Inner Nonlinear Waves and Inelastic Light Scattering of Fractional Quantum Hall States as Evidence of the Gravitational Anomaly

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We develop the quantum hydrodynamics of inner waves in the bulk of fractional quantum Hall states. We show that the inelastic light scattering by inner waves is a sole effect of the gravitational anomaly. We obtain the formula for the oscillator strength or mean energy of optical absorption expressed solely in terms of an independently measurable static structure factor. The formula does not explicitly depend on a model interaction potential.

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Introduction.—Excitations in the bulk of a fractional quantum Hall (FQH) state are neutral modes of density modulations. These modes are generally gapped. Evidence of collective modes was seen in inelastic light scattering [1,2]. The numerically obtained spectrum of small systems [3,4] also shows a dispersive branch of a collective excitation.

The experimental accessibility of the dispersion of neutral modes of FQH states calls for a better understanding of inner waves. There is a renewed interest in the subject. Some recent papers are in Ref. [5].

Here we show that inelastic light scattering by inner FQH waves is a sole effect of the gravitational anomaly. This observation gives a geometric interpretation to inner waves and also a new formula for the "oscillation strength" of optical absorption Δ_k.

The gravitational anomaly only recently entered the quantum Hall effect (QHE) literature (e.g., Ref. [6]). It is an elusive phenomenon which appeared as a higher-order gradient correction to bulk transport coefficients [7]. What would be the clean, experimentally accessible bulk effects of the gravitational anomaly? We argue that the gravitational anomaly governs one of the major observables in FQH, the inelastic light scattering.

A natural approach to studying inner waves is hydrodynamics. It goes back to the seminal paper [9] by Girvin, MacDonald, and Platzman (GMP). Our analysis is based on more recent development of FQH hydrodynamics [10] (see also [11]). As the GMP theory, the recent hydrodynamics approach has roots in a similarity between FQH states and a superfluid, but with the essential addition that the superfluid is rotating and incompressible.

We briefly describe the central point of this Letter. The correspondence [10] between the FQHE and a superfluid identifies electrons with quantized vortices in a fast rotating incompressible superfluid. Such hydrodynamics can be reformulated as the Helmholtz law (see, e.g., [12]): Vortices of an incompressible flow are frozen (or passively dragged by) the flow. Since vortices represent electrons, they could be probed by light. Then, the Helmholtz law forbids inelastic light scattering. Being perturbed by light, vortices instantaneously change the flow and remain frozen into a new flow. They cannot accelerate against the flow.

Our main observation is that the quantization subtly corrects the Helmholtz law through the gravitational anomaly. The inelastic light scattering is the effect of this correction.

The gravitational anomaly comes to the stage to prevent a quantization scheme from violating diffeomorphism invariance, the relabeling symmetry of the fluid. It is quite remarkable that optical probes directly test this fundamental symmetry.

The hydrodynamic description of inner FQHE waves faces a long-standing problem of the quantizing of incompressible hydrodynamics, specifically the flows with an extensive vorticity, the chiral flows. Accounting for the gravitational anomaly described below represents perhaps the first consistent quantization of incompressible flows, whose applications go beyond the QHE.

Before we proceed, an important comment about the spectrum of incompressible waves is in order. The GMP theory [9] adopted a variational approach initially developed by Feynman for the superfluid helium [13]. The GMP approach assumes that a certain two-body Hamiltonian $H = \sum_q V_q \rho_q \rho_{-q}$, where ρ_q is the electronic density mode, indeed delivers a FQH state. Then it assumes that excitations include a single-mode density modulation $|k\rangle = \rho_k|0\rangle$ and interprets the diagonal matrix element of the Hamiltonian $\Delta_k = (\langle k|H|k\rangle / \langle k|k\rangle)$ as a variational approximation to the excitation spectrum. The net result is expressed in terms of a model potential V_q.

