

## Qudit-Basis Universal Quantum Computation Using $\chi^{(2)}$ Interactions

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We prove that universal quantum computation can be realized—using only linear optics and  $\chi^{(2)}$  (three-wave mixing) interactions—in any  $(n + 1)$ -dimensional qudit basis of the  $n$ -pump-photon subspace. First, we exhibit a strictly universal gate set for the qubit basis in the one-pump-photon subspace. Next, we demonstrate qutrit-basis universality by proving that  $\chi^{(2)}$  Hamiltonians and photon-number operators generate the full  $\mathfrak{u}(3)$  Lie algebra in the two-pump-photon subspace, and showing how the qutrit controlled-Z gate can be implemented with only linear optics and  $\chi^{(2)}$  interactions. We then use proof by induction to obtain our general qudit result. Our induction proof relies on coherent photon injection or subtraction, a technique enabled by  $\chi^{(2)}$  interaction between the encoding modes and ancillary modes. Finally, we show that coherent photon injection is more than a conceptual tool, in that it offers a route to preparing high-photon-number Fock states from single-photon Fock states.

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*Introduction.*—Photons are promising information carriers for quantum computers, owing to the photon’s long room-temperature coherence time, high transmission speed, and high-fidelity preparation schemes [1–4], plus the availability of efficient photodetectors [5,6], and the scalable on-chip integration of linear and nonlinear optical components [7–10]. Architectures for optics-based quantum computation have gone through dramatic developments over the past two decades [11–16], but significant obstacles remain to be overcome.

Optics-based quantum computation depends on photon-photon interactions for the realization of a universal gate set. The lowest-order photon-photon interactions are described by unitary transformations of the form  $\hat{U} = \exp(-i\hat{L})$  that are generated by general two-wave mixing Hamiltonians,

$$\hat{L} \in \{(g\hat{a}\hat{b} + g^*\hat{a}^\dagger\hat{b}^\dagger), (g\hat{a}\hat{b}^\dagger + g^*\hat{a}^\dagger\hat{b})\}, \quad (1)$$

where  $g$  is a  $c$ -number, and  $\hat{a}^\dagger$  and  $\hat{b}^\dagger$  are photon-creation operators from different optical modes, so that  $[\hat{a}, \hat{b}^\dagger] = 0$ , or the same optical mode, for which  $[\hat{a}, \hat{b}^\dagger] = 1$ . Unitary transformations of this form can realize universal single-qubit rotations in the Fock-state basis but are not universal for quantum computation without some additional resource. To implement universal optics-based quantum computation, four-wave mixing (a  $\chi^{(3)}$  interaction) was previously considered to be the lowest-order optical nonlinearity that will suffice in this regard [13,17,18]. The inherent weakness of  $\chi^{(3)}$  interactions, however, has precluded their delivering the high-fidelity gates required to make optics-based quantum computation practical [19–21]. Linear-optical quantum

computation (LOQC) [14,22–24] circumvents the need for photon-photon interactions through postselection, but this approach comes with the need for a prohibitive number of perfect single-photon ancillae to cope with LOQC’s probabilistic nature and the ubiquitous photon loss [15,25–27].

One way to circumvent the weakness of photon-photon interactions is to employ the *lowest*-order nonlinearity that can provide universal quantum computation, viz., the  $\chi^{(2)}$  interaction whose three-wave-mixing Hamiltonians can be decomposed into linear combinations of the following terms [28]:

$$\hat{G}_1 = \frac{i\kappa}{2} [\hat{a}_s^\dagger\hat{a}_i^\dagger\hat{a}_p - \hat{a}_s\hat{a}_i\hat{a}_p^\dagger], \quad \hat{G}_2 = \frac{\kappa}{2} [\hat{a}_s^\dagger\hat{a}_i^\dagger\hat{a}_p + \hat{a}_s\hat{a}_i\hat{a}_p^\dagger]. \quad (2)$$

Here,  $\{\hat{a}_k^\dagger : k = s, i, p\}$  are the photon-creation operators of the interaction’s signal, idler, and pump modes, and the real-valued  $\kappa$  quantifies the interaction’s strength.

