

Tightening Quantum Speed Limits for Almost All States

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Conventional quantum speed limits perform poorly for mixed quantum states: They are generally not tight and often significantly underestimate the fastest possible evolution speed. To remedy this, for unitary driving, we derive two quantum speed limits that outperform the traditional bounds for almost all quantum states. Moreover, our bounds are significantly simpler to compute as well as experimentally more accessible. Our bounds have a clear geometric interpretation; they arise from the evaluation of the angle between generalized Bloch vectors.

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Quantum speed limits (QSLs) set fundamental bounds on the shortest time required to evolve between two quantum states [1–3]. The earliest derivation of the minimal time of evolution was in 1945 by Mandelstam and Tamm [4] with the aim of operationalizing the famous (but often misunderstood) *time-energy uncertainty relations* [5–9] $\Delta t \geq \hbar/\Delta E$, relating the standard deviation of energy with the time it takes to go from one state to another. QSLs were originally derived for the unitary evolution of pure states [10–12]; since then, they have been generalized to the case of mixed states [13–16], nonunitary evolution [17–19], and multipartite systems [20–23].

Extending their original scope, their significance has evolved from fundamental physics to practical relevance [24,25], defining the limits of the rate of information transfer [26] and processing [27], entropy production [28], precision in quantum metrology [29] and time scales of quantum optimal control [30–33]. For example, in Ref. [34], the authors use QSLs to calculate the maximal rate of information transfer along a spin chain; similarly, Reich *et al.* show that optimization algorithms and QSLs can be used together to achieve quantum control over a large class of physical systems [35]. In Refs. [36–38], QSLs are used to bound the charging power of nondegenerate multipartite systems, which are treated as batteries.

Combining the Mandelstam-Tamm result with the results by Margolus and Levitin, along with elements of quantum state space geometry [39], leads to a *unified QSL* [40]. It bounds the shortest time required to evolve a state ρ to another state σ by means of a unitary operator U_t generated by some time-dependent Hamiltonian H_t :

$$T_{\mathcal{L}}(\rho, \sigma) = \hbar \frac{\mathcal{L}(\rho, \sigma)}{\min\{E, \Delta E\}}, \quad (1)$$

where $\mathcal{L}(\rho, \sigma) = \arccos[\mathcal{F}(\rho, \sigma)]$ is the Bures angle, a measure of the distance between states ρ and σ ; $\mathcal{F}(\rho, \sigma) = \text{tr}[\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}]$ is the Uhlmann *root* fidelity [41,42];

$\rho_t = U_t \rho U_t^\dagger$, $E = (1/T) \int_0^T (\text{tr}[\rho_t H_t] - h_t^{(0)}) dt$ is the average energy, with $h_t^{(0)}$ being the ground-state energy of H_t ; and $\Delta E = (1/T) \int_0^T \sqrt{\text{tr}[\rho_t H_t^2] - \text{tr}[\rho_t H_t]^2} dt$ is the standard deviation [14] ($\hbar = 1$, here and in the following).

For pure states $\rho = |\psi\rangle\langle\psi|$ and $\sigma = |\phi\rangle\langle\phi|$, the Bures angle reduces to the Fubini-Study distance $d(|\psi\rangle, |\phi\rangle) = \arccos |\langle\psi|\phi\rangle|$ [39,43,44]. Under this condition, Eq. (1) is provably tight [40]. In the case of mixed states, on the other hand, the speed limit induced by the Bures metric is in general not tight.

In this Letter, we derive a tighter bound for the speed of unitary evolution. We propose the use of the angle between generalized Bloch vectors [45,46] as a distance for those elements of the state space that can be unitarily connected, and show that it induces an attainable bound for the unitary evolution of mixed qubits. However, this distance does not reduce to the Fubini-Study distance when pure states of dimension $N > 2$ are considered. We thus introduce another distance that reduces to the Fubini-Study distance for pure states, and derive the corresponding speed limit. Careful analysis of both newly introduced QSLs—analytical for qubits and numerical for higher dimensions—shows that they are tighter than the one derived from the Bures angle for the vast majority of states. We conclude with a unified bound for the speed limit of unitary evolution.

