

Gravity and the Spin-2 Planar Schrödinger Equation

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A Schrödinger equation proposed for the Girvin-MacDonald-Platzman gapped spin-2 mode of fractional quantum Hall states is found from a novel nonrelativistic limit, applicable only in $2 + 1$ dimensions, of the massive spin-2 Fierz-Pauli field equations. It is also found from a novel null reduction of the linearized Einstein field equations in $3 + 1$ dimensions, and in this context a uniform distribution of spin-2 particles implies, via a Brinkmann-wave solution of the nonlinear Einstein equations, a confining harmonic oscillator potential for the individual particles.

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Einstein's theory of general relativity ceases to be a theory of gravity when considered in a 3D spacetime (i.e., $2 + 1$ dimensions): there is no analog of the Newtonian force, nor gravitational waves. We call it a 3D gravity theory mainly because it shares with 4D general relativity the property of being a diffeomorphism invariant theory for a dynamical metric on spacetime, which makes it a useful “toy model” for considering how theories of this type might be compatible with quantum mechanics.

A simple modification of 3D general relativity known as new massive gravity (NMG) yields a parity-preserving 3D gravity theory that does admit gravitational waves, gravitational in the sense that the corresponding particle excitation of the quantum theory has spin 2, although these spin-2 gravitons are massive rather than massless [1]. Linearization about a Minkowski vacuum yields a free field theory that is equivalent to the 3D version of the massive spin-2 field theory proposed long ago by Fierz and Pauli [2]. There are various bimetric 3D gravity theories that have the same linearized limit [3,4], and NMG may itself be viewed as the simplest example, with an auxiliary tensor field as the second metric [5].

Although NMG has no real-world applications as a theory of gravity, it has potential applications in the world of condensed matter systems in $2 + 1$ dimensions. Naturally, these are typically nonrelativistic, so this motivates consideration of the nonrelativistic limit of NMG.

Nonrelativistic limits are notoriously more complicated than one would naively imagine, so it makes sense to first investigate the nonrelativistic limit of the 3D Fierz-Pauli (FP) theory. One might expect to find a Schrödinger equation for a nonrelativistic particle of spin 2.

As it happens, fractional quantum Hall states have a Girvin-MacDonald-Platzman (GMP) gapped spin-2 mode [6], and a particular Schrödinger equation has been proposed as an equation governing its dynamics [7,8]. Following suggestions of a geometrical interpretation of GMP states [9], this spin-2 planar Schrödinger equation was shown to emerge upon linearization of a particular nonrelativistic bimetric theory [10,11]. We should stress that these are space metrics rather than spacetime metrics, but an obvious question is whether this bimetric theory is the nonrelativistic limit of some relativistic bimetric theory, perhaps NMG. We do not answer this question, but we show that the Schrödinger equation proposed to describe the GMP mode is indeed a nonrelativistic limit of the 3D FP theory.

The standard way in which spin is incorporated into the (time-dependent) Schrödinger equation is via a multiplet of complex wave functions transforming in a representation of the rotation group. This implies an $SO(2)$ doublet for two space dimensions, and this is indeed what one finds from the standard nonrelativistic limit of the 3D FP theory for a complex tensor field, but the spin-2 Schrödinger equation proposed to describe the GMP mode has a single complex wave function. What we need, although only for $2 + 1$ dimensions, is a nonrelativistic limit for a real FP tensor field.

There is a problem with the nonrelativistic limit of real-field theories that propagate massive modes. This can be understood by considering the Klein-Gordon (KG) equation for a scalar field Φ of mass m . Including all factors of c and \hbar , this is

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$$\frac{1}{c^2}\ddot{\Phi} - \nabla^2\Phi + \left(\frac{mc}{\hbar}\right)^2\Phi = 0. \quad (1)$$

The $c \rightarrow \infty$ limit can be taken directly provided that the reduced Compton wavelength $\lambda = \hbar/(mc)$ is held fixed, but this yields a Yukawa equation (Laplace, if $1/\lambda = 0$), which is nondynamical. However, if Φ is complex, we can set

$$\Phi = e^{-\frac{i}{\hbar}(mc^2 - E_0)t}\Psi, \quad (2)$$

where E_0 is constant and Ψ is a new complex scalar field. The KG equation becomes

$$-\frac{1}{2mc^2}\left(i\hbar\frac{d}{dt} - E_0\right)^2\Psi - i\hbar\dot{\Psi} - \frac{\hbar^2}{2m}\nabla^2\Psi + E_0\Psi = 0, \quad (3)$$

and the $c \rightarrow \infty$ limit yields the Schrödinger equation

$$i\hbar\dot{\Psi} = H\Psi, \quad H = -\frac{\hbar^2}{2m}\nabla^2 + E_0. \quad (4)$$

