Scattering Amplitudes from Intersection Theory

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We use Picard-Lefschetz theory to prove a new formula for intersection numbers of twisted cocycles associated with a given arrangement of hyperplanes. In a special case when this arrangement produces the moduli space of punctured Riemann spheres, intersection numbers become tree-level scattering amplitudes of quantum field theories in the Cachazo-He-Yuan formulation.

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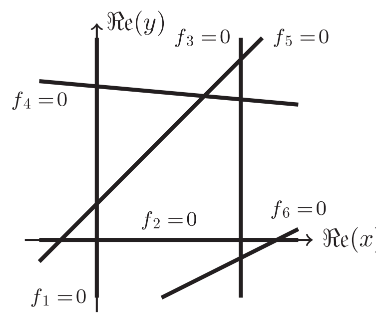
Introduction.—Over the past years, study of scattering amplitudes revealed many unexpected connections to geometric structures [1–4], allowing us to understand physical properties of quantum field theories—such as locality or unitarity—from a different perspective. At the same time, they equip us with new mathematical tools that vastly simplify practical calculations. In this work we unravel another connection to a branch of mathematics called *intersection theory* [5–8].

It has recently transpired that intersection theory plays an important role in string theory amplitudes, where, in particular, it provides a geometric interpretation of the Kawai-Lewellen-Tye (KLT) relations between open and closed string amplitudes, or—in the field-theory limit—Yang-Mills and Einstein gravity amplitudes [9,10]. Here, we show that analogous structures appear directly in scattering amplitudes of ordinary quantum field theories. We find that they can be understood as intersection numbers of the so-called *twisted cocycles* [6–8], which are certain families of differential forms.

It is instructive to start with an explicit example straightaway. Let us consider $\mathbb{CP}^2$ with inhomogeneous coordinates (x, y), dissected by six hyperplanes defined through linear equations $\{f_i = 0\}$. We can easily visualize the real section of this space with a concrete choice of hyperplanes, for instance,

$$f_1 = x,$$
$$f_2 = y,$$
$$f_3 = 1 - x,$$
$$f_4 = 1 - x/4 - y,$$
$$f_5 = 1/4 + x - y,$$
$$f_6 = 5/4 - x + 2y.$$



The space of our interest is the original manifold with these hyperplanes removed:

$$X = \mathbb{CP}^2 \Big\backslash \bigcup_{i=1}^{6} \{f_i = 0\}. \tag{1}$$

Associated to it, we can define a differential 1-form ω, called the *twist*, with logarithmic singularities along f's:

$$\omega = \sum_{i=1}^{6} \alpha_i d\log f_i = \underbrace{\left(\frac{\alpha_1}{x} + \frac{\alpha_3}{-1+x} + \frac{\alpha_4}{-4+x+4y} + \frac{\alpha_5}{1/4+x-y} + \frac{\alpha_6}{-5/4+x-2y}\right)}_{\omega_x} dx$$
$$+ \underbrace{\left(\frac{\alpha_2}{y} + \frac{\alpha_4}{-1+x/4+y} + \frac{\alpha_5}{-1/4-x+y} + \frac{\alpha_6}{5/8-x/2+y}\right)}_{\omega_y} dy, \tag{2}$$

where α's are constant coefficients adding up to zero. The twist 1-form fully characterizes the space X.

On this space we can introduce two differential forms, φ_L and φ_R. We choose them in such a way that they have logarithmic singularities on three of the hyperplanes defined above. For instance, we can take

$$\varphi_L = d\log\frac{f_2}{f_3} \wedge d\log\frac{f_3}{f_5} = \frac{5dx \wedge dy}{y(1-x)(1+4x-4y)}, \quad (3)$$

$$\varphi_R = d\log\frac{f_2}{f_3} \wedge d\log\frac{f_3}{f_6} = \frac{dx \wedge dy}{y(1-x)(5-4x+8y)}. \quad (4)$$

