Quasiperiodic Quantum Ising Transitions in 1D

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Unlike random potentials, quasiperiodic modulation can induce localization-delocalization transitions in one dimension. In this Letter, we analyze the implications of this for symmetry breaking in the quasiperiodically modulated quantum Ising chain. Although weak modulation is irrelevant, strong modulation induces new ferromagnetic and paramagnetic phases which are fully localized and gapless. The quasiperiodic potential and localized excitations lead to quantum criticality that is intermediate to that of the clean and randomly disordered models with exponents of $\nu = 1^+$ (exact) and $z \approx 1.9$, $\Delta_\sigma \approx 0.16$, and $\Delta_\gamma \approx 0.63$ (up to logarithmic corrections). Technically, the clean Ising transition is destabilized by logarithmic wandering of the local reduced couplings. We conjecture that the wandering coefficient w controls the universality class of the quasiperiodic transition and show its stability to smooth perturbations that preserve the quasiperiodic structure of the model.

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Sufficiently strong quasiperiodic modulation can drive a localization transition in one-dimensional wires, as was first shown by Azbel, Aubry, and André [1–4]. Insofar as the unmodulated wire is described by a critical Dirac theory, this suggests that strong modulation ought to be able to localize other quantum critical points. On the other hand, if the critical point mediates the development of long-range order, it must have an extended mode at zero energy. At the quantum Ising transition in the presence of *disorder*, this tension gives rise to infinite randomness physics [5–13]. In this Letter, we show that sufficiently strong smooth *quasiperiodic* modulation drives the quantum Ising transition into a new quasiperiodic (QP) Ising universality class. The properties of this universality class are intermediate between those of the clean and infinite randomness cases and are robust to smooth perturbations that preserve the QP structure.

The discovery and growth of quasicrystals [14–16] motivated the study of critical systems modulated by discrete quasiperiodic substitution sequences [17–25] including the quantum Ising chain [26–42]. However, recent optical lattice experiments naturally realize smooth quasiperiodic modulation [43–47]. While such modulation has been investigated in related models [1–4,48–63], Luck's analysis [34] of wandering showed smooth QP modulation to be perturbatively irrelevant at the quantum Ising transition. This deterred the further study of the strongly modulated regime, whose physics we here uncover.

The generic QP transverse field Ising model (TFIM) in one dimension has the Hamiltonian

$$H = -\frac{1}{2}\sum_j J(Qj)\sigma_j^x\sigma_{j+1}^x + \Gamma(Qj)\sigma_j^z, \tag{1}$$

where $J(\theta)$ and $\Gamma(\theta)$ are smooth 2π-periodic functions. The modulation is quasiperiodic if the wavelength $2\pi/Q$ is an irrational multiple of the lattice length $a = 1$. Our general results apply to a large class of irrational wave vectors (see Supplemental Material [64]); numerical results use the golden mean $Q/2\pi = \tau \equiv (1 + \sqrt{5})/2$. The QP model is best understood as the limit of a sequence of commensurate models with wave vectors $\tilde{Q} = 2\pi p/q$, for coprime integers p, q such that $p/q \to Q/2\pi$ [65]. The period q is then the finite length scale which controls scaling behavior.

Using the Jordan Wigner transformation, Eq. (1) maps on to a free Majorana chain [66]:

$$H = \frac{i}{2}\sum_j J(Qj)\gamma_{2j+1}\gamma_{2j+2} + \Gamma(Qj)\gamma_{2j}\gamma_{2j+1}, \tag{2}$$

where γ_i are Majorana fermion operators (for conventions and details, see Ref. [67]). For an open chain in the Ising ordered phase, there are exponentially localized zero modes bound to the system edges. The zero mode wave functions at the left edge is

$$\psi_{2j}^0 \propto \prod_{i<j}\left|\frac{\Gamma(Qi)}{J(Qi)}\right| \equiv \exp\left(\sum_{i<j}\delta(Qi)\right), \tag{3}$$

where $\delta(Qi) = \log|\Gamma(Qi)/J(Qi)|$ is the local reduced coupling. The equation $[\delta(Qi)] = 0$ determines the phase boundary, where $[\cdot]$ denotes spatial averaging. For QP modulation, the phase boundaries are independent of Q, as the spatial average $[\cdot]$ reduces to a phase average $[\cdot]_\theta$.