The efficiencies of  $\chi^{(2)}$  interactions have been steadily improving over the past decade [29–42]. Moreover, owing to the importance of  $\chi^{(2)}$  interactions in quantum state transduction for superconducting and ion-trap qubits, the platforms of interest for  $\chi^{(2)}$  interactions have expanded beyond traditional nonlinear crystals [37–42], bringing full utilization of their quantum dynamics closer to reality.

Coherent photon conversion, i.e.,  $\chi^{(2)}$  interactions defined in Eq. (2) in which the signal, idler, and pump modes are all quantum mechanical, was first proposed by Koshino [43], and later used by Langford *et al.* [29] to show how universal quantum computation can be realized with

that resource in the single-photon qubit basis. We refer to such interactions as *full-quantum*  $\chi^{(2)}$  interactions, to distinguish them from *pumped*  $\chi^{(2)}$  interactions, in which a nondepleting coherent-state pump reduces Eq. (2) to the two-wave interactions shown in Eq. (1). Langford *et al.*'s groundbreaking work, however, is not without drawbacks. Available schemes for correcting photon loss [11,44–46], viz., the dominant error in photonic quantum computation, require either measurement-based or  $\chi^{(3)}$  gates on the encoded basis. Thus, Ref. [29] does not provide a  $\chi^{(2)}$  approach that facilitates photonic quantum computation that is robust to photon loss.

In this Letter, and its companion paper [47], we show how the work of Langford *et al.* can be extended to a more natural computational basis for  $\chi^{(2)}$ -based quantum computation in which photon-loss errors can be addressed. More generally, we prove that  $\chi^{(2)}$  interactions plus linear optics can provide a strictly universal gate set for quantum computation in any  $(n+1)$ -dimensional qudit basis of the  $n$ -pump-photon subspace. Because any  $d$ -qudit unitary gate can be described by a Lie group element of  $SU[(n+1)^d]$ , the universality of a given class of Hamiltonians is directly related to that class's Lie algebra and the Lie group it generates via the exponential map [48]. Thus, we use Lie-algebra analysis to identify code subspaces that are closed under  $\chi^{(2)}$  Hamiltonian evolutions [50,51]. Our Lie-algebra analysis underlies the symmetry-operator formulation of qudit-basis error-correcting codes for photon-loss errors and the universal gate-set constructions in the encoded basis that we report in Ref. [47]. Hence, our proposal provides a  $\chi^{(2)}$  approach to photonic quantum computation that is robust to photon loss. We begin the development of our universality results with a summary of the linear optics and the  $\chi^{(2)}$  resources we shall employ. We follow with qubit and qutrit universality proofs, as preludes to our induction proof for the general qudit case.

*Linear optics and  $\chi^{(2)}$  resources.*—The linear optics resources we require are readily available: dichroic mirrors and phase shifters. The pumped  $\chi^{(2)}$  resource we require is quantum-state frequency conversion (QFC) [52–54], which converts a frequency- $\omega_{\text{in}}$  single-photon Fock state to a frequency- $\omega_{\text{out}}$  single-photon Fock state. The full-quantum  $\chi^{(2)}$  resources we require are second-harmonic generation (SHG), which converts a frequency- $\omega_{\text{in}}$  two-photon Fock state to a frequency- $2\omega_{\text{in}}$  single-photon Fock state; type-I phase-matched spontaneous parametric down-conversion (SPDC), which converts a frequency- $2\omega_{\text{in}}$  single-photon Fock state to a frequency- $\omega_{\text{in}}$  two-photon Fock state; and generalized sum-frequency generation (SFG $_{\theta}$ ), which accomplishes the state transformation

$$\text{SFG}_{\theta}|1, 1, 0\rangle = \cos(\theta)|1, 1, 0\rangle + \sin(\theta)|0, 0, 1\rangle, \quad (3)$$

where  $|n_s, n_i, n_p\rangle$  denotes a three-mode Fock state containing  $n_s$  frequency- $\omega_s$  signal photons,  $n_i$  frequency- $\omega_i$

idler photons, and  $n_p$  frequency- $\omega_p$  pump photons, with the pump's frequency satisfying  $\omega_p = \omega_s + \omega_i$ . (See the Supplemental Material [55] for more information.)