Attainability for mixed states.—Let $\rho = \sum_i \lambda_i |r_i\rangle\langle r_i|$ and $\sigma = \sum_i \lambda_i |s_i\rangle\langle s_i|$ be two mixed states with the same spectrum. Let $\rho' = \sum_i \lambda'_i |r_i\rangle\langle r_i|$ and $\sigma' = \sum_i \lambda'_i |s_i\rangle\langle s_i|$ be another pair of mixed states with the same degeneracy structure as ρ and σ , but different eigenvalues λ'_i . Any driving Hamiltonian that maps ρ to σ will map ρ' to σ' in the same amount of time, independent of their spectrum. On the other hand, the Bures angle is a continuous function of the spectrum of the mixed state; i.e., we could have $\mathcal{L}(\rho, \sigma) \approx 1$ while $\mathcal{L}(\rho', \sigma') \approx 0$. Even though the denominator of Eq. (1) may in principle also differ between

these two scenarios due to its state dependence [47], that bound cannot be tight for the case of mixed states. This observation is particularly evident in the case of mixed qubits, as exemplified in Fig. 1.

The poor performance of Eq. (1) stems from the construction of the Bures distance for mixed states, which relies on purifying state ρ_i to some $|\psi_i\rangle$ embedded in a larger Hilbert space $\mathcal{H} \otimes \mathcal{H}_B$, such that $\text{tr}_B[|\psi_i\rangle\langle\psi_i|] = \rho_i$. The distance \mathcal{L} between two states $\rho_1, \rho_2 \in \mathcal{S}(\mathcal{H})$ is defined as the minimal Fubini-Study distance between the pure states $|\psi_1\rangle, |\psi_2\rangle$, where the *minimum* is taken with respect to all possible unitaries that act on the elements of $\mathcal{H} \otimes \mathcal{H}_B$. However, tracing over \mathcal{H}_B , in general, turns unitary dynamics between $|\psi_1\rangle$ and $|\psi_2\rangle$ into nonunitary dynamics between ρ_1 and ρ_2 [50]. Consequently, the Bures metric does not necessarily select geodesics generated by unitary operations, even if ρ_1 and ρ_2 have the same spectrum.

The fact that Eq. (1) constitutes a loose bound for the speed of unitary evolution of mixed states is well known, and there are several proposals to tackle this problem [51–54]. In particular, Ref. [51] takes a geometric approach to obtain an infinite family of speed limits, whose properties, however, strongly depend on the choice of distance. As a trade-off for this freedom, the task of finding a distance that induces a tight bound for the unitary evolution of mixed states remains unsolved. Reference [53] proposes a bound that has a straightforward experimental interpretation, but that is obtained from an orbit-dependent distance, is valid only for nondegenerate states, and does not provide an explicit relation between the purity of the state and the minimal time of evolution.

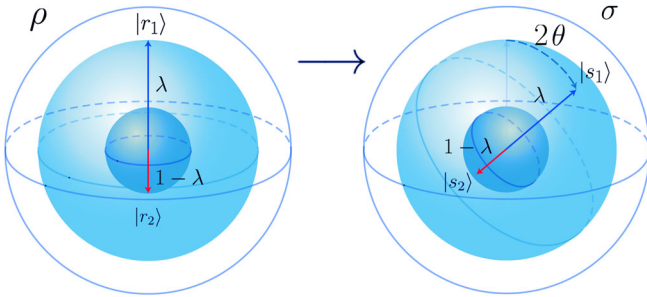


FIG. 1. Let ρ and σ be two mixed qubit states with the same spectrum, $\rho = \lambda|r_1\rangle\langle r_1| + (1-\lambda)|r_2\rangle\langle r_2|$, and $\sigma = \lambda|s_1\rangle\langle s_1| + (1-\lambda)|s_2\rangle\langle s_2|$, $\lambda \in (0, 1)$, $\lambda \neq 1/2$, where $\{|r_1\rangle, |r_2\rangle\}$ and $\{|s_1\rangle, |s_2\rangle\}$ are two orthonormal bases. The problem of unitarily evolving ρ to σ can be mapped to evolving $|r_1\rangle$ to $|s_1\rangle$ (or equivalently, $|r_2\rangle$ to $|s_2\rangle$). Equation (1) is tight for pure states; thus, any Hamiltonian that takes $|r_1\rangle$ to $|s_1\rangle$ will also take ρ to σ in the same time. For any Hamiltonian with bounded standard deviation $\Delta E \leq \mathcal{E}$, this time is bounded from below by θ/\mathcal{E} , where $\theta = d(|r_1\rangle, |s_1\rangle)$ is the distance between $|r_1\rangle$ and $|s_1\rangle$, i.e., half of the angle between the vectors associated with $|r_1\rangle$ and $|s_1\rangle$. However, Eq. (1) for the same constraint on the Hamiltonian suggests that $T_{\mathcal{L}} = \mathcal{L}(\rho, \sigma)/\mathcal{E}$, with $\mathcal{L}(\rho, \sigma) < \theta$ for every choice of $\lambda \neq 0, 1$ [see Eq. (9)], making the QSL unattainable for all mixed states.