Such an approach is justified for compressible fluids, like helium, where atomic density modulation is a linear wave. In this case, a single-mode $|k\rangle = \rho_k|0\rangle$ is a long-lived state. Contrary to GMP's major assumption, a single-mode state *does not* approximate a long-lived excitation of incompressible fluids, such as of the FQHE. A reason for it is that incompressible waves are essentially nonlinear. A single-mode state decays into multiple modes and does not have a spectrum, and Δ_k has no direct relation to true excitations, as it seems commonly accepted in the literature.

Still, we argue that Δ_k could be measured in optical absorption and give a new formula for Δ_k in terms of the structure factor. It refines the GMP formula which expresses Δ_k in terms of model potential V_q.

Correspondence between FQH states and fast rotating superfluid.—The analogy between Laughlin's states and a superfluid was suggested in Refs. [9,11] and developed to a correspondence in Ref. [10]. In short, a drift of vortices in a fast rotating superfluid and a motion of electrons in the FQH regime are governed by the same equations.

Fast rotating superfluid is a dense media of same sense vortices with a quantized circulation, which we denote by $2\pi\Gamma$. The total vorticity of the fluid is compensated by a solid rotation with a frequency Ω, such that the mean density of vortices is $\rho_0 = \Omega/(\pi\Gamma)$. We assume that the vortices are in a liquid phase (do not crystallize). The frequency of rotation Ω corresponds to the Larmor frequency $\Omega = eB/2m_*$ with an effective mass m_*. The "mass" is the only phenomenological parameter of the theory determined by the spectral gap. Its energy scale is the Coulomb interaction $\hbar\Omega \sim e^2/\ell$, where $\ell = \sqrt{\hbar/eB}$ is the magnetic length. Then vortices correspond to electrons if the vortex circulation in units of $\hbar/m_*$ is the inverse of the filling fraction and the gap in the spectrum is of the order of $\hbar\Omega$:

$$\Gamma = \left(\frac{\hbar}{m_*}\right)\nu^{-1}, \qquad \Omega = \frac{eB}{2m_*}. \tag{1}$$

This correspondence differs from that of GMP [9]. The authors of Ref. [9] referred to the work of Feynman [13], who considered atomic density modes of a compressible superfluid at rest. Rather, we discuss the modes of vorticity of a rotating incompressible superfluid [14].

We will measure the distance in units of magnetic length and the energy (the bulk gap) in units of the $2\hbar\Omega$, setting $\ell = \hbar = m_* = 1$. In these units, the mean density $\rho_0 = 1/(2\pi\Gamma) = \nu/2\pi$.

Helmholtz law.—The hydrodynamics of a 2D incompressible flow can be cast in the Helmholtz form: The material derivative of vorticity vanishes. If $\boldsymbol{u} = (u_x, u_y)$ is the velocity of a flow, $\omega = \boldsymbol{\nabla}\times\boldsymbol{u}$ is the vorticity, $D_t = \partial_t + \boldsymbol{u}\cdot\boldsymbol{\nabla}$ is the material derivative, and the fluid is incompressible $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$, then the Euler equation in the Helmholtz form reads

$$D_t\omega = 0. \tag{2}$$

In the context of FQH, vorticity is identified with the electronic density. In a rotating frame with no net vorticity, the correspondence reads

$$\rho(\boldsymbol{r}) = \rho_0 + \frac{1}{2\pi\Gamma}\omega(\boldsymbol{r}). \tag{3}$$

The velocity of the flow $\boldsymbol{u}$ does not have a measurable analog in the FQHE. It could be thought as a transversal part of the fictitious gauge field attaching a flux of magnetic field to electrons.

It is quite remarkable that essential features of Laughlin's states are encapsulated in the quantum Helmholtz equation. We will see some of it now.