Clearly, this procedure is not applicable for a real scalar field, and there is a group theoretical reason for this difficulty. The Bargmann symmetry group of the Schrödinger equation has one more generator than the Lorentz symmetry group of the KG equation, a central charge proportional to the mass m . This implies that the wave function provides only a projective representation of the Galilei group, so it must be complex, and hence the initial KG field must also be complex. The KG equation then has an additional U(1) phase invariance, so there is no longer a mismatch in the dimension of the relativistic and nonrelativistic symmetry groups.

A new nonrelativistic limit.—The same difficulty applies to real tensor fields, such as the symmetric traceless tensor field $f_{\mu\nu}$ of the spin-2 FP equations, traceless in the sense that $\eta^{\mu\nu}f_{\mu\nu} = 0$, where $\eta^{\mu\nu}$ is the inverse of the background Minkowski metric tensor. The FP equations comprise second-order dynamical equations and first-order subsidiary conditions:

$$[\square - (mc)^2]f_{\mu\nu} = 0, \quad \eta^{\mu\nu}\partial_\mu f_{\nu\rho} = 0, \quad (5)$$

where $\square \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$. Here we set $\hbar = 1$, in which case mc has dimensions of inverse length. Although the standard path to a nonrelativistic limit of these equations requires $f_{\mu\nu}$ to be complex, another nonrelativistic limit is possible for a Minkowski background of $2 + 1$ dimensions. In this 3D case, we have $\mu, \nu = 0, 1, 2$, the Minkowski metric matrix is $\text{diag}(-c^2, 1, 1)$, and $f_{\mu\nu}$ has five independent components parametrizing a scalar, vector, and traceless symmetric tensor of the SO(2) rotation group. The scalar is the real variable

$$f_{00} \equiv c^2(f_{11} + f_{22}), \quad (6)$$

while the vector and traceless symmetric tensor are, respectively, the complex variables

$$f[1] = f_{01} + if_{02}, \quad f[2] = \frac{1}{2}(f_{11} - f_{22}) + if_{12}. \quad (7)$$

In terms of these variables, the subsidiary conditions are

$$\dot{f}_{00} = c^2\Re[\bar{\partial}f[1]], \quad \dot{f}[1] = c^2\bar{\partial}f[2] + \frac{1}{2}\partial f_{00}. \quad (8)$$

As both $f[1]$ and $f[2]$ are complex, we may set

$$f[n] = e^{-i(mc^2 - E_0)t}\Psi[n], \quad n = 1, 2, \quad (9)$$

for new complex variables $\Psi[n]$. The dynamical equations for $f[n]$ are then solved to leading order as $c \rightarrow \infty$ if $\Psi[n]$ and $\dot{\Psi}[n]$ remain finite in this limit. Given this, the subsidiary conditions imply that

$$f_{00} = -m^{-1}\Im(\bar{\partial}f[1]) + \mathcal{O}(1/c^2), \\ f[1] = im^{-1}\bar{\partial}f[2] + \mathcal{O}(1/c^2). \quad (10)$$

Only $f[2]$ is independent, and its dynamical equation is

$$\frac{1}{c^2}\dot{f}[2] + [(mc)^2 - \nabla^2]f[2] = 0. \quad (11)$$

In terms of $\Psi[2]$, this equation takes the form (3) and its $c \rightarrow \infty$ limit is

$$i\dot{\Psi}[2] = H\Psi[2], \quad H = -\frac{1}{2m}\nabla^2 + E_0. \quad (12)$$

This Schrödinger equation is invariant under a symmetry group with one more generator than the Lorentz invariance group from which we started because the orbital angular momentum and the spin angular momentum are separately conserved in the $c \rightarrow \infty$ limit, with a spin rotation becoming a phase rotation by double the angle. In fact, this Schrödinger equation is identical to the Schrödinger equation for the spin-2 GMP mode of fractional quantum Hall states, as deduced in [11] from a nonrelativistic bimetric theory.