We now propose two distance measures for mixed states with the same fixed spectrum that do not suffer from the problems outlined above. The corresponding QSLs outperform the bound in Eq. (1) and are much simpler to compute, and to experimentally measure.

Generalized Bloch angle.—Any mixed state $\rho \in \mathcal{S}(\mathcal{H})$ can be represented as

$$\rho = \frac{1}{N} \left(\mathbb{1} + \sqrt{\frac{N(N-1)}{2}} \mathbf{r} \cdot \mathbf{A} \right), \quad (2)$$

where $N = \dim \mathcal{H}$, and $\mathbf{A} = (A_1, \dots, A_{N^2-1})$ is a set of operators that form a Lie algebra for $\text{SU}(N)$, such that $\text{tr}[A_i A_j] = 2\delta_{ij}$ [46]. The generalized Bloch vector (GBV) \mathbf{r} has to satisfy a set of relations in order to represent a state [46]. We define the subset $\mathcal{S}_{\Lambda}(\mathcal{H}) := \{\rho \in \mathcal{S}(\mathcal{H}) : \text{spec}(\rho) = \Lambda\}$ as the set of states with fixed spectrum Λ that can be unitarily connected. The function

$$\Theta(\rho, \sigma) = \arccos(\hat{\mathbf{r}} \cdot \hat{\mathbf{s}}) \quad (3)$$

is a distance for the elements of $\mathcal{S}_{\Lambda}(\mathcal{H})$ for any fixed spectrum Λ , where $\hat{\mathbf{r}}$ and $\hat{\mathbf{s}}$ are the GBVs associated with the states ρ and σ , respectively, normalized for their length $\|\hat{\mathbf{r}}\|_2 = \|\hat{\mathbf{s}}\|_2$ (see proof of Theorem 1). The angle Θ can be expressed as a function of ρ and σ , independently from the chosen Lie algebra, $\Theta(\rho, \sigma) = \arccos((\text{tr}[\rho\sigma] - 1/N)/(\text{tr}[\rho^2] - 1/N))$, using $\text{tr}[\rho\sigma] = (1 + (N-1)\mathbf{r} \cdot \mathbf{s})/N$. Note that the distance $\Theta(\rho, \sigma)$ does not depend on the basis chosen to represent the states, since the trace is basis independent.

Our first result is a bound on the speed of unitary evolution for the elements of $\mathcal{S}_{\Lambda}(\mathcal{H})$ with fixed spectrum Λ , derived from the distance Θ :

Theorem 1.—The minimal time required to evolve from state ρ to state σ by means of a unitary operation generated by the Hamiltonian H_t is bounded from below by

$$T_{\Theta}(\rho, \sigma) = \frac{\Theta(\rho, \sigma)}{Q_{\Theta}}, \quad \text{where} \quad (4)$$

$$Q_{\Theta} = \frac{1}{T} \int_0^T dt \sqrt{\frac{2\text{tr}[\rho_t^2 H_t^2 - (\rho_t H_t)^2]}{\text{tr}[\rho_t^2] - 1/N^2}}.$$

The proof, similar to that of other QSL results, is given in Sec. I of the Supplemental Material [55].

For pure states, we would like Eq. (4) to reduce to the unified bound (1) obtained from the Fubini-Study metric [43,44]. However, bound (4) satisfies this requirement only for qubits:

Remark 1.—Bound (4) does not reduce to the QSL induced by the Fubini-Study metric for pure states, except for qubits ($N = 2$).