*Universality in the qubit basis.*—The Lie group generated by  $\chi^{(2)}$  Hamiltonian evolutions is a subgroup of the unitary group  $U$ ; hence, it is compact. A compact Lie group, together with its generating Lie algebra, are completely reducible. This means that they can be written as a direct sum of irreducible representations over the state space  $\mathcal{H} \equiv \bigoplus_{n=1}^{\infty} \mathcal{H}_n$ , whose irreducible subspaces,  $\{\mathcal{H}_n\}$ , are labeled by their pump mode's maximum photon number  $n$ ; i.e., they are the  $n$ -pump-photon subspaces spanned by the three-mode Fock-state bases  $\{|0, 0, n\rangle, |1, 1, n-1\rangle, \dots, |n, n, 0\rangle\}$ . For qubit universality, we therefore encode in the one-pump-photon subspace  $\mathcal{H}_1$ , using the three-mode Fock states,

$$|\tilde{0}\rangle = |1, 1, 0\rangle, \quad |\tilde{1}\rangle = |0, 0, 1\rangle, \quad (4)$$

for our logical-qubit basis states. Here, the signal and idler are both at frequency  $\omega$  with orthogonal polarizations, the pump is at frequency  $2\omega$ , and all three share a common spatial mode. Universality is proved by the following theorem:

*Theorem 1.*—Universal quantum computation can be realized with  $\chi^{(2)}$  interactions and linear optics in any qubit basis of the one-pump photon subspace.

**Proof:** The  $\chi^{(2)}$  Hamiltonians,  $\hat{G}_1$  and  $\hat{G}_2$ , defined in Eq. (2) are proportional to the Pauli  $\hat{Y}$  and Pauli  $\hat{X}$  operators in the logical-qubit basis, which are universal for realizing single-qubit rotations. So, to complete our  $\chi^{(2)}$  universality proof for the logical-qubit basis in Eq. (4), it suffices for us to show that we can construct a controlled- $Z$  qubit gate for that basis [12], i.e., a gate (denoted  $\Lambda_2[Z]$  in what follows) that imparts a  $\pi$ -rad phase shift to the  $|\tilde{1}\rangle_c |\tilde{1}\rangle_t$  component of the joint state of the control (subscript  $c$ ) and target (subscript  $t$ ) qubits. Moreover, because  $\Lambda_2[Z]$  can be sandwiched between single-qubit  $\chi^{(2)}$  rotations to achieve the controlled- $Z$  function in any  $\mathcal{H}_1$  qubit basis, Theorem 1 will be proved once we have established how to realize  $\Lambda_2[Z]$ .

Figure 1 shows our optical circuit [59] for the  $\Lambda_2[Z]$  gate for the logical-qubit basis in Eq. (4). The control and target qubits enter on the upper and lower rails, respectively. QFC1 shifts the frequency of the control qubit's pump photon (if present) from  $2\omega$  to  $2\omega'$ , so that dichroic mirrors (DMs) are able to direct pump photons from the control and target qubits to the center rail's SFG $_{\pi}$  gate, where they serve as modes 1 (frequency  $\omega_1 \equiv 2\omega'$ ) and 2 (frequency  $\omega_2 = 2\omega$ ). This gate imparts a  $\pi$ -rad phase shift if and only if pump photons are present from both the control and target qubits. Thus, after another set of DMs restore the control and target pump photons to the top and bottom rails, respectively, the  $\Lambda_2[Z]$  gate—and hence the proof of Theorem 1—is completed by QFC2, which shifts the

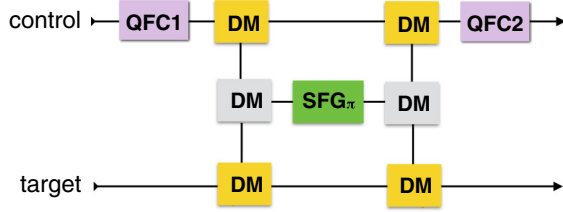


FIG. 1. Schematic for constructing the  $\Lambda_2[Z]$  gate in the logical-qubit basis [Eq. (4)] using  $\chi^{(2)}$  interactions and linear optics. QFC1 and QFC2: Quantum-state frequency conversions. DM: Dichroic mirror. SFG $_{\pi}$ : Generalized sum-frequency generation (3) with  $\theta = \pi$ .

frequency of the control qubit's pump photon (if present) from  $2\omega'$  to  $2\omega$ . Note that each  $\chi^{(2)}$  element in Fig. 1 acts on only one of its potentially excited bosonic-mode inputs; e.g., QFC1 affects its pump-mode input but neither its signal-mode input nor its idler-mode input. Such modal selectivity puts a burden on experimental realization. In particular, QFC1 and QFC2 will require a different nonlinear medium than will SFG $_{\pi}$ . This difficulty, however, may disappear once high-efficiency nondepleted  $\chi^{(3)}$  induced  $\chi^{(2)}$  interactions become available [29,34,35].

