Exploration of Fermi-Pasta-Ulam Behavior in a Magnetic System

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We study nonlinear spin motion in one-dimensional magnetic chains. We find significant differences from the classic Fermi-Pasta-Ulam (FPU) problem examining nonlinear elastic motion in a chain. We find that FPU behavior, the transfer of energy among low order eigenmodes, does not occur in magnetic systems with only exchange and external fields, but does exist if a uniaxial anisotropy is also present. The FPU behavior may be altered or turned off through the magnitude and orientation of an external magnetic field. A realistic micromagnetic model shows such behavior could be measurable.

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The Fermi-Pasta-Ulam [1] (FPU) problem was a catalytic study that simultaneously began numerical experiments and gave significant insight and stimulus to nonlinear physics. Performed more than 60 years ago, FPU addressed a fundamental issue in nonlinear physics through the study of large-amplitude vibrational motion of a one-dimensional linear chain. In a linear harmonic system, one expects that if energy is put into one eigenmode, the system will reach the ergodic limit; i.e., the energy will eventually be spread out equally through all the eigenmodes (through damping or small perturbations). What FPU found, in contrast, for a nonlinear system was very different. Energy added to one mode was transferred to nearby modes in frequency, but then the system would nearly completely return to the original mode as time progressed. Ultimately, the energy remained in a small number of modes, cycling between these modes with a time that typically was several thousand periods of the original motion.

The elastic FPU problem has created a vast area of research [2] with various explanations for the effects seen. The FPU recurrence was explained by Zabusky and Kruskal through soliton dynamics. [3] They showed that, in the continuum limit, the FPU problem was related to the Korteweg–de Vries equation and that a large amplitude periodic wave would decompose into solitons with different speeds. Collisions of the fast and slow solitons lead to a periodic reconstruction of the initial state.

Surprisingly, there are no theoretical studies of FPU-like behavior for magnetic systems despite fundamental differences between magnetic excitations and elastic vibrations. In contrast to the vibrational problem, spin waves have an inherent nonlinearity that is dependent on the precession angles of the spins. The magnetic system also has additional parameters, such as the magnitude and direction of an external magnetic field, that allow tunable variations in nonlinear behavior, a possibility not found in the elastic system. We do note that there has been one experimental study of FPU recurrence in a magnetic system employing an active-feedback-ring containing a YIG film [4] and other experimental and theoretical studies of nonlinear magnetic dynamics [5–20].

There is another essential difference between the magnetic and elastic systems. In the classic elastic problem the dispersion relation in the small wave vector limit is $\omega = ck$, so that doubling the wave vector k results in a doubled frequency of 2ω. This allows an initial mode to create doubled frequency modes in a resonant way (for a quadratic nonlinearity). In contrast, the magnetic system has a much more complex dispersion relation, which does not generally lead to resonance enhancement of other modes. With this as background, it is not obvious if FPU behavior can occur in a magnetic system.

In this Letter we explore whether FPU-like behavior can emerge for a set of spins in a quasi-one-dimensional system. We find the following: (i) FPU behavior does indeed occur in some magnetic systems—particularly systems with an effective uniaxial anisotropy. (ii) The FPU behavior may be altered or even turned on and off through the magnitude and orientation of an external magnetic field. (iii) Unlike the FPU problem, the magnetic system inherently contains both quadratic and cubic nonlinear terms in the equations of motion. The different types of nonlinearity lead to fundamental differences in the solutions, for example, whether the solutions preserve the spatial symmetry of the initial state [21].