The couplings in the simplest QP TFIM arise from a single tone:

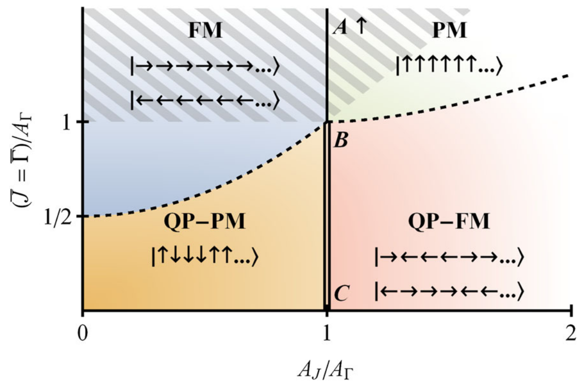

FIG. 1. Phase diagram. The hatched region defines the weakly modulated regime with no weak coupling [$J(Qi), \Gamma(Qi) > 0 \forall i$]. The usual gapped ferromagnetic (blue) and paramagnetic (green) phases appear in this regime, separated by a continuous transition in the clean Ising class (segment AB). At stronger modulation, we find two new modulated gapless phases: the QP-PM (yellow) and the QP-FM (red). The continuous transitions out of these phases (double and dashed lines) are in the new QP Ising class.

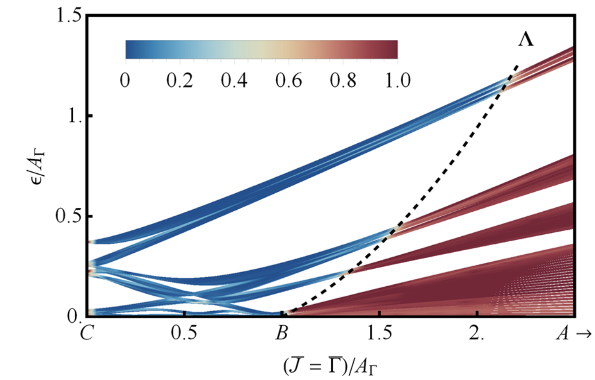

FIG. 2. Localization properties of the excitation spectrum on line ABC. The low-energy excitations on the segment AB are extended (red) up to a finite energy cutoff Λ, above which they are localized (blue). The cutoff Λ vanishes at the multicritical point B so that all finite energy excitations are localized on the segment BC. The color quantifies the scaling of the inverse participation ratio $I = \sum_i |\psi_i|^4 \sim q^{-a}$; $a = 0$ (1) for localized (extended) states. Parameters: $q = 233$, $\Delta = 42\pi/233$, and $\phi = \sqrt{2}$.

$$J(Qi) = \bar{J} + A_J \cos\left(Q(i + 1/2) + \phi\right),$$
$$\Gamma(Qi) = \bar{\Gamma} + A_\Gamma \cos(Qi + \phi + \Delta), \qquad (4)$$

where the phases ϕ and Δ shift the couplings with respect to the lattice. We highlight an interesting slice of the phase diagram in Fig. 1, where $\bar{J} = \bar{\Gamma}$. There are four phases. The usual gapped Ising PM and FM phases arise in the weakly modulated regime ($\bar{J} = \bar{\Gamma} > A_J, A_\Gamma$) at the top of the figure. Two new phases appear at strong modulation, when the couplings take both positive and negative signs: a QP-FM with modulated ferro- and antiferromagnetic correlations and a QP-PM with modulated transverse magnetization.

The two QP phases are gapless with localized excitations at all energies. Heuristically, this is a consequence of weak couplings (of the order of $1/q$) which occur when Qi in Eq. (4) samples near the zeros of $J(\theta)$ or $\Gamma(\theta)$. The weak couplings nearly cut the chain which localizes excitations on either side. In turn, excitations localized on the weak links have an arbitrarily low energy as $q \to \infty$. We note that the gapless excitations are not associated with rare regions, unlike in the Griffiths-McCoy phase of the disordered Ising chain [5,6,68].