Parity and time reversal.—As mentioned earlier, the FP equations for a complex tensor field allow a standard nonrelativistic limit. This leads to a parity-preserving pair of Schrödinger equations

$$i\dot{\Psi}[\pm 2] = H\Psi[\pm 2], \quad \Psi[\pm 2] = \varphi_{11} \pm i\varphi_{12}, \quad (13)$$

where φ_{11} and φ_{12} are the independent complex components of a complex symmetric traceless two-space tensor φ_{ij} . The wave functions $\Psi[\pm 2]$ are spin-2 helicity

eigenstates; by helicity, we mean the spin angular momentum, while spin is its absolute value. The point of this discussion is that parity reverses the sign of helicity and hence exchanges $\Psi[2]$ with $\Psi[-2]$. This can be seen from the following equivalence

$$\varphi_{ij} = \pm i \epsilon_{ik} \varphi_{kj} \Leftrightarrow \Psi[\mp 2] = 0. \quad (14)$$

If we choose to impose this constraint with the upper sign, then we are left with the single Schrödinger equation of (12). Furthermore, the condition $\Psi[-2] = 0$ identifies a spin rotation of angle θ with a shift of the phase of $\Psi[2]$ by angle 2θ , exactly as required.

As the real-field FP equations (5) are parity invariant, it follows that our new nonrelativistic limit of these equations must break parity. To see why, we observe that parity for the FP equations takes $f[2] \rightarrow \bar{f}[2]$, but one can see from (9) that the corresponding transformation of $\Psi[2]$ is not defined in the $c \rightarrow \infty$ limit. In contrast, time reversal, which takes $t \rightarrow -t$ and $f[2] \rightarrow \bar{f}[2]$ (since its action is antilinear) is defined in the $c \rightarrow \infty$ limit, so (12) must be, and is, time-reversal invariant.

The fact that (12) breaks parity suggests that a better starting point for this Letter might have been the parity violating equations of topologically massive gravity (TMG) [12], which has the square-root FP equations [13] as its linearized limit. Moreover, the self-duality condition (14) (for one choice of the sign) emerges naturally from the nonrelativistic limit of the complexified square-root FP equations, as detailed for the spin-1 case in [14]. However, it is not clear how the required complexification is to be implemented for the nonlinear TMG theory. This is not a problem for NMG because complexification is not required.

Generalized null reduction.—We now turn to a different derivation of the Schrödinger equation (12). It is well known that null reduction of a Lorentz invariant theory in a 5D Minkowski spacetime yields a Galilean invariant theory in 1+3 dimensions [15–17]. We seek some variant procedure that will take the linearized 4D Einstein field equations to the Schrödinger equation (12). A similar issue was addressed in [18] at the level of particle mechanics: the Hamilton-Jacobi equation for a nonrelativistic particle in d space dimensions was provided with an Eisenhart lift to $d+1$ dimensions. Here we propose a quantum version in which the planar Schrödinger equation (12) is lifted to the linearized 4D Einstein equations; in reverse, this becomes a generalized null reduction inspired by Scherk-Schwarz dimensional reduction [19]. In principle, the idea applies in any dimension, but it is only for a planar Schrödinger equation that one can lift to the real-field linearized Einstein equations.

Linearization of the 4D vacuum Einstein equations about a Minkowski vacuum with coordinates $\{x^m; m = 0, 1, 2, 3\}$ and Minkowski metric η_{mn} , yields the following equations for the metric perturbation tensor

$$\square h_{mn} - 2\partial_{(m} h_{n)} + \partial_m \partial_n h = 0, \quad (15)$$

where $h_m \equiv \eta^{pq} \partial_p h_{qm}$ and $h \equiv \eta^{mn} h_{mn}$. We shall choose light-cone coordinates for which $x^m = \{x^+, x^-, x^i\}$, where $i = 1, 2$ and $x^\pm = (x^3 \pm x^0)/\sqrt{2}$, and units for which $c = 1$, but we no longer set $\hbar = 1$.