To begin, we use a quasi one-dimensional model, a set of N exchange-coupled thin films, for the spin system as illustrated in Fig. 1(a). The films are effectively equivalent to a linear chain of spins, except with the addition of a demagnetizing field in the x direction. The equations of motion are given by

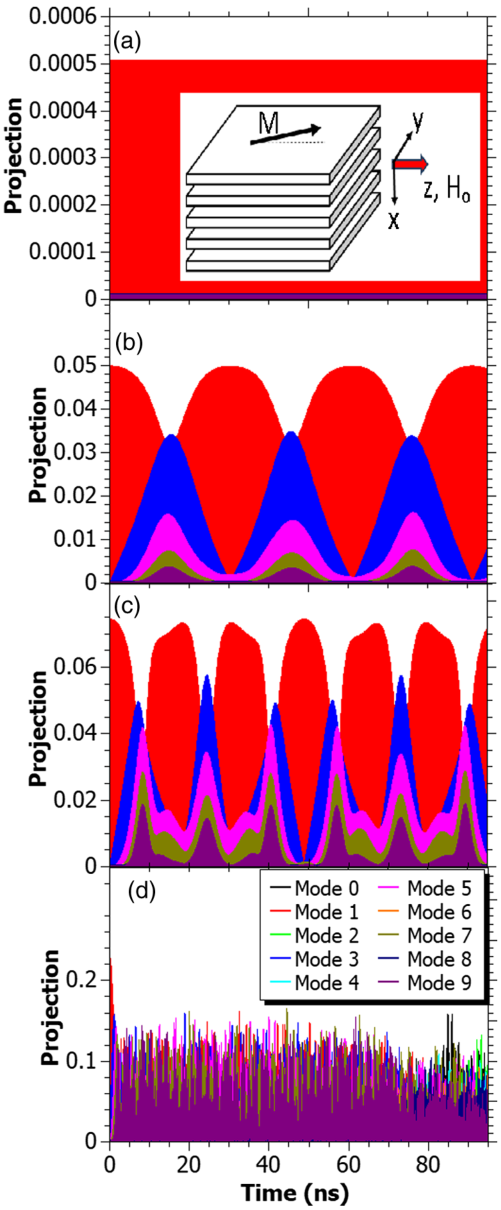

FIG. 1. Time evolution of the system started in the $n = 1$ state but with different initial amplitudes. (a) $A = 0.001$ and the system stays in the $n = 1$ mode. (b) $A = 0.1$ and the $n = 1$ mode redistributes to multiple other modes but repeatedly goes back into mode 1. (c) $A = 0.15$ and the simple periodic behavior seen in (b) has become much more complex (d) $A = 0.5$ and the time evolution shows strongly nonlinear behavior where the system is approaching ergodicity.

$$\frac{d\boldsymbol{M}(i)}{dt} = -|\gamma|\boldsymbol{M}(i) \times \{H_o\hat{z} - 4\pi M_x(i)\hat{x} + J[\boldsymbol{M}(i+1) + \boldsymbol{M}(i-1)]\}, \tag{1}$$

where $\boldsymbol{M}(i)$ is the magnetization for the ith film, $|\gamma| = 18.22$ rad/(ns kOe) is the gyromagnetic ratio, H_o is the applied magnetic field, $-4\pi M_x(i)\hat{x}$ is the demagnetizing field for a thin film, $\mathbf{M}$ is the magnetization with $|\mathbf{M}| = 1.75$ kG, and J is the exchange coupling constant where $JM = 20$ kOe.

The equations of motion for the components are

$$\frac{dM_x(i)}{dt} = -|\gamma|\{J[M_y(i)M_z(i+1) + M_y(i)M_z(i-1) - M_z(i)M_y(i+1) - M_z(i)M_y(i-1)] + M_y(i)H_o\}, \tag{2a}$$

$$\frac{dM_y(i)}{dt} = -|\gamma|\{J[M_z(i)M_x(i+1) + M_z(i)M_x(i-1) - M_x(i)M_z(i+1) - M_x(i)M_z(i-1)] - M_x(i)H_o - 4\pi M_z(i)M_x(i)\}, \tag{2b}$$

$$\frac{dM_z(i)}{dt} = -|\gamma|\{J[M_x(i)M_y(i+1) + M_x(i)M_y(i-1) - M_y(i)M_x(i+1) - M_y(i)M_x(i-1)] + 4\pi M_x(i)M_y(i)\}. \tag{2c}$$

The system is given an initial configuration and the equations are iterated forward in time numerically using Runge-Kutta integration. The time step is $\Delta t = 10^{-5}$ ns with typical run times over 350 ns. Individual runs with $\Delta t = 10^{-6}$ ns are used to check the results. As in the FPU study, we initially take the damping to be zero; however, the effects of damping are discussed at the end.