In this Letter, we focus on the phase boundary $A_J = A_\Gamma$, segment ABC in Fig. 1 (while the broader phase diagram is studied in Ref. [69]). All of the points on this line are Ising self-dual and accordingly critical. QP modulation is perturbatively irrelevant at the clean Ising transition [34]. Our numerics (not shown) confirm that all critical exponents in the weak modulation regime (segment AB, Fig. 1) are consistent with clean universality. The difference between the unmodulated model and the weakly modulated model becomes apparent only at a high energy: Figure 2 shows that the low-energy excitations are extended (red) up to a finite cutoff energy Λ, above which they become localized (blue). This mobility edge collapses ($\Lambda \to 0$) at the multicritical point B. On the segment BC, all finite energy excitations are localized, consistent with the localization of the adjacent QP-PM and QP-FM phases. This is our first qualitative indication that the critical properties of the QP and clean transitions are quite different.

Before turning to the detailed properties of the QP Ising transition, we briefly review single-parameter scaling. At clean critical points, coarse-grained observables are scale-free [70]. Single-parameter scaling posits that a single length scale and corresponding timescale diverge at the transition:

$$\xi \sim [\delta]^{-\nu}, \qquad \xi_t \sim \xi^z, \qquad (5)$$

where ν and z are the correlation length and dynamical critical exponents, respectively. These control the long-distance and long-time correlations in the vicinity of the critical point. For example,

$$[\langle \sigma_i^x(t)\sigma_{i+r}^x(0)\rangle] \sim \frac{1}{|r|^{2\Delta_\sigma}} \mathcal{C}(r/\xi, t/\xi_t), \qquad (6)$$

where Δ_σ is a scaling dimension and $\mathcal{C}$ a scaling function. These are both part of the universal data of the critical point. It is well known that the scaling ansatz holds at the clean Ising transition.

In the disordered and QP transitions, the scaling ansatz needs to be refined. The *spatially averaged* correlation functions satisfy scaling in the form of Eq. (6) with a single *mean* correlation length ξ. However, the *typical* correlation functions may decay on a shorter, but still divergent, length scale $\xi_{\mathrm{typ}} \sim [\delta]^{-\nu_{\mathrm{typ}}} \ll \xi$. Fisher [10] first emphasized this in the disordered case, where $\nu_{\mathrm{typ}} = 1$ while $\nu = 2$. In the QP case, we will find a much weaker logarithmic separation between $\xi_{\rm typ}$ and ξ.

An additional wrinkle for the disordered and QP Ising transitions is that they separate phases in which all excitations are localized. The localization length $\xi_{\rm loc}$ is a function of energy ϵ and deviation $[\delta]$ which must diverge as $[\delta], \epsilon \to 0$. By scaling, we can compute z from the energy dependence of $\xi_{\rm loc}$ and ν from its $[\delta]$ dependence. Henceforth, we drop the subscript from $\xi_{\rm loc}$, as it coincides with ξ where they are both defined.

We now turn to the analytic and numerical computation of the critical properties of the QP Ising transition.

Typical correlations.—We begin with the exponent that controls the decay of the zero energy wave function [Eq. (3)] across a region of size ℓ:

$$S_\ell(i) \equiv \log\left|\frac{\psi^0_{2(i+\ell)}}{\psi^0_{2i}}\right| = \sum_{j=i}^{i+\ell-1} \delta(Qj). \tag{7}$$

As the excitation mode with the longest localization length, this controls the decay of long-range spin-spin and fermion-fermion correlations. The typical correlation length follows immediately from evaluating the typical exponent controlling decay: $[S_\ell] = \ell[\delta] \sim \ell/\xi_{\rm typ}$. From Eq. (5), this implies $\nu_{\rm typ} = 1$.

Mean correlations.—The spatial modulation induces fluctuation in the exponent $S_\ell(i)$, which are characterized by the scale-dependent variance ("wandering"):

$$\sigma^2(S_\ell) = [S_\ell^2] - [S_\ell]^2. \tag{8}$$

If the wandering $\sigma > |[S_\ell]|$, then the system has a density of regions of size ℓ in which it is locally in the opposite phase. Thus, the spatially averaged correlations at this scale cannot determine the global phase; this generalizes the Harris-Luck instability argument [71,72] to the strong modulation regime. Furthermore, $\sigma(S_\xi) \sim |[S_\xi]|$ defines the mean correlation length ξ above which the global phase is determined. As $[\delta] \to 0$, ξ diverges faster than $\xi_{\rm typ}$ if the wandering grows with l.