The Helmholtz law reflects a geometric meaning of hydrodynamics: Incompressible flows are generated by a successive action of volume-preserving diffeomorphisms. In the QHE, this concept has been suggested in Ref. [15]. Therefore, FQH inner waves and the equivalent problem of a quantum hydrodynamics both are seen as a problem of the quantization of the group of volume-preserving diffeomorphisms. This group is generated by density mode operators $\bar\rho_k = \int e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\bar\rho(\boldsymbol{r})d^2\boldsymbol{r}$, with the algebra

$$[\bar\rho_k, \bar\rho_{k'}] = ie_{kk'}\bar\rho_{k+k'}, \tag{4}$$

with the structure constants $e_{kk'} = \boldsymbol{k}\times\boldsymbol{k}'$. On the torus, they are $e_{kk'} = 2e^{1/2(\boldsymbol{k}\cdot\boldsymbol{k}')}\sin[(1/2)\boldsymbol{k}\times\boldsymbol{k}']$. Here we used a bar to emphasize quantization as in Ref. [9]. The classical limit of (4) is the Poisson brackets of hydrodynamics [16].

Nonlinear waves.—A few important properties already follow from (2). A well-known fact is that the 2D incompressible hydrodynamics does not assume linear waves. In the language of the quantum theory, this means that single density modes are not long-lived states.

However, the Euler equation can be linearized about an inhomogeneous background. Example are Tkachenko linear modes of a vortex crystal [17]. If we impose a periodic density modulation $|k_0\rangle$, then on top of it there are linear waves $\bar\rho_{q-k_0}|k_0\rangle = \bar\rho_{q-k_0}\bar\rho_{k_0}|0\rangle$. This suggests that, in contrast to a single mode, the two-mode states do have a spectrum. This assertion agrees with the interpretation of the inelastic light scattering experiments of Pinczuk *et al.* [1] as a Raman type two-modes processes by Platzman and He [4]. We address the spectrum of inner waves elsewhere.

Another consequence mentioned already is that the Helmholtz law prohibits the absorption of light. We show how this problem is resolved by the quantization.

Quantization of Euler equation.—Quantization of the Euler equation meets essential difficulties. The advection term $\boldsymbol{u}\cdot\nabla\omega = \nabla(\boldsymbol{u}\cdot\omega)$ where two operators sit at the same point requires a regularization. The problem in a general setting has a long history of failures and is commonly considered nearly impossible. A scheme of regularization where points are split $\boldsymbol{u}[\boldsymbol{r}+(\epsilon/2)]\omega[\boldsymbol{r}-(\epsilon/2)]$ leads to inconsistencies. The difficulty is that the point-splitting distance itself depends on the flow $\epsilon[\boldsymbol{u}]$. Hence, a regularization scheme is specific to the flow and cannot be practical to all varieties of flows at once. However, if the flow consists of a dense media of vortices, the chiral flow, quantization could be achieved. In this case, a variable short-distance cutoff is the distance between vortices $\epsilon \sim 1/\sqrt{\rho}$.

We will use the complex notations. We denote the complex velocity by $u_z = u_x - iu_y$ and use the stream function ψ and the traceless part of the fluid momentum flux tensor $\Pi_{ij} = u_i u_j - (1/2)\delta_{ij}\boldsymbol{u}^2$. In complex coordinates, $u_z = 2i\partial_z\psi$ and $\Pi_{zz} = u_z u_z$. We will write the advection term as

$$\boldsymbol{u}\cdot\boldsymbol{\nabla}\omega = i[\partial_z^2\Pi_{\bar z\bar z} - \partial_{\bar z}^2\Pi_{zz}]. \tag{5}$$

Hence, we have to give a quantum meaning to u_z^2. For that, we recall a notion of the projected density operator.