These objects are examples of twisted cocycles, which, roughly speaking, are differential forms on X defined up to equivalence classes $\varphi \sim \varphi + \omega \wedge \xi$ for any $d\log$ form ξ. One can define an invariant pairing called the *intersection number* [11]. Its standard definition [6,7] reads

$$\langle \varphi_L, \varphi_R \rangle_\omega = \frac{1}{(2\pi i)^2} \int_X \iota_\omega(\varphi_L) \wedge \varphi_R, \quad (5)$$

where the map ι_ω turns φ_L into its compactly supported version, i.e., one that vanishes in a small neighbourhood of the hyperplanes $\{f_i = 0\}$. Note that the integrand would vanish if it was not for this map. Here we also remark that $\langle \varphi_L |$ and $| \varphi_R \rangle$ belong to different cohomologies, as will be discussed in the following section. A result of this calculation reveals a combinatorial formula [12]:

$$\langle \varphi_L, \varphi_R \rangle_\omega = \pm \sum_{\{f_i, f_j\} \in L,R} \frac{1}{\alpha_i \alpha_j} = \frac{1}{\alpha_2 \alpha_3}. \quad (6)$$

We review the meaning of the map ι_ω in Ref. [13], which also illustrates how factors of α_i arise in the denominators. The above result is a sum over all intersection vertices of the hyperplanes that are associated to both φ_L and φ_R. In our example, we have $L = (f_2, f_3, f_5)$ and $R = (f_2, f_3, f_6)$, which intersect at a single point $\{f_2 = 0\} \cap \{f_3 = 0\}$, and we inserted the correct sign [12]. An important feature of the above formula is that it is completely independent of the precise positions of the hyperplanes, as long as their arrangement is generic; i.e., no three f's intersect at a single point.

In this Letter we propose an alternative formula for computing intersection numbers as an integral localizing on the points (x^*, y^*) at which ω vanishes:

$$\langle \varphi_L, \varphi_R \rangle_\omega = \int dxdy \delta(\omega_x)\delta(\omega_y)\hat{\varphi}_L \hat{\varphi}_R$$
$$= \sum_{(x^*, y^*)} \det{}^{-1} \begin{bmatrix} \frac{\partial \omega_x}{\partial x} & \frac{\partial \omega_x}{\partial y} \\ \frac{\partial \omega_y}{\partial x} & \frac{\partial \omega_y}{\partial y} \end{bmatrix} \hat{\varphi}_L \hat{\varphi}_R |_{(x,y)=(x^*,y^*)}, \quad (7)$$

where we used $\varphi = \hat{\varphi} dx \wedge dy$. Here, δ functions should be understood as multidimensional residue prescriptions around the zeros of ω. Remarkably, this formula evaluates to the rational function of α's [Eq. (6)] for any choice of φ_L and φ_R, and does so in a highly nontrivial manner.

Readers familiar with scattering amplitudes literature will notice a resemblance of Eq. (7) to the Cachazo-He-Yuan (CHY) formulas [32,33]. This is not a coincidence. In fact, CHY formalism uses a particular, singular, arrangement of hyperplanes, for example,

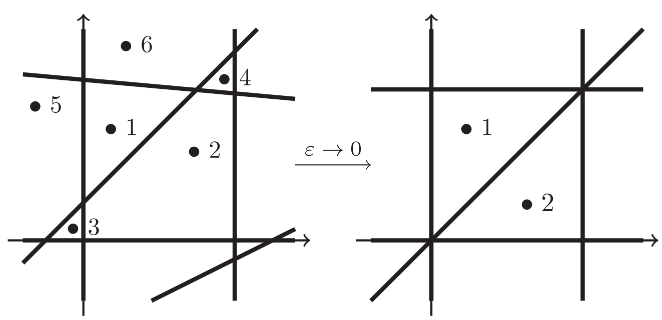
$$\{x = 0\} \cup \{y = 0\} \cup \{1 - x = 0\}$$
$$\cup \{1 - y = 0\} \cup \{x - y = 0\},$$

with the last hyperplane at infinity, such that the resulting space X is the moduli space of punctured Riemann spheres, in this case $X = \mathcal{M}_{0,5}$. Equation (6) can no longer be used directly, as the arrangement of hyperplanes is not generic. Nevertheless, the new formula Eq. (7) is still valid. Let us see how this comes about.