For disordered chains, the exponent S_ℓ undergoes a random walk so that $\sigma \sim \sqrt{\ell}$. In the QP chain, the long-range correlations of the spatial modulation produce a more complicated nonmonotonic wandering (see Supplemental Material [64]). In particular, there are exponentially separated special lengths ℓ (the convergents of $Q/2\pi$) at which σ is anomalously small. Nevertheless, for typical large ℓ, the wandering σ^2 is very close to its Cesaro average:

$$\frac{1}{\ell}\sum_{\ell'=1}^{\ell} \sigma^2(S_{\ell'}) \sim \begin{cases} c & \text{if } |J(\theta)|, |\Gamma(\theta)| > 0, \\ w \log \ell & \text{otherwise.} \end{cases} \tag{9}$$

The two cases in Eq. (9) are physically distinguished by the presence of weak couplings and correspond to segments AB and BC in Fig. 1, respectively. Here, c is an l-independent constant, and we pithily dub w the *logarithmic wandering coefficient* (see Supplemental Material [64] for the derivation). Generically, this coefficient depends only on the wave vector Q and number and order of the zeros within a period of the coupling functions. We conjecture that w uniquely parametrizes the family of QP Ising transitions.

The correlation length exponent follows immediately from the coarse-grained wandering described by Eq. (9). On segment AB, $\nu = 1$ and the mean and typical correlations do not separate. This is consistent with AB being in the clean Ising universality class [34]. On segment BC, the mean correlation length is logarithmically enhanced:

$$\xi \sim [\delta]^{-1} \log^{1/2}(1/[\delta]) \tag{10}$$

compared to $\xi_{\rm typ}$ (i.e., "$\nu = 1^+$").

Dynamical exponent.—The dynamic properties show more dramatic signatures of the change in universality. Treating the secular equation of the Hamiltonian (2) to leading order in the wandering of S_ℓ, we find

$$z \approx 1 + w. \tag{11}$$

This follows from estimating the scaling of the bandwidth of the lowest band with period q (see Supplemental Material [64]) [73]. For the golden mean $Q/2\pi = \tau$, the wandering coefficient $w = (2\pi^2/15\sqrt{5}\log\tau) \approx 1.2$ [74], which produces an estimate of $z \approx 2.2$.

This estimate of z neglects spatial correlations of the wandering, higher-order moments and the deterministic deviations of $\sigma(S_\ell)$ from its Cesaro average. We are thus unable to detect multiplicative logarithmic corrections to the dynamical scaling which are suggested by Eq. (10). All results which follow are valid only up to the possibility of such corrections.

Figure 3 shows three different numerical measurements of z which collectively verify both single-parameter scaling and universality. Figures 3(a) and 3(b) probe dynamical scaling through the ϕ, Δ averaged integrated density of states $n(\epsilon) \sim \epsilon^{1/z}$ at asymptotically vanishing and finite energy scales, respectively. With periodic modulation q, the maximum energy ϵ_0 of the lowest miniband satisfies $n(\epsilon_0) = 1/q$. This implies $\epsilon_0 \sim q^{-z}$; Fig. 3(a) confirms this power law holds with exponent $z \approx 1.9$ for system sizes over 5 orders of magnitude up to $q \approx 10^6$. Figure 3(b) shows that the same exponent governs the scaling of $n(\epsilon)$ with ϵ up to a finite energy. Here, $n(\epsilon)$ is extracted from the histogram of energy levels from 10^4 diagonalizations at size $q = 4181$ across sampled values of ϕ, Δ. Both panels collapse data from a series of points along the BC phase boundary, consistent with universality.

