi^+) + \cdots,\end{aligned} \tag{10}$$

where α is an unknown LEC. In the second line, we have expanded the Lagrangian to the first power of the pion field; the first term corresponds to the neutron-proton mass splitting, while the second corresponds to the P-odd pion-nucleon coupling. The fact that they share the same LEC α implies a relation between these two quantities:

$$F_\pi h^1_\pi \approx -\frac{(\delta m_N)_{4q}}{\sqrt{2}}, \tag{11}$$

where $(\delta m_N)_{4q}$ is the neutron-proton mass splitting induced by $\mathcal{L}^w_{\rm PC}$:

$$(\delta m_N)_{4q} = \frac{1}{m_N}\langle p|\mathcal{L}^w_{\rm PC}(0)|p\rangle = -\frac{1}{m_N}\langle n|\mathcal{L}^w_{\rm PC}(0)|n\rangle. \tag{12}$$

Equation (11) is the central result of our Letter, and it is derived from the LO ChPT. Next, we consider the NLO effects due to both one-loop diagrams and LECs to this tree-level relation. The relevant one-particle irreducible loop diagrams are depicted in Fig. 1. Meanwhile, one also needs to compute the renormalization of F_π as well as the wave function renormalization of the pion field. They are given by

$$\delta(F_\pi) = -\frac{I_e}{F_\pi^2} + \frac{M_\pi^2}{F_\pi} l_4, \qquad \sqrt{Z_\pi} - 1 = \frac{I_e}{3F_\pi^2} - \frac{M_\pi^2}{F_\pi^2} l_4, \tag{13}$$

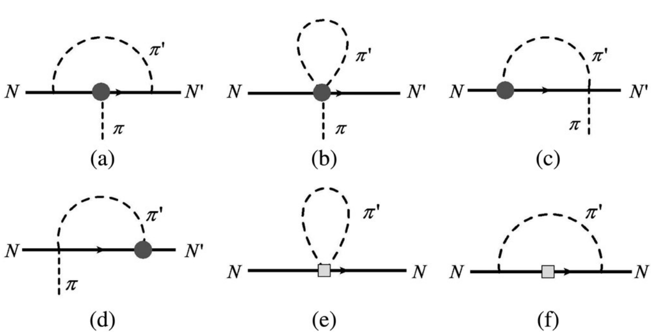

FIG. 1. One-loop diagrams that contribute to h^1_π [(a)–(d)] and $(\delta m_N)_{4q}$ [(e),(f)]. The black dot and gray box denote the $\Delta I = 1$ $NN\pi$ and NN weak vertex insertion, respectively. Self-energy diagrams are not shown explicitly.

where l_4 is the well-known $O(p^4)$ LEC in the SU(2)-mesonic chiral Lagrangian [60]. The total one-loop correction to the left- and right-hand side of Eq. (11), including both the one-particle irreducible and wave function renormalization contributions, reads (the one-loop correction to h^1_π was previously calculated in Ref. [61])

$$\begin{aligned}\delta(F_\pi h^1_\pi)_{\rm loop} &= \left(\frac{g_A^2}{F_\pi^2}I_a - \frac{1}{F_\pi^2}I_e + \delta Z_N\right)F_\pi h^1_\pi, \\ \delta((\delta m_N)_{4q})_{\rm loop} &= \left(\frac{g_A^2}{F_\pi^2}I_a - \frac{1}{F_\pi^2}I_e + \delta Z_N\right)(\delta m_N)_{4q},\end{aligned} \tag{14}$$

where

$$I_a = -\frac{3M_\pi^2}{64\pi^2}\left(R_\pi + \frac{2}{3}\right), \qquad I_e = \frac{M_\pi^2}{16\pi^2}R_\pi, \tag{15}$$

with $R_\pi = [2/(d-4)] + \gamma_E - \ln(4\pi) - 1 + \ln(M_\pi^2/\mu^2)$ and μ the renormalization scale, are the loop functions defined in Ref. [50], and $Z_N = 1 + \delta Z_N$ is the nucleon wave function renormalization whose explicit form does not concern us. From Eq. (15), one observes that the loop corrections to both sides of Eq. (11) simply result in a common multiplicative factor, so the matching relation is unaltered by one-loop corrections. The results above are obviously incomplete, because one needs to introduce the NLO weak chiral Lagrangian simultaneously in order to absorb the ultraviolet divergences in the loop diagrams as well as to render the final expressions scale independent. Such a Lagrangian involves a single insertion of the quark mass matrix. There are only two independent terms at this order [59]:

$$\mathcal{L}^w_{\rm tot,NLO} = \tilde{c}_1 \bar{N}\{\chi_+, X_R\}N + \tilde{c}_2 {\rm Tr}(\chi_+)\bar{N}X_R N. \tag{16}$$