Normal ordering and quantization.—States on the lowest Landau level (LLL), and also flows of rotating superfluid, are realized as Bargmann space [9,18]. It is a space of holomorphic functions with the inner product $\langle g|f\rangle = \int e^{-(1/2)|z|^2} g^*(\bar z) f(z) dz d\bar z$. The density operators acting in the Bargmann space obeying the algebra (4) are realized by the normally ordered operator $\bar\rho_k = \sum_i e^{-(i/2)kz_i^\dagger} e^{-(i/2)\bar k z_i}$, where k is a complex wave vector and $z_i^\dagger = 2\partial_{z_i}$. GMP called it a projected (onto LLL) density operator. It is organized such that a state $|k\rangle = \bar\rho_k|0\rangle$ is holomorphic and, hence, belongs to LLL. It is also chiral $\bar\rho_k^\dagger = \bar\rho_{-k}$. Similarly, the two-mode operator that entered the momentum flux tensor on the Bargmann space is represented by a normal ordered string:

$$\overline{\rho_k\rho_{k'}} = \sum_{i,j} e^{-(i/2)kz_i^\dagger} e^{-(i/2)k'z_j^\dagger} e^{-(i/2)k^* z_i} e^{-(i/2)k'^* z_j}. \tag{6}$$

The projected density modes generate coherent states of LLL and also states of rotating superfluid if z_i is a coordinate of a vortex.

We denote the Wick contraction $\overset{\frown}{AB} = \overline{AB} - \bar A\,\bar B$ and compute $\overset{\frown}{u_z u_z}$. The contraction of two density modes follows from (6):

$$\overset{\frown}{\rho_k\rho_{k'}} = \bar\rho_{k+k'}(1 - e^{\frac{1}{2}\boldsymbol{k}\cdot\boldsymbol{k}'}). \tag{7}$$

The next step is to express the momentum flux tensor on the Bargmann space through the generators $\bar\rho_k$. We get insight by computing it for the ground state where the density is uniform $\rho_k = N\delta_{k0}$ and there is no flow.

Equivalently, the contraction of two stream functions is

$$\overset{\frown}{\psi(\boldsymbol{r})\psi}(\boldsymbol{r}') = \frac{2\pi}{\nu}\int e^{i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}\left(\frac{1-e^{-(1/2)k^2}}{k^4}\right)\frac{d^2k}{(2\pi)^2}. \tag{8}$$

Now we can compute $\overset{\frown}{\mathrm{u}_z(\boldsymbol{r})\mathrm{u}_z}(\boldsymbol{r}') = -4\partial_z\partial_{z'}\overset{\frown}{\psi(\boldsymbol{r})\psi}(\boldsymbol{r}')$. In the hydrodynamic limit ($|r-r'| \gg \ell$), $\overset{\frown}{\mathrm{u}_z(\boldsymbol{r})\mathrm{u}_z}(\boldsymbol{r}') \sim (z-z')^{-2}$. As $r \to r'$, the net result is zero due to the rotational symmetry. The effect of short-distance regularization does not show up.

Gravitational anomaly in hydrodynamics.—Now we extend these calculations when u_z^2 is sandwiched between two flow states with a nonuniform density. In this case, the cutoff as $\epsilon[\mathbf{u}]$ is nonuniform. The result follows from the geometric interpretation of the fluid flow. In this picture, the distance between particles (vortices) is interpreted as a metric $ds^2 = \rho|dz|^2$ of an auxiliary evolving Riemann surface. The scalar curvature of this surface is

$$\mathcal{R} = -4\rho^{-1}\partial_z\partial_{\bar z}\log\rho. \tag{9}$$

The distance between particles is invariant under a change of coordinates or by relabeling particles. In hydrodynamics, this fictitious symmetry is typically applied to fluid atoms. In our approach, it is a relabeling symmetry of vortices. We want to keep it in quantization.