We give the proof in Sec. II of the Supplemental Material [55]. The reason why Θ does not conform with the Fubini-Study distance for pure states of arbitrary dimension is that

the group of rotations on the GBVs does not correspond to the group of unitary operators on states. When going from initial to final state, unitary evolution avoids the forbidden regions of the generalized Bloch sphere, whereas rotations would go straight through these regions, underestimating the distance between the considered states.

In order to derive a speed limit that conforms with the QSL for pure states regardless of the dimension of the system, we introduce the distance

$$\Phi(\rho, \sigma) = \arccos \left(\sqrt{\frac{\text{tr}[\rho\sigma]}{\text{tr}[\rho^2]}} \right) \quad (5)$$

for the elements of $\mathcal{S}_\Lambda(\mathcal{H})$ for any fixed spectrum Λ , which reduces to the Fubini-Study distance for the case of pure states. If states of different purity were considered, neither Θ nor Φ would be distances, since the symmetry and triangle inequality properties would be lost. As with Θ , we derive a bound on the speed of unitary evolution from distance Φ :

Theorem 2.—The minimal time required to evolve from state ρ to state σ by means of a unitary operation generated by the Hamiltonian H_t is bounded from below by

$$T_\Phi(\rho, \sigma) = \frac{\Phi(\rho, \sigma)}{Q_\Phi}, \quad \text{where} \quad (6)$$

$$Q_\Phi = \frac{1}{T} \int_0^T dt \sqrt{\frac{\text{tr}[\rho_t^2 H_t^2 - (\rho_t H_t)^2]}{\text{tr}[\rho_t^2]}}.$$

The proof can be carried out using arguments similar to those for the proof of Theorem 1; see Sec. III of the Supplemental Material [55]. Remarkably, the bound expressed in Eq. (6) reduces to the Mandelstam-Tamm bound for pure states, since Φ reduces to the Fubini-Study distance and Q_Φ reduces to ΔE [56].

In contrast to bound (1), the two QSLs derived here account for both the energetics of the dynamics and the purity of the driven state. The latter is accounted for by the denominators of Q_Θ and Q_Φ , while the term in the numerators, $\sqrt{\text{tr}[\rho_t^2 H_t^2 - (\rho_t H_t)^2]}$, is a lower bound on the instantaneous standard deviation of the Hamiltonian H_t [57].

It is worth highlighting that the bounds derived from Θ and Φ are significantly easier to compute than the one expressed in Eq. (1) for the case of mixed states, since no square root of density operators needs to be calculated, and thus no eigenvalue problem needs to be solved. More specifically, in order to compute the Bures angle, one needs to perform two matrix multiplications and two matrix square roots, whereas only two matrix multiplications are needed to compute Θ or Φ [58]. Accordingly, distances Θ and Φ can be experimentally estimated more efficiently than the Bures angle, which involves the evaluation of the root fidelity between the two considered states, and is harder to obtain than their overlap. The latter can be determined by means of a controlled-swap circuit

[61–63]. Finally, not only are our bounds simpler to compute and measure, they also outperform Eq. (1), as we will show next.

Attainability of new bounds.—We now study the bounds presented in Eqs. (4) and (6), and compare them to that in