The standard null reduction is achieved by requiring $\partial_- h_{mn} = 0$. Instead, we proceed on the assumption that ∂_- is invertible, in which case we may impose the light-cone gauge condition $h_{m-} = 0$, for which

$$\begin{aligned} h &= h_{ii}, & h_+ &= \partial_- h_{++} + \partial_i h_{i+}, \\ h_- &= 0, & h_i &= \partial_- h_{i+} + \partial_j h_{ij}. \end{aligned} \quad (16)$$

The equation for h_{m-} reduces to $\partial_- (h_m - \partial_m h) = 0$, which implies that $h_m = 0$ and $h = 0$; these equations imply that

$$\partial_- h_{++} = -\partial_i h_{i+}, \quad \partial_- h_{i+} = -\partial_j h_{ij}, \quad (17)$$

and also that the linearized Einstein equations reduce to $\square h_{mn} = 0$. As ∂_- is assumed invertible, we may solve for the auxiliary variables h_{i+} and h_{++} . This leaves only the traceless part of h_{ij} , which satisfies the 4D wave equation, implying the propagation of transverse waves with two independent polarizations. So far, this is standard light-cone gauge fixing.

Next, we define

$$\Psi[1] = h_{1+} + i h_{2+}, \quad \Psi[2] = h_{11} + i h_{12}. \quad (18)$$

The auxiliary variable equations (17) are now

$$\partial_- h_{++} = -\Re(\bar{\partial} \Psi[1]), \quad \partial_- \Psi[1] = -\bar{\partial} \Psi[2], \quad (19)$$

and the wave equation for the traceless transverse metric perturbation is

$$2\partial_- \partial_+ \Psi[2] = -\nabla^2 \Psi[2]. \quad (20)$$

We now propose to effect a new null reduction by setting

$$\partial_- \Psi[n] = i(m/\hbar) \Psi[n], \quad n = 1, 2, \quad (21)$$

for positive mass m ; the factor of \hbar is needed here on dimensional grounds. Equations (19) now imply that

$$\begin{aligned} h_{++} &= -(\hbar/m) \Im(\bar{\partial} \Psi[1]) + \text{const}, \\ \Psi[1] &= i(\hbar/m) \bar{\partial} \Psi[2]. \end{aligned} \quad (22)$$

These equations are analogous to the subsidiary equations for the 3D spin-2 FP equations in the form of (10). As in that case, only $\Psi[2]$ is independent, and it satisfies

$$-\frac{\hbar^2}{2m}\nabla^2\Psi[2] = i\hbar\dot{\Psi}[2], \quad (23)$$

where $\dot{\Psi} \equiv \partial_+\Psi$. This is the Schrödinger equation (12), with $E_0 = 0$ but we address this below.

One may again ask how it is that the parity invariance of our starting point is not reflected in the end result, and in this case, the answer is that parity is broken by the choice of sign for the mass m appearing in (21). If we had supposed m to be negative, then we would have had to take the complex conjugate of (23) to arrive at a standard Schrödinger equation for $\bar{\Psi}[2]$, but it follows from the definition of $\Psi[2]$ in (18) that $\bar{\Psi}[2] = \Psi[-2]$.

We have now provided two distinct gravity interpretations of the planar Schrödinger equation that have appeared in the context of the spin-2 GMP mode of fractional quantum Hall states. Our interpretation of it as a non-relativistic limit of the 3D FP equations is closest to geometrical proposals in the condensed-matter literature, but our derivation from the 4D linearized Einstein equations provides a more direct link to gravity. In both cases, the enabling feature is the fact that the relevant subgroup of the Lorentz group is $U(1)$; this is the rotation group in 3D and the transverse rotation group in 4D, and Wigner’s “little group” in both cases, because in 3D the spin-2 particle is massive, whereas in 4D it is massless.

Gravity and the Schrödinger potential.—So far, we have discussed the planar Schrödinger equation only for a free spin-2 particle or spin-2 GMP mode in the condensed matter context. The gravity origin of this equation becomes useful when we consider how these particles might interact. In this relativistic context, each particle will produce a gravitational field that is felt by all the others, and we can approximate the effect on any individual particle by some collective background spacetime metric; each particle then moves freely in this background. We may anticipate that this mean-field type of approximation will result in some potential for the Schrödinger Hamiltonian.