*Universality in the qutrit basis.*—For qutrit universality, we encode in  $\mathcal{H}_2$  using the three-mode Fock states

$$|\tilde{0}\rangle = |1, 1, 1\rangle, \quad |\tilde{1}\rangle = |2, 2, 0\rangle, \quad |\tilde{2}\rangle = |0, 0, 2\rangle, \quad (5)$$

for our logical-qutrit basis states. Here, the signal and idler have frequency  $\omega$  and are orthogonally polarized, while the pump has frequency  $2\omega$ , and all three share a common spatial mode. These states can be prepared by type-II phase-matched SPDC in the two-pump-photon subspace [36], and are naturally confined to this subspace under  $\chi^{(2)}$  interactions. It follows that restricting linear combinations of the  $\chi^{(2)}$  Hamiltonians,  $\hat{G}_1, \hat{G}_2$ , the modal photon-number operators,  $\{\hat{N}_k \equiv \hat{a}_k^\dagger \hat{a}_k : k = s, i, p\}$ , and the nested commutators of these operators to the two-pump-photon subspace  $\mathcal{H}_2$  constitutes a Lie algebra  $\mathfrak{g}$ . The Lie group  $H$  associated with  $\mathfrak{g}$  is found from the exponential map  $\exp : \mathfrak{g} \rightarrow H$ , where for each group element  $\hat{h} \in H$ ,  $\exists \hat{E} \in \mathfrak{g}$  and  $t \in \mathbb{R}$  such that  $\hat{h} = \exp(it\hat{E})$ . For simplicity, in all that follows, we set  $\kappa = 1$  in the Hamiltonians  $\hat{G}_1$  and  $\hat{G}_2$ . We begin our universality demonstration with a theorem about  $\mathfrak{g}$ .

*Theorem 2.*—The Lie algebra  $\mathfrak{g}$  is  $\mathfrak{u}(3)$ .

**Proof:** First, we prove that  $\mathfrak{u}(3) \subseteq \mathfrak{g}$ . From the original  $\chi^{(2)}$  Hamiltonians  $\hat{G}_1$  and  $\hat{G}_2$ , we can obtain all transformations generated by linear combinations of  $\hat{G}_1, \hat{G}_2, \hat{N}_s, \hat{N}_i, \hat{N}_p$  and their nested commutators. Using the vector  $\mathbf{v}^T \equiv [v_0 v_1 v_2]$  to represent the qutrit  $|\psi\rangle = v_0|1, 1, 1\rangle + v_1|2, 2, 0\rangle + v_2|0, 0, 2\rangle$ , we obtain the matrix representations

$$\hat{G}_1 = \frac{i}{2} [\hat{a}_s^\dagger \hat{a}_i^\dagger \hat{a}_p - \hat{a}_s \hat{a}_i \hat{a}_p^\dagger] = \frac{-i}{2} \begin{bmatrix} 0 & 2 & \sqrt{2} \\ -2 & 0 & 0 \\ -\sqrt{2} & 0 & 0 \end{bmatrix}, \quad (6)$$

$$\hat{G}_2 = \frac{1}{2} [\hat{a}_s^\dagger \hat{a}_i^\dagger \hat{a}_p + \hat{a}_s \hat{a}_i \hat{a}_p^\dagger] = \frac{1}{2} \begin{bmatrix} 0 & 2 & \sqrt{2} \\ 2 & 0 & 0 \\ \sqrt{2} & 0 & 0 \end{bmatrix}, \quad (7)$$

$$\hat{G}_3 = i[\hat{G}_1, \hat{G}_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (8)$$

$$\hat{G}_4 = i[\hat{G}_2, \hat{G}_3] = 3 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (9)$$

$$\hat{G}_5 = i[\hat{G}_3, \hat{G}_1] = 3i \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (10)$$