To proceed, we notice that in the hydrodynamic limit the contraction of stream functions (8) is the Green function of the Laplace operator

$$\overset{\frown}{\psi(\boldsymbol{r})\psi}(\boldsymbol{r}') = \frac{\pi}{\nu}G(\boldsymbol{r},\boldsymbol{r}'). \tag{10}$$

It is natural to assume that in a flow state the contraction is the Green function of the Laplace-Beltrami operator in the metric $\rho|dz|^2$. Then the problem is reduced to a covariant regularization of the Green function as $r \to r'$. Such a regularization identifies the short-distance cutoff with the geodesic distance $d(\boldsymbol{r},\boldsymbol{r}')$. With this prescription, we define the Wick contraction of the momentum flux tensor $\Pi_{zz} = u_z^2$ as a limit:

$$\overset{\frown}{u_z u_z} = \frac{4\pi}{\nu}\lim_{r\to r'}\partial_z\partial_{z'}\left[G(\boldsymbol{r},\boldsymbol{r}') + \frac{1}{2\pi}\log d(\boldsymbol{r},\boldsymbol{r}')\right]. \tag{11}$$

The result of this limit is known: It is the Schwarzian of the metric (Supplemented Material [19])

$$\overset{\frown}{u_z u_z} = \frac{1}{6\nu}\left(\partial_z^2\log\rho - \frac{1}{2}(\partial_z\log\rho)^2\right). \tag{12}$$

Then the contraction of the advection term (5) is expressed through the curvature (9)

$$\overset{\frown}{\boldsymbol{u}\cdot\boldsymbol{\nabla}\omega} = \frac{1}{96\pi}\boldsymbol{\nabla}\mathcal{R}\times\boldsymbol{\nabla}\omega. \tag{13}$$

This is the main result of the quantization [20]. We can now treat the hydrodynamics as a field theory, with a constant cutoff, independent of the flow. With the help of (13), we obtain

$$D_t\bar{\rho} = \frac{1}{96\pi}\nabla\mathcal{R}\times\nabla\bar{\rho}. \tag{14}$$

If waves are small and long, $\mathcal{R}\approx -\rho_0^{-2}\Delta\rho$, the correction to the Helmholtz law could be treated in the harmonic and long-wave approximation:

$$D_t\bar{\rho}_k = \frac{\pi}{24\nu^2}\sum_q q^2(\boldsymbol{k}\times\boldsymbol{q})\bar{\rho}_q\bar{\rho}_{k-q}. \tag{15}$$

Deviation from Helmholtz law.—The implication of quantum corrections is that the Helmholtz law held for quantum operators does not hold for their matrix elements: The material derivative for the projected density mode (14) does not vanish. Acceleration of particles against the flow appears as quantum corrections, but, as we will see, it is the only source for the light scattering. The universal departure from the Helmholtz law is our main result.

Hamiltonian.—Now we are in a position to determine the Hamiltonian which together with the brackets (4) yields Eq. (14). Here we present the result, leaving calculations to Supplemental Material [19].

We write the Hamiltonian in terms of mean density of the flow, denoting it also by ρ. Separating classical and quantum contribution $H=\int(\mathcal{H}-\hbar S)\rho_0 d^2\boldsymbol{r}$ and restoring units we obtain

$$\mathcal{H} = \frac{m_*}{2}(\boldsymbol{u}^2 + 2\boldsymbol{u}\cdot\boldsymbol{u}_0 - \pi\Gamma^2\rho\log\rho), \tag{16}$$

$$S = -\pi\Gamma\left(\rho\log\rho + \frac{1}{96\pi}(\nabla\log\rho)^2\right). \tag{17}$$

The first two terms in the classical part (16) are the kinetic and centrifugal energies, $\nabla\times\boldsymbol{u}_0 = 2\Omega$. The last term in (16) regularizes the divergency of the kinetic energy at vortex cores. It was known in the theory of superfluid since the 1961 paper of Kemoklidze and Khalatnikov [21]; see, also, recent Ref. [22]. This term is the Casimir invariant, whose Poisson bracket with all local fields vanishes. It does not show in equations of motion (14) but enters the current [Eq. (3) of [19]] as a divergence-free term.