We can organize the coefficients of a particular arrangement into a matrix:

$$C = \begin{array}{c} \begin{array}{cccccc} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & \frac{\varepsilon}{4} & \frac{4+\varepsilon}{4} \\ 1 & 0 & -1 & -\frac{\varepsilon}{4} & 1 & -\varepsilon \\ 0 & 1 & 0 & -1 & -1 & 2\varepsilon \end{bmatrix} \end{array} \begin{array}{c} 1 \\ x \\ y \end{array}, \quad (8)$$

giving $f_i = c_{1i} + c_{2i}x + c_{3i}y$. We set it up such that $\varepsilon = 1$ yields the original arrangement, which deforms into the singular one as $\varepsilon \to 0$. A sign of singularity is that several 3×3 minors of C vanish in this limit. Keeping parameters αconstant for example, with $\alpha_i > 0$ for $i = 1, 2, \ldots, 5$, the hyperplanes and zeros of ω behave as follows:



The hyperplane $\{f_6 = 0\}$ moved to infinity. Out of the six zeros of ω, only 1 and 2 survive at finite positions. Both 3 and 4 get trapped between three hyperplanes and eventually cease to be zeros of ω since (0,0) and (1,1) are not a part of the manifold X. Similarly, 5 and 6 shoot off to infinity. This can be easily verified from the explicit form of the twist in the strict $\varepsilon \to 0$ limit:

$$\tilde{\omega} = \left(\frac{s_{12}}{x} + \frac{s_{24}}{x-1} + \frac{s_{23}}{x-y}\right)dx + \left(\frac{s_{13}}{y} + \frac{s_{34}}{y-1} + \frac{s_{23}}{y-x}\right)dy, \quad (9)$$

where we made an identification of α's with specific Mandelstam invariants, $s_{ab} = (k_a + k_b)^2$, involving ingoing lightlike momenta k_a. Note that it preserves the condition $\sum_{i=1}^{6} \alpha_i = 0$ due to momentum conservation. Using a pair of twisted cocycles, for example,

$$\tilde{\varphi}_L = d\log\frac{f_1}{f_5} \wedge d\log\frac{f_5}{f_4} = -\frac{dx \wedge dy}{x(x-y)(y-1)}, \quad (10)$$

$$\tilde{\varphi}_R = d\log\frac{f_2}{f_5} \wedge d\log\frac{f_5}{f_3} = \frac{dx \wedge dy}{y(y-x)(x-1)}, \quad (11)$$

we can evaluate their intersection number at this singular arrangement via Eq. (7), giving

$$\langle \tilde{\varphi}_L, \tilde{\varphi}_R \rangle_{\tilde{\omega}} = \frac{1}{s_{23}} \left(\frac{1}{s_{12} + s_{13} + s_{23}} + \frac{1}{s_{24} + s_{34} + s_{23}} \right),$$

which is indeed an example of a biadjoint scalar partial amplitude [33,34]. The limit $\varepsilon \to 0$ needs to be taken *before* performing the integration Eq. (7). Evaluating it at finite ε yields identically zero in agreement with Eq. (6), since $L = (f_1, f_5, f_4)$ and $R = (f_2, f_5, f_3)$ have no intersection points in a generic arrangement. In general, intersection numbers change discontinuously depending on the topology, but not geometry, of the arrangement.

With this example we illustrated how the new prescription Eq. (7) provides a way of calculating intersection numbers even at singular hyperplane arrangements, such as the ones giving rise to scattering amplitudes. Let us now flesh out details of this construction in its full generality.

General formula.—In general, let us consider a generic arrangement of k hyperplanes on $\mathbb{CP}^m$. They are described with

$$f_i = c_{1i} + \sum_{a=2}^{m+1} c_{ai}\sigma_a, \quad (12)$$

where σ_a for $a = 2, 3, \ldots, m+1$ are the inhomogeneous coordinates on $\mathbb{CP}^m$. This corresponds to a point in the Grassmannian, $C \in \mathrm{Gr}(m+1, k)$. A given arrangement is nonsingular if all maximal minors of C are nonvanishing. The resulting manifold is $X = \mathbb{CP}^m \setminus \cup_{i=1}^{k} \{f_i = 0\}$, and the twist 1-form ω is defined as in Eq. (2), giving

$$\omega = \sum_{a=2}^{m+1} \left(\sum_{i=1}^{k} \frac{\alpha_i c_{ai}}{c_{1i} + \sum_{b=2}^{m+1} c_{bi}\sigma_b} \right) d\sigma_a, \quad (13)$$

with $\sum_{i=1}^{k} \alpha_i = 0$ and α's sufficiently generic.