To explore this idea in the context of generalized null reduction, we must start from some solution of the full 4D Einstein field equations: $G_{mn} = 8\pi G_N T_{mn}$, where G_{mn} is the Einstein tensor, G_N is Newton’s constant, and T_{mn} is some specified source tensor (we again set $c = 1$). Given a four-metric that solves these equations, we linearize about it to find the following equations for the metric perturbation tensor

$$0 = D^2 h_{mn} - 2D_{(m} h_{n)} + D_m \partial_n h + 2[R_{p(m} h_{n)}^p + R_{pmnq} h^{pq}], \quad (24)$$

where D is the covariant derivative with respect to the affine connection for which $R^p{}_{mnq}$ is the Riemann tensor and R_{mn} is the Ricci tensor, and

$$h_m = g^{pq} D_p h_{qm}, \quad h = g^{mn} h_{mn}. \quad (25)$$

We choose as our background the particular Brinkmann-wave metric

$$ds^2 = 2dx^+ dx^- + 2v(x^+, \mathbf{x})(dx^+)^2 + d\mathbf{x} \cdot d\mathbf{x}, \quad (26)$$

which also played a role in the Eisenhart lift of [18] and earlier in [20].

The function v is independent of x^- , which ensures that ∂_- is a null Killing vector field; in particular, constant v yields Minkowski spacetime. The only nonzero components of the affine connection, up to symmetry, are

$$\Gamma_{++}^- = \partial_+ v, \quad \Gamma_{++}^i = -\partial_i v, \quad \Gamma_{i+}^- = \partial_i v, \quad (27)$$

and the only nonzero components of the curvature and Ricci tensors, up to symmetries, are

$$R_{+ij+} = \partial_i \partial_j v, \quad R_{++} = \nabla^2 v, \quad (28)$$

where ∇^2 is the Laplacian on the transverse two space. The background Einstein equations are satisfied if

$$\nabla^2 v = 8\pi G_N T_{++}, \quad (29)$$

where T_{++} must be the only nonzero component of T_{mn} .

As before, we impose the gauge condition $h_{m-} = 0$. The Ricci tensor term in (24) is then zero. The function v does not enter into the expressions for h_m and h in the light-cone gauge, and neither does it enter into the dynamical equation for h_{m-} , so we still have $h_m = h = 0$, and the resulting equations (17), while the dynamical equations reduce to

$$D^2 h_{mn} + 2R_{mpqn} h^{pq} = 0. \quad (30)$$

We need consider only the equation for h_{ij} , which is

$$2\partial_- \partial_+ h_{ij} - 2v \partial_-^2 h_{ij} + \nabla^2 h_{ij} = 0. \quad (31)$$

Only the traceless part of h_{ij} is nonzero, and we can trade this for $\Psi[2]$ as before. Imposing the generalized null-reduction condition (21), we again recover Eqs. (22) determining the auxiliary fields in terms of $\Psi[2]$, while the equation for $\Psi[2]$ again becomes the spin-2 planar Schrödinger equation, but now with Hamiltonian

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V(t, \mathbf{x}), \quad V = mv, \quad (32)$$

where $t = x^+$. One solution of (29) for zero source yields the linear potential $V = m\mathbf{g} \cdot \mathbf{x}$, which is naturally interpreted as the result of a constant acceleration \mathbf{g} .

The simplest nonzero source is $T_{++} = \rho$, for constant ρ . In this case, the general rotationally invariant solution of (29) for positive v is

$$v = \frac{1}{2}\omega^2|\mathbf{x} - \mathbf{x}_0|^2, \quad \omega^2 = 8\pi G_N \rho. \quad (33)$$

This yields the Hamiltonian for a planar harmonic oscillator of angular frequency ω , which confines the particle to a region centered on the arbitrary point with coordinates \mathbf{x}_0 . The natural interpretation is that of a constant uniform distribution of spin-2 particles with each particle occupying an area $\hbar/(m\omega)$.

An obvious question is whether this result can also be found from the nonrelativistic limit of some interacting extension of the 3D spin-2 FP theory, such as NMG. In this context, the potential has an interpretation within Newton-Cartan geometry as the time component of the gauge-potential one form associated with the central charge of the Bargmann algebra [21,22]; a gauge transformation preserving the form of this potential shifts v by a function of t , which corresponds to the freedom to redefine the wave function by a t -dependent phase factor. However, it is not clear to us at present how this modification can be implemented in the context of the new nonrelativistic limit described here that avoids complexification of the FP field.

Finally, we should mention that a study by Vasiliev [23] of relativistic conformal field theories in their “unfolded” formulation led to a holographic-dual Schrödinger equation in one lower dimension, and a proposed twistor transform interpretation that was conjectured to be related to what we have called, following [18,20], the Eisenhart lift.

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