$$\hat{G}_6 = \frac{1}{2} (i[\hat{G}_1, \hat{G}_4] + i[\hat{G}_5, \hat{G}_2]) = \frac{3}{4} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad (11)$$

$$\hat{G}_7 = i[\hat{G}_4, \hat{G}_2] = \frac{3i}{4} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad (12)$$

$$\hat{G}_8 = \frac{1}{2} (1 - \hat{N}_p) = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad (13)$$

$$\hat{G}_9 = \frac{1}{2} \left( \frac{\hat{N}_s + \hat{N}_i}{2} + \hat{N}_p \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (14)$$

for all the independent generators, where the second equalities apply in the two-pump-photon subspace  $\mathcal{H}_2$ . It is then straightforward to verify that the Gell-Mann matrices arising from linear combinations of the above generators are

$$\hat{\lambda}_1 = \hat{G}_4/3, \quad \hat{\lambda}_2 = -\hat{G}_5/3, \quad (15)$$

$$\hat{\lambda}_3 = 2\hat{G}_8 + \hat{G}_3, \quad \hat{\lambda}_4 = \sqrt{2}(\hat{G}_2 - \hat{G}_4/3), \quad (16)$$

$$\hat{\lambda}_5 = \sqrt{2}(\hat{G}_1 - \hat{G}_5/3), \quad \hat{\lambda}_6 = 4\hat{G}_6/3, \quad (17)$$

$$\hat{\lambda}_7 = 4\hat{G}_7/3, \quad \hat{\lambda}_8 = (\hat{G}_3 + 6\hat{G}_8)/\sqrt{3}. \quad (18)$$

Gell-Mann matrices are one representation of the complete set of linearly independent generators for the  $\mathfrak{su}(3)$  Lie algebra. Together with  $\hat{G}_9$ , they form the complete set of generators for  $\mathfrak{u}(3)$ , proving that  $\mathfrak{u}(3) \subseteq \mathfrak{g}$ .

We complete our proof of Theorem 2 by showing that  $\mathfrak{g} \subseteq \mathfrak{u}(3)$ . Because the two-pump-photon subspace  $\mathcal{H}_2$  is closed under  $\mathfrak{g}$ , every Lie group element  $\hat{h} \in H$  is generated by an  $\hat{E} \in \mathfrak{g}$  via  $\hat{h} = \exp(it\hat{E})$  for some  $t \in \mathbb{R}$ . As  $\exp(it\hat{E})$  is a unitary transformation in the two-pump-photon subspace, we have  $H \subset U(3)$ . Furthermore, this condition holds if and only if  $\mathfrak{g} \subseteq \mathfrak{u}(3)$ , thus finishing Theorem 2's proof.

References [60–62] show that if operators  $\{\hat{G}_k\}$  and their nested commutators generate the Lie algebra  $\mathfrak{u}(3^m)$ , they can then be used to construct a universal set of unitaries  $U_k(t) = \exp(-it\hat{G}_k)$  in the  $m$ -qutrit subspace. Setting  $m = 1$ , we have the following claim:

*Claim 1.*—Universal single-qutrit rotations can be realized with  $\chi^{(2)}$  interactions.

Universal qutrit computation entails not only universal single-qutrit unitary gates but also universal two-qutrit unitary transformations in  $\mathcal{H}_2^{\otimes 2}$ , so we need the following theorem:

*Theorem 3.*—Universal qutrit quantum computation can be realized with  $\chi^{(2)}$  interactions and linear optics in any qutrit basis of the two-pump-photon subspace.

**Proof:** From Claim 1, we know that arbitrary  $U(3)$  qutrit rotations can be realized with  $\chi^{(2)}$  interactions. It is also known [12,63–65] that a universal single-qutrit gate set plus a controlled-Z gate for the logical-qutrit basis in Eq. (5)—denoted  $\Lambda_3[Z]$ —are universal for qutrit computation in any qutrit basis of the two-pump-photon subspace  $\mathcal{H}_2$ .