We extract $\xi^{-1}(\epsilon)$ from a least squares fit to the relationship

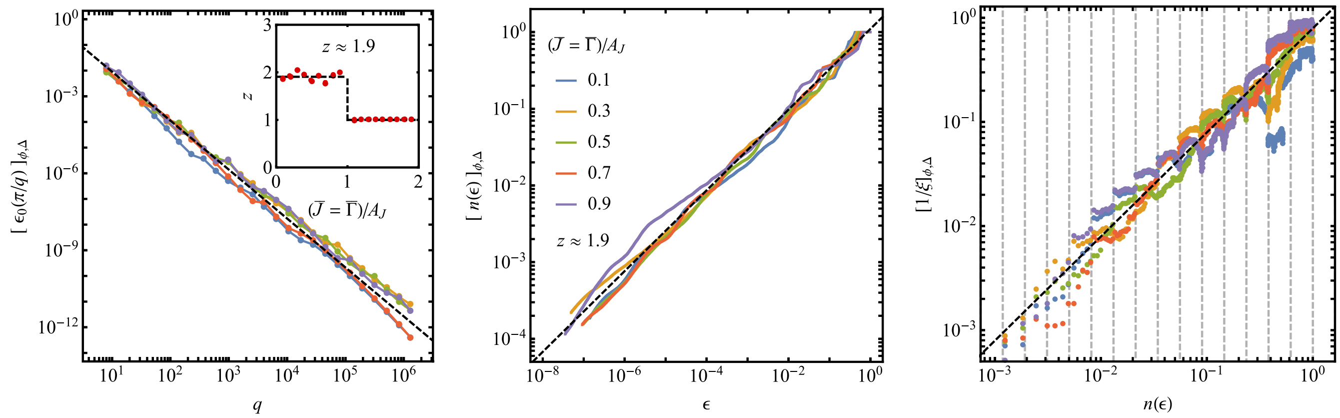

FIG. 3. Dynamical scaling at the QP Ising transition. (Left) The maximum energy of the lowest band $[\epsilon_0]_{\phi,\Delta}$ scales as a power law q^{-z} over 5 orders of magnitude (mean over 5000 ϕ, Δ samples at each Fibonacci length). (Inset) Least squares fit exponent z as a function of the parameter along the phase boundary. (Center) The integrated density of states $[n(\epsilon)]_{\phi,\Delta} \sim \epsilon^{1/z}$ over 7 orders of magnitude in energy at the largest size available ($q = 4181$). (Right) The inverse localization length $[1/\xi(\epsilon)]_{\phi,\Delta}$ is linearly proportional to $n(\epsilon)$, consistent with single-parameter scaling. The deviations from the central trend show sharp features at the log-periodically spaced convergents of the golden ratio τ (vertical dashed lines). In all panels, the measurements are shown at five different values of $\bar{\Gamma}/A_\Gamma$ on segment BC of Fig. 1, indicating universality. Standard errors are smaller than the point size; deviations from power law trends are deterministic and due to the QP modulation.

$$\log[|\psi_i(\epsilon)\bar{\psi}_{i+r}(\epsilon)|] = -r\xi^{-1}(\epsilon) + \text{const}, \tag{12}$$

where $\psi_i(\epsilon)$ is the eigenmode at energy ϵ for systems of size $q = 1597$. We again see evidence of universality along the phase boundary.

In all three panels in Fig. 3, the visible deviation from pure power laws reflects deterministic modulation. The phase averaging of various quantities reduces the deviations from the central trends but does not completely suppress them. We expect deviations from pure power laws due to rare values of l at which $\sigma(S_\ell)$ deviates significantly from its Cesaro mean [see Eq. (9)]. These special values are marked by dashed lines in Fig. 3(c), where they correlate with atypically delocalized excitations.

The presence of these special points leads us to conjecture that the single-parameter scaling forms, e.g., in Eq. (6), hold up to a nonuniversal multiplicative $O(1)$ function. That is, the scaling form provides the *envelope* for these $O(1)$ fluctuations. A consequence of this hypothesis is that the critical exponents are well defined as $q \to \infty$ but the convergence of finite-size numerical estimates is only $O(1/\log q)$. This is consistent with the scatter in the inset in Fig. 3(a).

Scaling dimensions.—The equal time correlators at the QP Ising transition decay with a faster power law than at the clean Ising transition but slower than at infinite randomness. Figure 4 shows the excellent finite-size scaling collapse of the mean equal time spin correlator $[\langle\sigma_i^x\sigma_{i+r}^x\rangle]_{i,\phi,\Delta}$ at the QP transition. Using data from different points on the QP transition line, we extract $\Delta_\sigma \approx 0.16$ [see Eq. (6)]. We find a similarly enhanced value of the scaling dimension of the Majorana fermions $\Delta_\gamma \approx 0.63$ (data not shown). In contrast, for the clean TFIM $\Delta_\sigma = 0.125$, $\Delta_\gamma = 0.5$, and for the random TFIM $\Delta_\sigma = (3 - \sqrt{5})/4 \approx 0.19$, $\Delta_\gamma \approx 1.1$ [10,75].