To examine the question raised by FPU, i.e., whether the system becomes ergodic or whether there is a persistent exchange of energy between the lower eigenstates, we follow the FPU example and start the system in a low order magnetic linear eigenstate. An initial eigenstate of order n is given by

$$\frac{M_y(i, t=0)}{M} = A\cos\left(\frac{\pi n[i-1]}{N-1}\right), \qquad M_x(i, 0) = 0,$$
$$M_z(i, 0) = \sqrt{M^2 - M_x(i, 0)^2 - M_y(i, 0)^2}, \tag{3}$$

where A is the amplitude of the initial excitation, i is the site index, and n is the mode number. The spins at the outer edges of the chain are unpinned, and each only has one nearest neighbor interacting via exchange. We characterize the resulting motion by projecting the motion of the magnetization at any time t onto a set of linear eigenstates.

$$a_n(t) = \frac{1}{N}\left|\sum_{i=1}^{N}\frac{M_y(i,t)}{M}\cos\left(\frac{\pi n[i-1]}{N-1}\right)\right|, \tag{4}$$

where the mode number is again given by n.

As an example, if we take an initial state with $n = 1$, the total number of sites is $N = 512$, and $H_o = 5$ kOe. To characterize the time evolution [22], we project the configuration at any time onto the linear eigenmodes. If the initial amplitude A is small, the magnetic dynamics shows a characteristically linear behavior in that it stays in the initial state, $n = 1$, as seen in Fig. 1(a). In contrast, with a larger initial amplitude [23], [Fig. 1(b)], the inherent nonlinearity of the magnetic system becomes important and there is a recurring energy transfer from the initial mode to other modes in the system and back to the initial mode [24,25]. As the initial amplitude is increased further, Fig. 1(c), the simple periodic behavior found in Fig. 1(b) has disappeared, and the resulting motion is much more complex. The final case, Fig. 1(d), shows the system has crossed the threshold separating FPU-like behavior from the expected chaotic behavior [26] and the onset of ergodicity [27]. This behavior is similar to that found in the elastic FPU problem, but now in a magnetic system. We were unable to find FPU-like behavior in systems with only exchange fields and an external magnetic field. The FPU behavior found above required an effective uniaxial anisotropy, in this case provided by the demagnetizing field.

A significant difference between the elastic and magnetic FPU systems is the nature of the nonlinearity. In the elastic FPU problem, the nonlinearity is essentially added in an arbitrary way. The FPU paper added both quadratic and cubic displacement terms to the elastic equations of motion. With only a cubic term present, the symmetry of the initial mode (odd or even about the midpoint) is preserved for all the generated modes. When quadratic terms are present, there is no symmetry requirement. In contrast, the magnetic system has inherent nonlinearity. To find the lowest order terms in the nonlinearity one expands m_z as

$$M_z(i) \cong M\left[1 - \frac{1}{2}\left(\frac{M_x(i)^2 + M_y(i)^2}{M^2}\right) + \cdots\right] \quad (5)$$

in the equations of motion. One finds that Eqs. (2a) and (2b) have a cubic nonlinearity while Eq. (2c) has a quadratic nonlinearity. Our numerical results emphasize that both terms play an important role. At lower amplitudes we find that the cubic terms play the primary role, and symmetry is preserved. For example, a system initially in mode 1 (a mode odd about the midpoint of the chain) primarily decays into modes 3, 5, and 7 [see Fig. 1(b)], which are all odd as well. For larger amplitudes [see Fig. 1(d)] both even and odd modes are generated. We note that the most important cubic terms are those associated with the demagnetizing fields, again emphasizing the importance of these terms for the magnetic case.