On this space we introduce the mth twisted cohomology group [7]:

$$H^m(X, \nabla_\omega) = \{\varphi | \nabla_\omega \varphi = 0\} / \{\nabla_\omega \xi\}, \quad (14)$$

where $\nabla_\omega = d + \omega \wedge$ is the connection and ξ is any smooth $(m-1)$-form on X. The dimension of this group is $d = \binom{k-2}{m}$. Its elements are called twisted cocycles. One choice of a basis is the one constructed from cocycles of the form

$$\varphi_L = d\log\frac{f_{L(1)}}{f_{L(2)}} \wedge d\log\frac{f_{L(2)}}{f_{L(3)}} \wedge \cdots \wedge d\log\frac{f_{L(m)}}{f_{L(m+1)}}$$
$$= \hat{\varphi}_L d\sigma_2 \wedge d\sigma_3 \wedge \cdots \wedge d\sigma_{m+1}, \quad (15)$$

for example, with $1 = L(1) < L(2) < \cdots < L(m+1) < k$ [12]. It is known that an arbitrary twisted cocycle can be expressed in a logarithmic basis [7], such as the one above. Similarly, the dual mth twisted cohomology is defined with the connection $\nabla_{-\omega}$, whose basis can be chosen to be the same as in Eq. (15).

Intersection numbers are normally computed using the definition Eq. (5) with normalization $1/(2\pi i)^m$ for twisted cocycles φ_L and φ_R in the original and dual cohomologies, respectively; see Ref. [13]. In generic arrangements they evaluate, up to an overall sign, to [12]

$$\langle \varphi_L, \varphi_R \rangle_\omega = \pm \sum_{\{f_1, f_2, \ldots, f_m\} \in L, R} \frac{1}{\alpha_1 \alpha_2 \cdots \alpha_m}. \quad (16)$$

In singular cases, the above expression requires careful evaluation using blowups; see Ref. [13].

Here, we give an alternative formula as an integral localizing on the zeros of ω:

$$\langle \varphi_L, \varphi_R \rangle_\omega = \frac{1}{(-2\pi i)^m} \oint_{\bigwedge_{a=2}^{m+1} \{|\omega_a| = \epsilon\}} \frac{\varphi_L \hat{\varphi}_R}{\prod_{a=2}^{m+1} \omega_a}. \quad (17)$$

The above formula is valid even at singular hyperplane arrangements. We used a more precise notation in terms of a multidimensional residue around the zeros of Eq. (13), in place of δ functions localizing the integral like in the example Eq. (7).

Proof.—Intersection numbers of twisted cocycles satisfy twisted period relations [6]:

$$\langle \varphi_L, \varphi_R \rangle_\omega = \frac{1}{(2\pi i)^m} \sum_{\alpha, \beta = 1}^{d} \int_{\mathcal{A}_\alpha} e^{\int \omega} \varphi_L \mathbf{H}^{-1}_{\beta\alpha} \int_{\mathcal{B}_\beta} e^{-\int \omega} \varphi_R. \quad (18)$$

By $\exp \int \omega$ we denote the multivalued function $\prod_{i=1}^{k} f_i^{\alpha_i}$ with some choice of a branch. We have two sets of d twisted *cycles* $\{\mathcal{A}_\alpha\}$ and $\{\mathcal{B}_\beta\}$ forming bases of their respective homology groups. Here, $\mathbf{H}$ is the intersection matrix, whose entries are the intersection numbers of these cycles [6–8]. Since integrals in the above expression do not generically converge at the same time, they are to be understood in terms of their analytic continuation. In order to use localization arguments, however, we need to define appropriate bases of cycles which fix this problem.