The QP critical correlations are observed on length scales $r < q$; for $r > q$, the system is actually periodic and we recover clean Ising correlations [76]. In Fig. 4, this is presaged by the small upturn near $r = q$.

Discussion.—Weak quasiperiodic modulation is well known to be perturbatively irrelevant at the clean Ising transition [34]. We have shown that sufficiently strong modulation destabilizes this transition and drives the TFIM into a new spatially modulated QP Ising transition. Like in

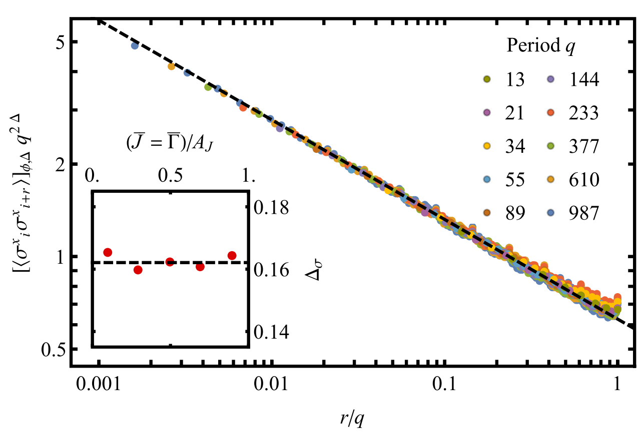

FIG. 4. Finite-size scaling of spin correlations. (Main) The average spin correlations $q^{2\Delta_\sigma}[\langle\sigma_i^x\sigma_{i+r}^x\rangle]_{i,\phi,\Delta}$ collapse when plotted versus fractional separation r/q for critical dimension $\Delta_\sigma \approx 0.16$ at $(\bar{J} = \bar{\Gamma})/A_J = 0.5$. (Inset) Least median deviation fit exponent Δ_σ is stable along segment BC consistent with universality.

the infinite randomness case, the low-energy excitations are localized throughout the critical fan, although with a power law diverging localization length as $\epsilon \to 0$. The exponents of the QP Ising transition lie between their clean and disordered counterparts. The most dramatic signatures of this transition are in the localized dynamics and larger specific heat as compared to the clean case.

Our results rely on the emergence of logarithmic wandering with coefficient w describing the dominant long-distance fluctuations of the order. We conjecture that w controls the universal content of a family of QP Ising transitions. As w is only a function of wave number Q and the number and order of the zeros of $J(\theta), \Gamma(\theta)$, it follows that the critical properties are insensitive to smooth perturbations which preserve the wave number. This is investigated in Ref. [69]. We have provided numerical evidence for this universality by varying couplings along the boundary BC.

Remarkably, logarithmic wandering arises *without weak couplings* when $J(\theta)$, $\Gamma(\theta)$ have step discontinuities. Technically, this follows from the $1/k$ tails in the Fourier transform of $\delta(\theta)$. As the size of the steps controls w, we can realize a large family of QP Ising transitions with tunable exponents in such models. The quasiperiodic substitution sequences studied in Refs. [26–39,42] correspond to choosing $J(\theta)$, $\Gamma(\theta)$ to be certain square waves. Though in these models there is no concomitant localization of excitations, *mutatis mutandis*, our analysis applies: Generically, these models have the logarithmic wandering of Eq. (9) and power law criticality intermediate to the clean and random cases [34,69,72].

The stability of the QP Ising transitions to the introduction of interactions is an open question. On the one hand, interactions which effectively lift weak couplings could destroy the log wandering. On the other hand, the example of step modulation suggests that weak couplings are not strictly necessary for modified criticality.