As stated, the magnetic system allows a dynamic tuning by varying the strength or direction of an external field. We explore the effect of changing the strength of the external magnetic field in Fig. 2. We set the initial condition to contain 95% of the 0th mode and 5% of the mode $n = 1$, with $N = 512$ spins and an amplitude of $A = 0.1$. (Use of 100% of the $n = 0$ mode leads to numerical issues

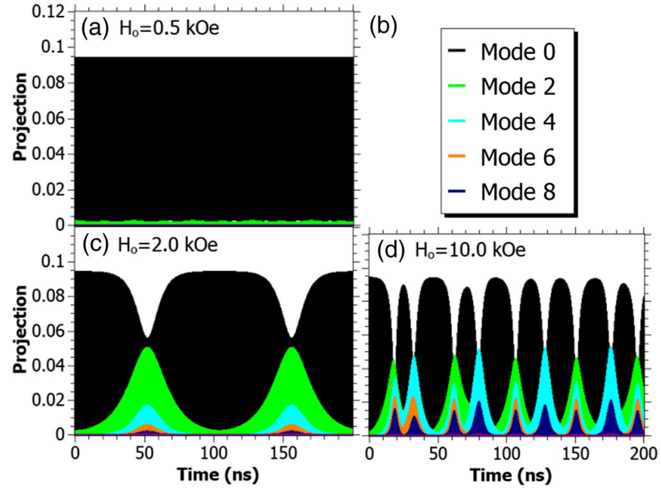

FIG. 2. Mode projections for identical systems with different applied fields. The initial state contains 95% $n = 0$ and 5% $n = 1$ and $A = 0.1$ The time evolution in (b) and (c) shows FPU-like behavior.

where the results depend on the initial conditions and time step because of the extremely high symmetry of the initial state [28].)

With the external field along the z axis and at the smallest field, Fig. 2(a), the system simply remains in the initial state. As the field is increased, Figs. 2(b) and 2(c), FPU-like behavior emerges in that the initial state transfers to other states in a simple periodic pattern where only low-order even modes are involved. When the field is increased to 10 kOe, Fig. 2(c), the onset of FPU-like behavior occurs earlier. There is a noticeably faster interchange between the modes at higher magnetic fields. The results imply that changing an external field can serve as a stabilizer or moderator. This is similar to the induction period previously found by Hirooka and Saito in the elastic case [29,30]. However, in the magnetic system, changing the external field strength effectively mediates the nonlinearity, in comparison to the introduction of an artificial nonlinearity parameter used in the earlier work by Hirooka and Saito.

In Fig. 3 we explore the effect of changing the direction of the applied magnetic field. The results in Fig. 3(a) are qualitatively similar to those seen in Fig. 1(c), even though the field is increased to $H_o = 25$ kOe. A field with the same magnitude applied along x in Fig. 1(b) (out-of-plane field) is sufficient to saturate the system along the x axis. The resulting behavior, as seen in Fig. 3(b), is no longer periodic, although there are still times where the system reverts almost entirely into the $n = 1$ mode. Thus, the FPU behavior is significantly tunable by the direction of the static applied field.

Although this is a theoretical Letter, we discuss the possibility of experimental measurements. The calculations above assume, as in the original FPU paper, that damping is unimportant. Real magnetic systems, of course, have