The  $\Lambda_3[Z]$  gate realizes the unitary transformation  $\Lambda_3[Z]|\tilde{j}\rangle_c|\tilde{k}\rangle_t = (-1)^{\delta_{j2}\delta_{k2}}|\tilde{j}\rangle_c|\tilde{k}\rangle_t$  for states in  $\mathcal{H}_2$ , where  $\delta_{uv}$  is the Kronecker delta. Figure 2 shows how this gate can be realized using  $\chi^{(2)}$  interactions and linear optics. The control and target qubits enter on the upper and lower rails, respectively, where second-harmonic generators (SHGs) convert two-photon Fock-state pumps at frequency  $2\omega$  to a single-photon Fock state at frequency  $4\omega$ . The shaded block labeled  $\Lambda_2[Z]$  is the same gate shown in Fig. 1, *except* that

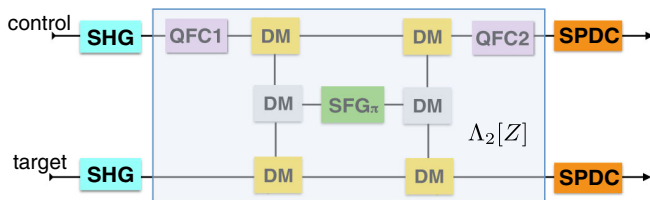


FIG. 2. Schematic for constructing the  $\Lambda_3[Z]$  gate in the logical-qutrit basis (5) using  $\chi^{(2)}$  interactions and linear optics. SHG: Second-harmonic generation.  $\Lambda_2[Z]$ : The optical circuit from Fig. (1) with modifications described in the text. SPDC: Type-I phase-matched spontaneous parametric down-conversion.

(i) its QFC1 converts a frequency- $4\omega$  single-photon Fock state to a frequency- $4\omega'$  single-photon Fock state, (ii) its first set of DMs route the frequency- $4\omega'$  photon (if present) from the upper rail and the frequency- $4\omega$  photon (if present) from the lower rail to the  $\text{SFG}_\pi$  block on the center rail, (iii) its  $\text{SFG}_\pi$  block is arranged to apply a  $\pi$ -rad phase shift to the state  $|1, 1, 0\rangle$ , whose first two entries are the photon numbers of its frequency- $4\omega'$  and frequency- $4\omega$  inputs, (iv) its second set of DMs return the frequency- $4\omega'$  and frequency- $4\omega$  photons to the upper and lower rails, respectively, and (v) its QFC2 converts a frequency- $4\omega'$  single-photon Fock state to a frequency- $4\omega$  single-photon Fock state. The SPDC blocks then complete the  $\Lambda_3[Z]$  gate—by converting frequency- $4\omega$  single-photon Fock states (if present) to frequency- $2\omega$  two-photon Fock states—because the  $\Lambda_2[Z]$  block has imparted a  $\pi$ -rad phase shift to the  $|\tilde{2}\rangle_c|\tilde{2}\rangle_t$  component of the original input state. Together with Claim 1, the  $\Lambda_3[Z]$  construction proves Theorem 3.

*Universality in the  $(n + 1)$ -dimensional qudit basis.*—The culmination of our  $\chi^{(2)}$  universality work is the following theorem:

*Theorem 4.*—Universal qudit quantum computation can be realized with  $\chi^{(2)}$  interactions and linear optics in any  $(n + 1)$ -dimensional basis of the  $n$ -pump-photon subspace.

**Proof:** Our proof is by induction. We have already shown that Theorem 4 holds for  $n = 1$  and  $n = 2$ . The induction proof is completed by assuming that Theorem 4 holds for  $n = m$ , and then showing that it holds for  $n = m + 1$ . The details appear in Ref. [55]. Here we just note that they involve a Lie-group result [51,66] and coherent photon injection or subtraction. Coherent photon injection and subtraction are full-quantum  $\chi^{(2)}$  interactions between the encoded modes and ancillary modes. Although used as a conceptual tool in the proof of Theorem 4, coherent photon injection has independent merit owing to its enabling preparation of high-photon-number Fock states from single-photon Fock states. Thus, we devote the next section to its description.

