Generalized Hardy's Paradox

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Here, we present the most general framework for n-particle Hardy's paradoxes, which include Hardy's original one and Cereceda's extension as special cases. Remarkably, for any $n \geq 3$, we demonstrate that there always exist generalized paradoxes (with the success probability as high as $1/2^{n-1}$) that are stronger than the previous ones in showing the conflict of quantum mechanics with local realism. An experimental proposal to observe the stronger paradox is also presented for the case of three qubits. Furthermore, from these paradoxes we can construct the most general Hardy's inequalities, which enable us to detect Bell's nonlocality for more quantum states.

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Introduction.—Hardy's paradox is an important all-versus-nothing (AVN) proof of Bell's nonlocality, a peculiar phenomenon that has its roots deep in the famous debate raised by Einstein, Podolsky, and Rosen (EPR) in 1935 [1]. Hardy's original proof [2,3], for two particles, has been considered as "the simplest form of Bell's theorem" and "one of the strangest and most beautiful gems yet to be found in the extraordinary soil of quantum mechanics" [4]. To date, a number of experiments have been carried out to confirm the paradox in two-particle systems [5–13]; theoretically, Hardy's paradox has been generalized from the two-qubit to a multiqubit family [14]. The two-particle Hardy's paradox can be stated in an inspiring way as follows [15]: In any local theory, if the events $A_2 < B_1$, $B_1 < A_1$, and $A_1 < B_2$ never happen, then, naturally, the event $A_2 < B_2$ must never happen. According to quantum theory, however, two-particle entangled states and local projective measurements exist that break down these local conditions; that is, in terms of probabilities,

$$P(A_2 < B_1) = P(B_1 < A_1) = P(A_1 < B_2) = 0, \quad \text{and} \quad P(A_2 < B_2) > 0,$$

where the last condition evidently conflicts with the prediction of local theory, leading to a paradox. In [14], the author showed that, for the n-qubit Greenberger-Horne-Zeilinger (GHZ) state the maximal success probability (i.e., the last condition above) can reach $\{1 + \cos[\pi/(n-1)]\}/2^n$.

Moreover, a quantum paradox can be naturally transformed to a corresponding Bell's inequality. For instance, the paradox mentioned above can be associated to the following Hardy's inequality $P(A_2 < B_2) - P(A_2 < B_1) - P(B_1 < A_1) - P(A_1 < B_2) \leq 0$, which is equivalent to Zohren and Gill's version [16] of the Collins-Gisin-Linden-Massar-Popescu inequalities (i.e., tight Bell's inequalities for two arbitrary d-dimensional systems, and the inequality becomes the Clause-Horne-Shimony-Holt inequality for $d = 2$) [17]. See, also, [18] for a connection between Hardy's inequality and Wigner's argument.

Demonstrating the conflict between quantum mechanics and local theories has had a long history ever since the EPR paper. It has brought out many important contributions to both physical foundations and applications, particularly introducing the concept of entanglement, viewed as "the characteristic trait of quantum mechanics" that distinguishes quantum theory from classical theory [19]. Among many others, the most important breakthrough was due to Bell who put the debate of the conflict on firm, physical ground in a statistical manner [20], and it has been regarded as "the most profound discovery of science" [21]. The Clause-Horne-Shimony-Holt (CHSH) inequality [22], serving as a revised version of Bell's original one, has been adopted to reveal nonlocality in various experiments, ranging from Aspect's experiment [23] in 1981 to some very recent loophole-free Bell-experiment tests [24–26]. On the other hand, differing from the statistical violation of inequalities, the AVN proof of nonlocality allows us to demonstrate the contradiction in an elegant, logical paradox, such that its experimental practice will be, in principle, simplified to a single-run operation. Among various AVN proofs, the GHZ paradox [27] has been carried out experimentally based on entangled photons [28]. In spite of that, it applies to three-particle systems [27] or more [29,30], but has, so far, defied any two-particle formulation.

Therefore, Hardy's paradox, with post-selections taken into consideration, stands out among the others, since (i) it applies to the two-party scenario, (ii) it can be generalized to multiparty and high-dimensional scenarios [14] (hereafter, we would like to refer to Cereceda's version of the n-qubit Hardy's paradox or inequality as the standard Hardy's paradox or inequality, to distinguish it from the most general ones that we shall present in this Letter), and (iii) inequalities constructed based on it allow us to detect more entangled states and provide a key element to prove Gisin's theorem [31,32]—which states that any entangled pure state violates Bell's inequality [33]. The GHZ paradox does not share most of these merits (see, also, the Mermin-Ardehali-Belinskii-Klyshko inequality [34–36], which was also a kind of generalization of CHSH inequality to n qubits, but was not violated by all pure entangled states, not even by all the generalized GHZ states).

In what follows, we present our main results on the generalized Hardy's paradoxes and their corresponding inequalities, along with an experimental proposal to observe the stronger quantum paradoxes in a three-qubit system.