The quantum part (17) also consists of two terms. The first term is a quantum correction to the Kemoklidze-Khalatnikov term. The second term represents the effect of the gravitational anomaly.

Static structure factor.—Now we check that the Helmholtz equation (2) and its consequences (14)–(17) encode the independently known long wave expansion of the static structure factor $s_k = (1/N)\langle 0|\rho_{-k}\rho_k|0\rangle$. This check justifies the hydrodynamic approach.

According to the theory of linear response, the structure factor appears in the harmonic approximation of the Hamiltonian as a rigidity of density modes (see [19]):

$$H \approx \frac{1}{2N}\sum_{q\neq 0} s_q^{-1}\overline{\rho_{-q}\rho_q}. \tag{18}$$

We compute the inverse structure factor by expanding (16) and (17). The result is

$$s_q^{-1} = \frac{2}{q^2} - \left(\frac{1}{2\nu} - 1\right) + [s_q^{-1}]_+, \tag{19}$$

where $[s_q^{-1}]_+$ is the part of the expansion which consists of positive powers of q. The leading term in $[s_q^{-1}]_+$ followed from the last term in (17) is the effect of the gravitational anomaly:

$$[s_q^{-1}]_+ = \frac{q^2}{24\nu} + \mathcal{O}(q^4). \tag{20}$$

Inverting (19), we obtain the first three terms of the small q expansion of the structure factor

$$s_q = \frac{q^2}{2} + \frac{q^4}{8\nu}(1-2\nu) + \frac{q^6}{8\nu^2}\left(\frac{3}{4}-\nu\right)\left(\frac{1}{3}-\nu\right) + . \tag{21}$$

Each of the three terms in (21) is independently known, has a universal meaning, and reflects symmetries of the electronic fluid. The term q^2 corresponds to the kinetic energy $\frac{1}{2}\boldsymbol{u}^2$, q^4 corresponds to the $\rho\log\rho$ term in (16) and (17) and is referred as the "compressibility" sum rule. Finally, the q^6 term represents the gravitational anomaly. It was first obtained in Ref. [23] directly from Laughlin's wave function. In equivalent forms, it appeared in Ref. [24]. There is no reason to think that higher terms, but the first three, are universal.

Using (21), we obtain the projected structure factor

$$\bar{s}_k = \frac{1}{N}\langle 0|\bar{\rho}_{-k}\bar{\rho}_k|0\rangle.$$

From (7), we have $\bar{s}_q = s_q - (1-e^{-(1/2)q^2})$. Hence,

$$\bar{s}_q = (1-\nu)\frac{q^4}{8\nu}\left(1+\frac{1}{6\nu}(3-10\nu)q^2\right) + \cdots. \tag{22}$$

Harmonic approximation.—We can now express the correction to the Helmholtz law in terms of the structure factor. Let us compute $[H,\bar{\rho}_k]$ with the Hamiltonian (18). The first term in the expansion of s_q^{-1} (19) gives the material derivative, and the second does not contribute. The correction to the Helmholtz law is due to the positive part of the expansion (19), whose leading term is the gravitational anomaly (20). We obtain a refine form of the Eq. (15) valid at all k:

$$D_t\bar{\rho}_k = \frac{\pi}{\nu}\sum_q e_{kq}[s_q^{-1}]_+\bar{\rho}_q\bar{\rho}_{k-q}. \tag{23}$$

Optical absorption by nonlinear waves.—Absorption occurs when light accelerates particles against the flow, i.e., due to a departure from the Helmholtz law.