*Coherent photon injection.*—The coherent photon injection used in our universality proof is a generalization of a result from Hubel *et al.* [67]. To illustrate how it works, suppose we start with the qubit-basis state  $|\tilde{1}\rangle = |0, 0, 1\rangle$  from Eq. (4) with the goal of generating the qutrit-basis state  $|\tilde{2}\rangle = |0, 0, 2\rangle$  from Eq. (5). Coherent photon injection accomplishes this task as follows: We adjoin the  $|0, 0, 1\rangle$  system with an ancillary pump mode (photon-creation operator  $\hat{a}_{p'}^\dagger$ ) that has the same frequency as, but is orthogonally polarized to, the pump mode of  $|0, 0, 1\rangle$ . We then turn on the  $\chi^{(2)}$  interaction  $\hat{G}_{2a} = [\hat{a}_s^\dagger\hat{a}_i^\dagger\hat{a}_{p'} + \hat{a}_s\hat{a}_i\hat{a}_{p'}^\dagger]$  between the original signal-idler modes and the ancillary pump mode to realize the transformation  $e^{i\pi\hat{G}_{2a}/2}|0, 0, 1\rangle|1\rangle_a = |1, 1, 1\rangle|0\rangle_a$ . This coherent photon injection has transformed the qubit-basis state  $|\tilde{1}\rangle = |0, 0, 1\rangle$  in the one-pump-photon subspace to the qutrit-basis

state  $|\tilde{1}\rangle = |1, 1, 1\rangle$  in the two-pump-photon subspace. A qutrit-basis  $\chi^{(2)}$  gate can now rotate  $|\tilde{1}\rangle$  to  $|\tilde{2}\rangle$  in the two-pump-photon subspace [55]. Insofar as the pump mode is concerned, this overall procedure has converted a single-photon Fock-state input to a two-photon Fock-state output. The injection process can now be repeated to transform  $|0, 0, 2\rangle$  to  $|1, 1, 2\rangle$ , after which a  $\chi^{(2)}$ -enabled rotation in the three-pump-photon subspace will yield  $|0, 0, 3\rangle$ . In this manner, high-photon-number Fock states can be prepared using only single-photon sources and full-quantum  $\chi^{(2)}$  interactions.

*Conclusions.*—We have shown that universal optics-based quantum computation using only linear optics and  $\chi^{(2)}$  interactions is possible in any  $(n + 1)$ -dimensional qudit basis of the  $n$ -pump-photon subspace, with the natural basis being the three-mode Fock states  $\{|0, 0, n\rangle, |1, 1, n - 1\rangle, \dots, |0, 0, n\rangle\}$  of frequency- $\omega$ , orthogonally polarized signal and idler modes, and a frequency- $2\omega$  pump mode, all of which share a common spatial mode. Our work extends the usual gate-model universality to the universality of  $\chi^{(2)}$  Hamiltonian interactions in their irreducible subspaces. Such extension facilitates error correction for photon loss by providing a symmetry-operator formalism for hardware-efficient quantum error correction [47]. Moreover, Lie algebraic understanding of  $\chi^{(2)}$  interactions opens a path for defining an Abelian group that would enable fault-tolerant quantum computation that is robust to photon loss and physical rotation errors. To reach the end of that path, however, will require technology development.

The resources required for our qudit-basis  $\chi^{(2)}$  quantum computation are single photon sources and linear optics, plus  $\chi^{(2)}$  interactions. High-quality linear optics (dichroic mirrors and phase shifters) are already available, and high-efficiency quantum-state frequency conversion (the pumped  $\chi^{(2)}$  interaction we need) have been demonstrated. But because currently available or demonstrated single-photon sources and full-quantum  $\chi^{(2)}$  (SHG, SFG $_{\pi}$ , and SPDC) interactions fall short of what our architectures require, continued advances in these technologies must occur before our quantum computation proposals become practical. There is some reason for optimism in this regard; e.g., the efficiency of the  $\chi^{(2)}$  nonlinearity has been improved from  $10^{-7}$  [29] to  $10^{-1}$  [68] in less than a decade. Furthermore, state-of-the-art experimental realizations of strong  $\chi^{(2)}$  interactions—including in solid-state circuits [37], flux-driven Josephson parametric amplifiers [38,39,42], superconducting resonator arrays [40,41], non-depleted four-wave-mixing-induced three-wave mixing in photonic microstructured fibers [29,34,35],  $\chi^{(2)}$  interactions inside ring resonators [69], and nonlinear interactions in frequency-degenerate double-lambda systems [70]—are closing the gap between theory and practical applications of full-quantum  $\chi^{(2)}$  interactions.

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