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[1] S. Aubry and G. André, Ann. Isr. Phys. Soc. **3**, 133 (1980).
[2] M. Y. Azbel, Phys. Rev. Lett. **43**, 1954 (1979).
[3] D. R. Hofstadter, Phys. Rev. B **14**, 2239 (1976).
[4] P. G. Harper, Proc. Phys. Soc. London Sect. A **68**, 874 (1955).
[5] B. M. McCoy and T. T. Wu, Phys. Rev. **176**, 631 (1968).
[6] B. M. McCoy and T. T. Wu, Phys. Rev. **188**, 982 (1969).
[7] S.-k. Ma, C. Dasgupta, and C.-k. Hu, Phys. Rev. Lett. **43**, 1434 (1979).
[8] R. Shankar and G. Murthy, Phys. Rev. B **36**, 536 (1987).
[9] D. S. Fisher, Phys. Rev. Lett. **69**, 534 (1992).
[10] D. S. Fisher, Phys. Rev. B **51**, 6411 (1995).
[11] D. S. Fisher, Physica (Amsterdam) **263A**, 222 (1999).
[12] O. Motrunich, S.-C. Mau, D. A. Huse, and D. S. Fisher, Phys. Rev. B **61**, 1160 (2000).
[13] F. Iglói and C. Monthus, Phys. Rep. **412**, 277 (2005).
[14] D. Shechtman, I. Blech, D. Gratias, and J. W. Cahn, Phys. Rev. Lett. **53**, 1951 (1984).
[15] D. Levine and P. J. Steinhardt, Phys. Rev. Lett. **53**, 2477 (1984).
[16] R. Merlin, K. Bajema, R. Clarke, F.-Y. Juang, and P. K. Bhattacharya, Phys. Rev. Lett. **55**, 1768 (1985).
[17] I. I. Satija and M. M. Doria, Phys. Rev. B **38**, 5174 (1988).
[18] J. Q. You, X. Zeng, T. Xie, and J. R. Yan, Phys. Rev. B **44**, 713 (1991).
[19] J. Vidal, D. Mouhanna, and T. Giamarchi, Phys. Rev. Lett. **83**, 3908 (1999).
[20] J. Hermisson, J. Phys. A **33**, 57 (2000).
[21] K. Hida, Phys. Rev. Lett. **86**, 1331 (2001).
[22] P. Tong and M. Zhong, Phys. Rev. B **65**, 064421 (2002).
[23] K. Hida, J. Phys. Soc. Jpn. **74**, 57 (2005).
[24] A. P. Vieira, Phys. Rev. Lett. **94**, 077201 (2005).
[25] A. P. Vieira, Phys. Rev. B **71**, 134408 (2005).
[26] M. M. Doria and I. I. Satija, Phys. Rev. Lett. **60**, 444 (1988).
[27] F. Iglói, J. Phys. A **21**, L911 (1988).
[28] C. Tracy, J. Phys. A **21**, L603 (1988).
[29] H. A. Ceccatto, Phys. Rev. Lett. **62**, 203 (1989).
[30] M. K. Kolář, M. Ali, and I. I. Satija, Phys. Rev. B **40**, 11083 (1989).
[31] V. Benza, Europhys. Lett. **8**, 321 (1989).
[32] V. G. Benza, M. Kolá, and M. K. Ali, Phys. Rev. B **41**, 9578 (1990).
[33] Z. Lin and R. Tao, Phys. Rev. B **46**, 10808 (1992).
[34] J. Luck, J. Stat. Phys. **72**, 417 (1993).
[35] L. Turban, F. Iglói, and B. Berche, Phys. Rev. B **49**, 12695 (1994).
[36] U. Grimm and M. Baake, arXiv:cond-mat/9604116.
[37] J. Hermisson, U. Grimm, and M. Baake, J. Phys. A **30**, 7315 (1997).
[38] F. Iglói, L. Turban, D. Karevski, and F. Szalma, Phys. Rev. B **56**, 11031 (1997).
[39] F. Iglói, D. Karevski, and H. Rieger, Eur. Phys. J. B **1**, 513 (1998).
[40] J. Hermisson and U. Grimm, Phys. Rev. B **57**, R673 (1998).
[41] F. J. Oliveira Filho, M. S. Faria, and A. P. Vieira, J. Stat. Mech. 2012, P03007 (2012).
[42] W. N. Yessen, in *Annales Henri Poincaré* (Springer, New York, 2014), Vol. 15, pp. 793–828.