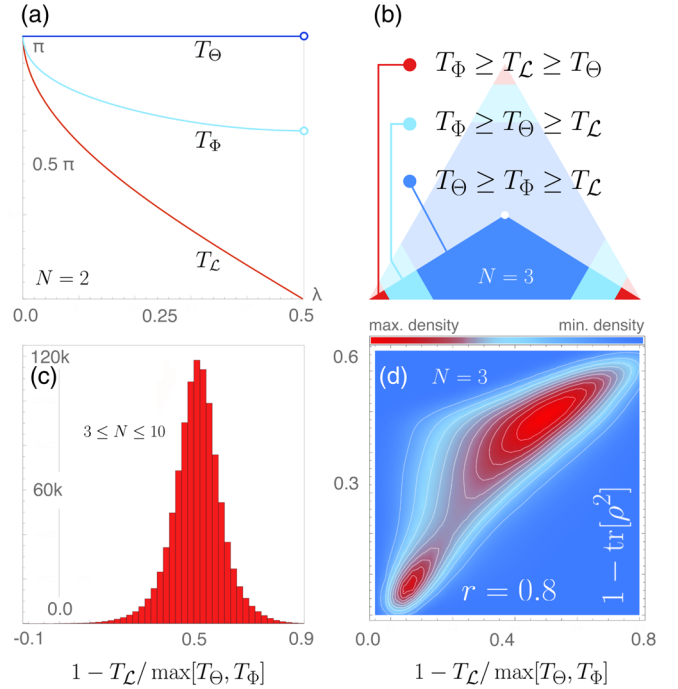


FIG. 2. (a) Bounds $T_\mathcal{L}$ [Eq. (1)], T_Φ [Eq. (6)], and T_Θ [Eq. (4)], as a function of the eigenvalue λ , for two mixed and antipodal qubit states $\rho = \lambda|r_1\rangle\langle r_1| + (1-\lambda)|r_2\rangle\langle r_2|$ and $\sigma = \lambda|r_2\rangle\langle r_2| + (1-\lambda)|r_1\rangle\langle r_1|$. The unitary evolution is generated by the Hamiltonian $H = e^{i\varphi}|r_1\rangle\langle r_2| + \text{H.c.}$ Bounds are symmetric with respect to $\lambda = 1/2$. The same hierarchy holds for nonantipodal mixed qubit states. Bound T_Θ is always attainable. (b) For $N = 3$ (qutrits), the hierarchy between the three bounds expressed with three regions of the polytope defined by the spectrum $\{\lambda_1, \lambda_2, \lambda_3\}$ of states ρ and σ , as indicated in the legend. The corners represent pure states, while the center represents the maximally mixed state. The shape of the regions reflects a specific choice of H , ρ , and σ ; similar features are common to those of any pair of states. For the case of qutrits, $T_\mathcal{L}$ is never larger than $\max[T_\Theta, T_\Phi]$ (see Sec. IV of the Supplemental Material [55]). (c) Evaluation of $1 - T_\mathcal{L} / \max[T_\Theta, T_\Phi]$ as a measure of the tightness of the new bounds, for $3 \leq N \leq 10$, with a sample size of 10^6 . For every run, a different Haar-random state and a different Hamiltonian are generated. $T_\mathcal{L}$ can be larger than $\max[T_\Theta, T_\Phi]$, but only for 0.1% of the sampled states, and only with a difference of 1% with respect to the largest of the new bounds. (d) Density plot of 10^5 qutrit states, sampled approximately uniformly in terms of purity. The axes show numerical estimations of $1 - T_\mathcal{L} / \max[T_\Theta, T_\Phi]$ (horizontal) and $1 - \text{tr}[\rho^2]$ (vertical). We obtained the Pearson correlation coefficient $r = 0.8$ for the linear model between $1 - \text{tr}[\rho^2]$ and $1 - T_\mathcal{L} / \max[T_\Theta, T_\Phi]$. Bounds T_Θ and $T_\mathcal{L}$ coincide for pure states (bottom left), as shown analytically, and differ for increasingly mixed states (top right). This behavior qualitatively extends to N -dimensional systems.

Eq. (1) for the same choice of initial state ρ and Hamiltonian H_t . For the case of mixed qubits, we calculate all three bounds analytically: Take $\rho = \lambda|r_1\rangle\langle r_1| + (1 - \lambda)|r_2\rangle\langle r_2|$ as the initial state, and $H = e^{i\varphi}|r_1\rangle\langle r_2| + \text{H.c.}$ as the Hamiltonian, where $\varphi \in [0, 2\pi]$ is a phase. The chosen Hamiltonian generates the optimal unitary evolution for any choice of final state $\sigma = \lambda|s_1\rangle\langle s_1| + (1 - \lambda)|s_2\rangle\langle s_2|$, for $|s_1\rangle = \cos\theta|r_1\rangle + e^{i\varphi}\sin\theta|r_2\rangle$. The bounds read

$$T_{\Theta}(\rho, \sigma) = \theta, \quad (7)$$

$$T_{\Phi}(\rho, \sigma) = \arccos\left(\sqrt{\frac{1 + k^2 \cos 2\theta}{1 - k^2}}\right)\sqrt{\frac{1 - k^2}{2k^2}}, \quad (8)$$

$$T_{\mathcal{L}}(\rho, \sigma) = \arccos(F_+(\theta, \lambda) + F_-(\theta, \lambda)), \quad (9)$$

where $F_{\pm}(\theta, \lambda) = \frac{1}{2}\sqrt{1 + k^2 c_{2\theta} \pm 2kc_{\theta}\sqrt{1 - k^2 s_{\theta}^2}}$, with $c_x = \cos x$, $s_x = \sin x$, and $k = 1 - 2\lambda$.