Their contributions are

$$\begin{aligned}\delta(h^1_\pi)_{\rm LEC} &= -\frac{8\sqrt{2}}{F_0}B_0\bar{m}(\tilde{c}_1 + \tilde{c}_2), \\ \delta((\delta m_N)_{4q})_{\rm LEC} &= 16B_0\bar{m}(\tilde{c}_1 + \tilde{c}_2),\end{aligned} \tag{17}$$

where $\bar{m} = (m_u + m_d)/2$. We find that the quantities $\delta(h^1_\pi)_{\rm LEC}$ and $\delta((\delta m_N)_{4q})_{\rm LEC}$ also satisfy the matching relation in Eq. (11). Therefore, the soft-pion theorem relating h^1_π to $(\delta m_N)_{4q}$ is protected against all corrections of NLO, including both the one-loop and LEC contributions and without assuming isospin symmetry. Hence, we expect the accuracy of such a relation to be better than $(M_\pi/\Lambda_\chi)^2 \sim 1\%$ when the light quark masses take the physical values.

In the conventional approach, the h^1_π coupling is extracted from the hadronic matrix element involving the nucleon-pion state. Such a hadronic matrix element can be calculated nonperturbatively using lattice QCD. However, it would cause three complexities.

(i) The nucleon-pion state is a rescattering state. As a result, the hadronic matrix element calculated in the finite lattice box suffers from a power-law finite-volume effect. Only after the appropriate finite-volume correction [62] can the hadronic matrix element in the finite volume be related to the physical one in the infinite volume.

(ii) Because of the inequality of the energies of the on-shell nucleon and the nucleon-pion state, the weak four-quark operator involves an energy insertion. As the injected energy must exceed $E \geq M_\pi + m_n - m_p$, the LO effect of the energy insertion scales as $\sqrt{m_q}$ and may dominate over the loop correction in the ChPT. This LO contamination can be removed by the antisymmetric combination of the forward $(p \to n\pi)$ and backward $(n\pi \to p)$ transitions [59], but the higher-order terms $\sim m_q$ still remain. Although the systematical effect associated with the nonzero energy insertion vanishes in the chiral limit, it still complicates the lattice calculation, as nonzero quark masses are used in the simulation.

(iii) Although a three-quark interpolating operator can be used to create the nucleon-pion state in the S wave [44], it is known from lattice QCD that the overlapping amplitude between a three-quark operator and a two-hadron state can be significantly suppressed [63]. To gain a better precision, it is desirable to use a nucleon-pion interpolating operator to build the correlation function. However, it would make the quark contractions more complicated and the calculation more expensive.

By relating the P-violating hadronic matrix element to the P-conserving one using Eq. (11), one can reduce a nucleon-pion state to a single nucleon state. As a consequence, all of the three complexities mentioned above disappear, and the calculation is much simplified. Considering the fact that the only existing lattice calculation performed at $M_\pi \simeq$ 389 MeV yields a result with $\sim$50% statistical uncertainty [44], it is an important intermediate step to study the P-conserving matrix element as an alternative. Using the Feynman-Hellmann method proposed in Ref. [64], one can calculate $(\delta m_N)_{4q}$ using the correlation functions of a single time variable, which simplifies the procedure to remove the excited-state contamination.

Finally, the soft-pion theorem shown here also brings benefits to other diagram-based analysis of h_π^1 such as the Dyson-Schwinger equation and the partially quenched ChPT. The former, for example, computes hadronic matrix elements by evaluating "loop diagrams" of quark and gluons with insertions of fully dressed vertices and propagators obtained by solving integral equations. The disappearance of the pion in the external state greatly reduces the number of diagrams, in particular, those involving contractions between the quarks in the nucleon and the pion. The latter is able to isolate diagrams with specific contractions in a given hadronic matrix element through calculations of tree and loop diagrams in the ChPT with an extended flavor sector [65–67]. Based on our theorem, the objects of interest are translated to baryon mass parameters, so the number of loop diagrams is much smaller (which can be seen from Fig. 1), making the calculation more tractable.

In summary, we demonstrate that the h_π^1, induced by $\Delta I = 1$ parity-odd four-quark operators resulting from the exchange of the Z boson as well as QCD running, can be related to the neutron-proton mass splitting $(\delta m_N)_{4q}$ induced by a corresponding set of $\Delta I = 1$, parity-even four-quark operators. The matching relation is established as a soft-pion theorem $F_\pi h_\pi^1 = -(\delta m_N)_{4q}/\sqrt{2}$, which is protected against any correction of NLO in the chiral expansion. Therefore, instead of h_π^1, one may study $(\delta m_N)_{4q}$, which is a much simpler hadronic matrix element due to the disappearance of the pion from the external state. Such matching brings about benefits to both lattice and other nonperturbative QCD calculations of h_π^1. We hope that our finding could serve as a new starting point for the next round of theoretical investigations, which could be directly contrasted to the upcoming experimental results, and thus shed new lights on the many unresolved puzzles in hadronic weak interactions.

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