Following the Picard-Lefschetz prescription [35], we choose bases of twisted cycles to be the paths of steepest descent and ascent of $\exp\int\omega$ on the same branches, denoted by $\{\mathcal{J}_\alpha\}$ and $\{\mathcal{K}_\beta\}$, respectively. By definition, each of them passes through exactly one critical point $\sigma_a^{(\alpha)}$ of $\exp\int\omega$, or equivalently a zero of ω. Therefore, cycles intersect only at these points and the intersection matrix becomes an identity matrix, $\mathbf{H}_{\alpha\beta} = \delta_{\alpha\beta}$, giving

$$\langle\varphi_L, \varphi_R\rangle_\omega = \frac{1}{(2\pi i)^m}\sum_{\alpha=1}^{d}\int_{\mathcal{J}_\alpha} e^{\int\omega}\varphi_L \int_{\mathcal{K}_\alpha} e^{-\int\omega}\varphi_R. \tag{19}$$

Cycles $\{\mathcal{K}_\alpha\}$ are now paths of steepest *descent* of $\exp -\int\omega$. Crucially, this means that each integral on the right-hand side of the above equation converges. Let us rescale $\omega\to\tau\omega$ and take $\tau\to\infty$. In this limit, each integral localizes on precisely one of the d critical points:

$$\lim_{\tau\to\infty}\langle\varphi_L, \varphi_R\rangle_{\tau\omega} = \frac{1}{(-\tau)^m}\sum_{\alpha=1}^{d}\det{}^{-1}\left[\frac{\partial^2\int\omega}{\partial\sigma_a\partial\sigma_b}\right]\hat{\varphi}_L\hat{\varphi}_R\Big|_{\sigma_a=\sigma_a^{(\alpha)}},$$

where the exponential factors cancel out between the two integrals for each critical point, and $\partial\int\omega/\partial\sigma_a = \omega_a$ is single valued. Since the number of critical points equals the dimension of the homology group d [7], all zeros of ω are counted.

On the other hand, intersection numbers $\langle\varphi_L, \varphi_R\rangle_{\tau\omega}$ are known to scale homogeneously as τ^{-m}, provided that φ_L and φ_R are expressed in a logarithmic basis [12], cf. Eq. (16). We therefore conclude that the above localization formula is exact in τ and hence we can set $\tau = 1$. Expressing the result in terms of a multidimensional residue, this proves our claim Eq. (17).

Scattering amplitudes as intersection numbers.—Let us now consider a special case in which the arrangement of hyperplanes produces the moduli space of n-punctured Riemann spheres, $X = \mathcal{M}_{0,n}$. The dimension of X is $m{=}n{-}3$, and $k = n(n-3)/2 + 1$ hyperplanes are given by

$$\bigcup_{a=2}^{n-2}\{\sigma_a = 0\}\bigcup_{a=2}^{n-2}\{\sigma_a - 1 = 0\}\bigcup_{2\le a<b\le n-2}\{\sigma_a - \sigma_b = 0\},$$

with the last one located at infinity. We introduce three coordinates $(\sigma_1, \sigma_{n-1}, \sigma_n) = (0, 1, \infty)$ and choose the coefficients α for hyperplanes $\{\sigma_a - \sigma_b = 0\}$ to be Mandelstam invariants s_{ab}. For massless kinematics they add up to zero by momentum conservation. There exists a special kinematic region with all s_{ab} except for $s_{1,n-1}$ being positive [36], where all $(n-3)!$ zeros of ω lie in distinct chambers in the real section of the moduli space [37].

The dimension of the cohomology group d undergoes a huge reduction compared to a generic arrangement, from $\binom{n(n-3)/2-1}{n-3}$ to $(n-3)!$ in this singular limit. It also gains an enhanced SL$(2,\mathbb{C})$ redundancy, $\sigma_a\to(A\sigma_a+B)/(C\sigma_a+D)$, with $AD - BC = 1$.