Consider an acoustic wave imposed through the Hall bar as in an experiment [2]. It creates a state $|k\rangle = \bar{\rho}_k|0\rangle$. In solids, the optical absorption measures the differential intensity $S_k(\omega) = (1/N)\langle k|\delta(H - \hbar\omega)|k\rangle$ and the integrated intensity $\bar{s}_k = \hbar \int S_k(\omega)d\omega = (1/N)\langle k|k\rangle$, the projected static structure factor. Another object of interest is the oscillation strength, the first moment of the intensity

$$\bar{f}_k = \int \omega S_k(\omega)d\omega = \frac{1}{N}\langle k|H|k\rangle = \frac{i}{2N}\langle 0|\dot{\bar{\rho}}_k\bar{\rho}_{-k}|0\rangle \quad (24)$$

and the mean energy $\Delta_k = \bar{f}_k/\bar{s}_k = \langle k|H|k\rangle/\langle k|k\rangle$.

In fluids, intensity must be written in a coordinate system moving with the fluid. This means that the time derivative in (24) is the material derivative

$$\bar{f}_k = \frac{1}{2Ni}\langle 0|[D_t\bar{\rho}_k, \bar{\rho}_{-k}]|0\rangle. \quad (25)$$

Hence, only the rhs of (23) enters (25).

Typically, $S_k(\omega)$ features an asymmetric peak supported by the curve $\hbar\omega = \Delta_k$, rudimentarily interpreted as a spectrum of excitations. Such an interpretation will be valid, would $\bar{\rho}_k|0\rangle$ be a long-lived state, as happens in a compressible fluid. As we commented above, in the FQHE, the state $\bar{\rho}_k|0\rangle$ is short-lived.

Interpretation aside, we compute $\bar{f}_k$. Equation (23) reduces (25) to $\langle 0|\bar{\rho}_{-k}\bar{\rho}_q\bar{\rho}_{k-q}|0\rangle$, which we compute with the help of the algebra (4). We express the result in terms of $\tilde{s}_k = (1-\nu)^{-1}e^{k^2/2}\bar{s}_k$ and in units $\hbar^2/(\pi m_*\ell^2)$ and use the structure constants (4) for the torus:

$$\Delta_k = \tilde{s}_k^{-1}\int \sin^2\left(\frac{1}{2}\boldsymbol{k}\times\boldsymbol{q}\right)e^{-(q^2/2)}[s_q^{-1}]_+(\tilde{s}_q - \tilde{s}_{k-q})d^2q.$$

Contrary to (4.15) of Ref. [9], our formula does not explicitly depend on a model interaction. It is expressed only through an independently measured structure factor. We emphasize that beyond terms in (22) the structure factor depends on details of the material and so as the mean energy Δ_k.

Magnetoroton minimum.—Both $\bar{f}_k$ and $\bar{s}_k$ and their ratio Δ_k feature a broad asymmetric peak at $k\ell \sim 1$.

At $k \to 0$, $\tilde{s}_k \sim (k^4/8\nu)$ and $[s_k^{-1}]_+ \sim (k^2/24\nu)$. At $k \to \infty$, $\tilde{s}_k = 1$ and $[s_k^{-1}]_+ = (1/2\nu)$. Hence, the mean energy Δ_k smoothly interpolates between

$$\Delta_{k=0} = 4\nu\int q^2[s_q^{-1}]_+(\nabla_q^2\tilde{s}_q)e^{-(q^2/2)}d^2q \quad (26)$$

and

$$\Delta_{k\to\infty} = \int [s_q^{-1}]_+(\tilde{s}_q + 1)e^{-(q^2/2)}d^2q. \quad (27)$$

Numerically, the evaluation of Δ_k from model Hamiltonians [3,4,9] also shows a minimum. GMP called it the magnetoroton minimum. However, it is unclear whether it has a universal meaning. The minimum relies on features of $\bar{s}_k$ beyond its universal part (19)–(22).

A sequence of minima in optical absorption are reported in Ref. [2] for fractions other than Laughlin's. It is not clear whether they are related to the GMP minimum for Laughlin's states.

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[19] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.120.086601 for the derivations of (i) the Hamiltonian (14) and current, (ii) the relation of the linear response theory, the static structure factor to the hydrodynamics (21), (iii) the short distance expansion of the Green function (11).

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