Note that these bounds are independent of the relative phase φ , as we expect, and only depend on the distance $\theta = d(|r_1\rangle, |s_1\rangle)$ between the basis elements, and on the value of λ . Bound T_{Θ} is tight and attainable and does not depend on the spectrum Λ . A simple plot of the bounds shows that $T_{\Theta} \geq T_{\Phi} \geq T_{\mathcal{L}}$ [see Fig. 2(a)]. The three bounds coincide for pure states $\lambda = 0, 1$ and for the trivial case of $\theta = 0$.

In the general case of higher dimensions (up to $N = 10$), we study the tightness of bounds T_{Θ} and T_{Φ} , for time-independent trajectories, numerically [see Figs. 2(b) and 2(c)] [64]. The new bounds T_{Θ} and $T_{\mathcal{L}}$ coincide for pure states (as analytically shown above), and the difference between $\max\{T_{\Theta}, T_{\Phi}\}$ and $T_{\mathcal{L}}$ grows with decreasing purity [see Fig. 2(d)]. Despite the fact that T_{Θ} and T_{Φ} are larger than $T_{\mathcal{L}}$ for the vast majority of cases, there are some exceptional regions where the latter can be larger than the new bounds, such as along some degenerate subspaces, which form a subset of measure zero of $\mathcal{S}_{\Lambda}(\mathcal{H})$. In the absence of a strict hierarchy between these bounds, we cast our main result in the form of a unified bound

$$T_{\text{QSL}}(\rho, \sigma) = \max\{T_{\mathcal{L}}, T_{\Theta}, T_{\Phi}\}, \quad (10)$$

where $T_{\mathcal{L}}$, T_{Θ} , and T_{Φ} are given by Eqs. (1), (4), and (6), respectively.

Conclusions.—In this Letter, we have addressed the problem of the attainability of quantum speed limits for the unitary evolution of mixed states. We first showed that the conventional bound given in Eq. (1) is not generally tight for mixed states, because the Bures distance is not a suitable choice under the assumption of unitary evolution.

We have proposed two new distances between those elements of state space with the same spectrum—i.e., those that can be unitarily connected—and derived the corresponding QSLs. The first distance coincides with the angle between the GBVs and induces a tight and attainable speed limit for the case of mixed qubit states, but it does not

reduce to the unified bound in Eq. (1) for pure states of arbitrary dimension. The second distance is designed to conform for the case of pure states, while being as similar as possible to the generalized Bloch angle. These bounds arise from the properties of state space, when mixed states are represented as GBVs, providing thus a simple geometric interpretation. We have shown that the bounds obtained by these two distances are tighter than the conventional QSL given in Eq. (1) for the vast majority of states. Moreover, our new bounds are always easier to compute, as well as easier to measure experimentally.

Beyond its fundamental relevance, our result provides a tighter, and hence more accurate bound on the rate of information transfer and processing in the presence of classical uncertainty. For instance, the computational speed of a quantum computer that works between mixed states would be bounded by Eq. (10), rather than Eq. (1). The latter bound would wrongly suggest that, in order to speed up computation, one could simply add noise, reducing the purity of the considered states, with the effect of reducing the time required to evolve between them. This paradoxical situation is now ruled out by our new bound, which demonstrates that in the proximity of maximally mixed states, the time required to perform any unitary evolution is finite and comparable to the time required to perform the evolution between pure states.

There is a natural trade-off between the tightness of a QSL and its computational complexity. The ideas presented in this Letter open the door to finding a distance based on the explicit geometric structure of (mixed) state space. Such a distance would allow for the derivation of a QSL that is guaranteed to be tight, but at the same time easy to compute. It also remains open to apply the ideas developed here for the case of nonunitary dynamics. Such a generalization would require modifying our proposed distances such that they accommodate changes in purity. Operationally meaningful QSLs for open dynamics would be of great practical importance to both theorists and experimentalists alike; however, developing them would require a careful analysis of the resource accounting implicit in the choice of different distances.

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