[43] G. Roati, C. DErrico, L. Fallani, M. Fattori, C. Fort, M. Zaccanti, G. Modugno, M. Modugno, and M. Inguscio, Nature (London) **453**, 895 (2008).
[44] B. Deissler, M. Zaccanti, G. Roati, C. DErrico, M. Fattori, M. Modugno, G. Modugno, and M. Inguscio, Nat. Phys. **6**, 354 (2010).
[45] M. Schreiber, S. S. Hodgman, P. Bordia, H. P. Lüschen, M. H. Fischer, R. Vosk, E. Altman, U. Schneider, and I. Bloch, Science **349**, 842 (2015).
[46] P. Bordia, H. P. Lüschen, S. S. Hodgman, M. Schreiber, I. Bloch, and U. Schneider, Phys. Rev. Lett. **116**, 140401 (2016).
[47] H. P. Lüschen, P. Bordia, S. S. Hodgman, M. Schreiber, S. Sarkar, A. J. Daley, M. H. Fischer, E. Altman, I. Bloch, and U. Schneider, Phys. Rev. X **7**, 011034 (2017).
[48] D. J. Thouless, Phys. Rev. B **28**, 4272 (1983).
[49] M. Kohmoto, Phys. Rev. Lett. **51**, 1198 (1983).
[50] M. Kohmoto, B. Sutherland, and C. Tang, Phys. Rev. B **35**, 1020 (1987).
[51] T. Janssen and J. Los, Phase Transitions **32**, 29 (1991).
[52] D. Thouless and Y. Tan, J. Phys. A **24**, 4055 (1991).
[53] H. Hiramoto and M. Kohmoto, Int. J. Mod. Phys. B **06**, 281 (1992).
[54] J. H. Han, D. J. Thouless, H. Hiramoto, and M. Kohmoto, Phys. Rev. B **50**, 11365 (1994).
[55] J. Biddle, B. Wang, D. J. Priour, Jr., and S. Das Sarma, Phys. Rev. A **80**, 021603 (2009).
[56] S. Ganeshan, J. H. Pixley, and S. Das Sarma, Phys. Rev. Lett. **114**, 146601 (2015).
[57] S. Gopalakrishnan, Phys. Rev. B **96**, 054202 (2017).
[58] S. Iyer, V. Oganesyan, G. Refael, and D. A. Huse, Phys. Rev. B **87**, 134202 (2013).
[59] V. Mastropietro, Phys. Rev. Lett. **115**, 180401 (2015).
[60] M. Tezuka and N. Kawakami, Phys. Rev. B **85**, 140508 (2012).
[61] W. DeGottardi, D. Sen, and S. Vishveshwara, Phys. Rev. Lett. **110**, 146404 (2013).
[62] I. I. Satija, arXiv:1705.09130.
[63] T. Devakul and D. A. Huse, Phys. Rev. B **96**, 214201 (2017).
[64] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.120.175702 for derivation of the logarithmic wandering, Eq. (9), when $Q/2\pi$ is a badly approximable number, and for estimation of the dynamical exponent z.
[65] Optimal values of p and q are given by the continued fraction approximants to $Q/2\pi$.
[66] E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. (N.Y.) **16**, 407 (1961).
[67] A. Chandran and C. R. Laumann, Phys. Rev. X **7**, 031061 (2017).
[68] R. B. Griffiths, Phys. Rev. Lett. **23**, 17 (1969).
[69] P. Crowley, A. Chandran, and C. Laumann (to be published).
[70] N. Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group* (Addison-Wesley, Reading, MA, 1992).
[71] A. B. Harris, J. Phys. C **7**, 1671 (1974).
[72] J. M. Luck, Europhys. Lett. **24**, 359 (1993).
[73] We measure all energies relative to $\epsilon = 0$.
[74] D. Speyer, Mathematics Stack Exchange http://math.stackexchange.com/q/2418521.
[75] F. Iglói and I. A. Kovács, Phys. Rev. B **97**, 094205 (2018).
[76] The transition to Ising criticality for $r > q$ occurs only if the system is clean Ising critical. Because of $O(1/q)$ finite-size corrections to the clean Ising critical boundaries, this is not automatically satisfied. For $q\Delta - p\pi \in 2\pi\mathbb{Z}$, the QP Ising boundary is a clean Ising boundary. These Δ form a dense set in $[0, 2\pi]$ in the QP limit.