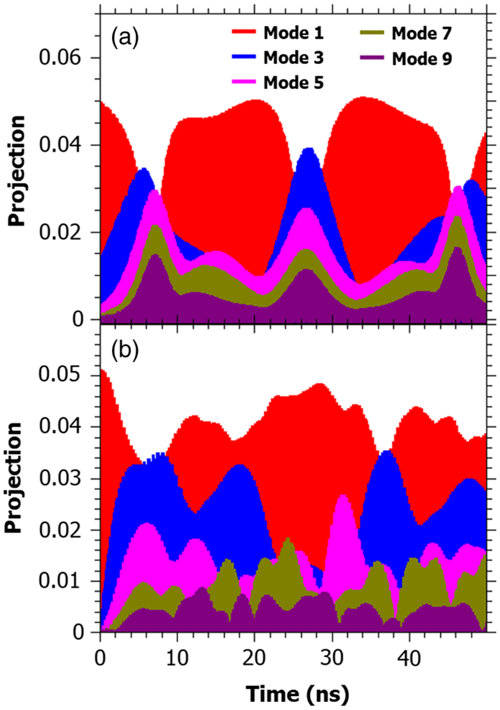

FIG. 3. The projection of the dynamic state onto the linear eigenmodes of the system when (a) the field is applied in plane, and (b) the field is applied out of plane. The applied field is $H_o = 25$ kOe, and $A = 0.1$.

damping, characterized by a dimensionless parameter α, which can play an important role. Recently Heusler compounds have been investigated [31] with values of α near 0.001. Values of α near 10^{-4} have been measured in CoFe alloys [32]. Bulk samples of yttrium iron garnet (YIG) have very low damping $\alpha = 10^{-5}$ and even thin YIG films [33,34] can have values on the order of $\alpha = 2.7 \times 10^{-4}$ or 0.9×10^{-4}. With this in mind, Fig. 4 shows results for an $n = 1$ initial state with a damping value of $\alpha = 10^{-4}$ [all other parameters are the same as in Fig. 1(b)]. Despite

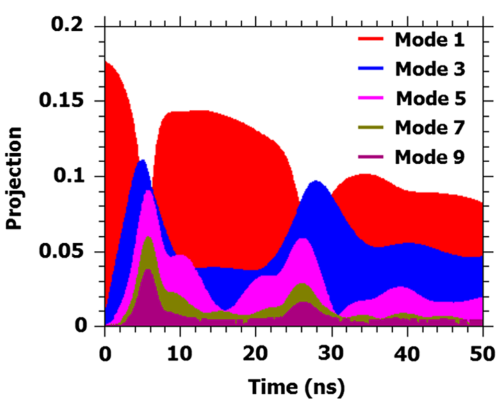

FIG. 4. The projection of the dynamic state onto the linear eigenmodes of the system when damping ($\alpha = 10^{-4}$) is present. The remaining parameters are $H_o = 5$ kOe, the initial state is $n = 1$ with an amplitude of 0.25.

the damping, there is a clear FPU-like interchange of energy between the lowest modes [all odd modes, as in Fig. 1(b)].

The system considered here, as illustrated in Fig. 1(a), is not the only possible configuration for experimentally studying magnetic FPU-like behavior. We have also studied whether FPU-like behavior can occur in a narrow stripe with a realistic micromagnetic model. The parameters used were $A = 0.1$, $M = 1.7$ kG, $A_{ex} = 2.5 \times 10^{-6}$ erg/cm, $H_o = 25$ kOe in the x direction, $\Delta t = 5 \times 10^{-5}$ ns, and $\alpha = 1 \times 10^{-5}$. We used cell configurations of $1 \times 1 \times 256$ (256 cells in the z direction) and cell sizes of $5 \times 5 \times 5$ nm. Here, as previously, the anisotropy is introduced by the demagnetizing fields, although through a different orientation. Starting in the $n = 1$ mode, we found an interchange in energy between the $n = 1$ and $n = 3$ modes with time, similar to that shown in Fig. 1(b). The recurrence time was about 10 ns.

In summary, magnetic systems can have FPU-like behavior under certain conditions—one of the transverse directions is effectively a hard axis. The magnetic FPU behavior is highly tunable, in contrast to the elastic case. Finally, we note that the study of the FPU problem in elastic systems lead to an important connection between solitons and FPU behavior. This leads one to speculate that the study of magnetic FPU behavior may also lead to an enhanced understanding of solitons in realistic magnetic systems.

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