A basis of twisted cocycles can be written using Parke-Taylor forms [10] for the $(n-3)!$ permutations α:

$$\begin{aligned}\mathrm{PT}(\alpha) &= d\log\frac{\sigma_{1,\alpha(2)}}{\sigma_{\alpha(2),\alpha(3)}}\wedge\cdots\wedge d\log\frac{\sigma_{\alpha(n-3),\alpha(n-2)}}{\sigma_{\alpha(n-2),n-1}}\\ &= (-1)^n\frac{d\sigma_{\alpha(2)}\wedge d\sigma_{\alpha(3)}\wedge\cdots\wedge d\sigma_{\alpha(n-2)}}{\sigma_{1,\alpha(2)}\sigma_{\alpha(2),\alpha(3)}\cdots\sigma_{\alpha(n-2),n-1}},\end{aligned} \tag{20}$$

where $\sigma_{ab} = \sigma_a - \sigma_b$. The twist 1-form ω becomes a linear combination of *scattering equations* [38], E_a:

$$\omega = \sum_{a=2}^{n-2}\left(\sum_{\substack{b=1\\ b\neq a}}^{n}\frac{s_{ab}}{\sigma_{ab}}\right)d\sigma_a = \sum_{a=2}^{n-2}E_a d\sigma_a. \tag{21}$$

With these assignments, intersection numbers Eq. (17) become scattering amplitudes in the CHY formulation [32,33,39]. Physically, the twist 1-form Eq. (21) translates between singularities of the S matrix and boundaries of the moduli space. Quantum field theory whose amplitudes are being computed depends on the choice of φ_L and φ_R. For instance, the ingredient $Pf'\Psi$ defined in Ref. [32] can be expanded in the basis of twisted cocycles, and the pairings

$$\langle Pf'\Psi, Pf'\Psi\rangle_\omega, \qquad \langle\mathrm{PT}(\alpha), Pf'\Psi\rangle_\omega, \qquad \langle\mathrm{PT}(\alpha), \mathrm{PT}(\beta)\rangle_\omega$$

give amplitudes of Einstein gravity, Yang-Mills theory, and biadjoint scalar, respectively.

In this case, Eq. (18) reduces to the so-called *chiral* KLT relation [9,40,41], and τ, which is a rescaling parameter of the Mandelstam invariants, can be identified with the inverse string tension α'. In particular, this proves the following two statements.

(i) The result of chiral KLT is a field-theory scattering amplitude in the CHY prescription. This provides mathematical foundations for more physical considerations coming from string theory [42–47].

(ii) Since the $\alpha'\to 0$ limit of a closed string amplitude is unaffected up to a sign by the $\alpha'\to-\alpha'$ replacement on one side of the KLT relation, the field-theory limit of a closed string amplitude is given by the CHY formula.

Recall that both φ_L and φ_R are elements of the twisted cohomology groups, and, in particular, are required to have only logarithmic singularities on the boundaries of the moduli space. Similarly, Mandelstam invariants s_{ab} entering ω are required to add up to zero, making the above results valid only for massless external states.

Outlook.—Let us put the results of this Letter into a broader perspective. In Ref. [10] we found that twisted cycles and cocycles associated to the moduli space $\mathcal{M}_{0,n}$ play a special role in scattering amplitudes. Three types of pairings calculate the following classes of amplitudes:

Pairing	Class
$\langle$cocycle, cocycle$\rangle$	Closed string, CHY
[cycle, cocycle$\rangle$	Open string
[cycle, cycle]	Inverse KLT kernel

In this Letter we studied intersection numbers of twisted cocycles, which fall into the first class. Within it, the difference between closed string and CHY-type amplitudes comes from a different choice of the dual cohomology group; see Ref. [10] and references therein.

Every twisted cocycle has a corresponding cycle, whose boundaries coincide with logarithmic singularities of the former. For instance, Parke-Taylor forms [Eq. (20)] map to associahedra tiling the moduli space [10,48]. Intersection numbers of both cycles and cocycles can then be described using adjacency properties of the associahedra [10,49], or their linear combinations [50–53].

One of the advantages of this way of thinking is that it allows for geometric understanding of relations between different amplitudes, in particular the KLT relations [9,10]. They can be summarized using convenient bra-ket notation; see Ref. [13].

It is natural to expect that similar interpretation in terms of intersection numbers can be made at higher loops or for specific theories in four dimensions, especially given recent evidence that field-theory loop integrands can be obtained from genus-zero Riemann surfaces [54,55] and obey KLT formulas [56]. The additional challenge is to consider nongeneric kinematics on top of singular hyperplane arrangements.

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