Generalized Hardy's paradox.—For simplicity, we shall use the notations in [32] to formalize the generalized n-qubit Hardy's paradox. Consider a system composed of n qubits that are labeled with the index set $I_n = \{1, 2, \ldots, n\}$. For the kth qubit, we choose two observables $\{a_k, b_k\}$ that take binary values $\{0, 1\}$ in the local realistic model. Let us denote $a_\alpha = \prod_{k\in\alpha} a_k$ and $\bar{b}_\alpha = \prod_{k\in\alpha} \bar{b}_k$ with $\bar{b}_k = 1 - b_k$ for an arbitrary subset $\alpha \subseteq I_n$, $\bar{k} = I_n/k$ for arbitrary $k \in I_n$ and $\bar{\alpha} = I_n/\alpha$. Moreover, we denote $|\alpha|$ as the size of the subset α, and abbreviate the probability $p(x = 1, y = 1, \ldots)$ as $p(xy\ldots)$.

We now present the following theorem:

Theorem 1.—For any given sizes $|\alpha|$ and $|\beta|$ $(2 \le |\alpha| \le n, 1 \le |\beta| \le |\alpha|)$ satisfying the constraint $|\alpha| + |\beta| \le n + 1$, then in the local hidden variable (LHV) model, the following zero-probability conditions

$$p(b_\alpha a_{\bar{\alpha}}) = p(\bar{b}_\beta a_{\bar{\beta}}) = 0, \quad \forall\ \alpha, \beta \in I_n,$$

must lead to the following zero-probability condition

$$p(a_{I_n}) = 0.$$

Proof of Theorem 1.—Note that the above equations are all linear for the LHV model which is a convex polytope, whose extreme points are the deterministic LHV model. Thus, we only need to prove this theorem for the deterministic LHV model, that is,

$$b_\alpha a_{\bar{\alpha}} = \bar{b}_\beta a_{\bar{\beta}} = 0, \quad \forall\ \alpha, \beta \in I_n$$

must lead to the following zero-probability condition

$$a_{I_n} = 0.$$

We shall prove it by *reductio ad absurdum*. Suppose $a_{I_n} \neq 0$, then, in the deterministic LHV model, one directly obtains

$$b_\alpha = \bar{b}_\beta = 0, \quad \forall\ \alpha, \beta \in I_n,$$

which implies at least one of $|\beta|$ observables b_k's arbitrarily chosen from the set $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ must take the value "1"—namely, in the set $\mathcal{B}$, we have at least $n - (|\beta| - 1)$ observables equal to 1—and which, similarly, implies at least one of $|\alpha|$ observables b_k's arbitrarily chosen from the set $\mathcal{B}$ must take the value "0"—namely, in the set $\mathcal{B}$ we have at least $n - (|\alpha| - 1)$ observables equal to 0. Hence, at most, $(|\alpha| - 1)$ observables b_k's equal 1. This yields $(|\alpha| - 1) \ge n - (|\beta| - 1)$, i.e., $|\alpha| + |\beta| \ge n + 2$, in contradiction to the constraint $|\alpha| + |\beta| \le n + 1$. ■

For the sake of convenience, we label the generalized paradox as the $[n; |\alpha|, |\beta|]$ scenario. It can be verified, directly, that the standard Hardy's paradox is the $[n; n, 1]$ scenario by taking $|\alpha| = n$, $|\beta| = 1$. Nevertheless, quantum mechanics gives a different prediction that the success probability $p(a_{I_n})$ can be nonzero, thus, resulting in a generalized Hardy's paradox, stated as:

Theorem 2.—For the generalized GHZ state, by choosing appropriate quantum projective measurements on n qubits, the success probability $p(a_{I_n})$ is always greater than zero, and, for any $n \ge 3$, we can always have a stronger quantum paradox in comparison to the standard Hardy's paradox.*Proof of Theorem 2.*—Quantum mechanically, let us consider the generalized GHZ state

$$|\Psi\rangle_{g\mathrm{GHZ}} = h_0|0\ldots0\rangle + h_1|1\ldots1\rangle,$$

with $h_0 = |h_0| \ge 0$, $h_1 = |h_1| e^{i\theta_h}$ (The usual GHZ state corresponds to $|h_0| = |h_1| = 1/\sqrt{2}$, $\theta_h = 0$). We always assume the measurements a_i's, b_i's, and $\bar{b}_i$'s for the n observers are in the direction $a_0|0\rangle + a_1 e^{i\theta_a}|1\rangle$, $b_0|0\rangle + b_1 e^{i\theta_b}|1\rangle$, and $b_1|0\rangle + b_0 e^{i(\theta_b+\pi)}|1\rangle$, respectively, by direct calculation, we then obtain

$$\begin{aligned} p(b_\alpha a_{\bar{\alpha}}) &= |b_0^{|\alpha|} a_0^{n-|\alpha|} |h_0| + b_1^{|\alpha|} a_1^{n-|\alpha|} |h_1| e^{i\vartheta}|^2, \\ p(\bar{b}_\beta a_{\bar{\beta}}) &= |b_1^{|\beta|} a_0^{n-|\beta|} |h_0| + b_0^{|\beta|} a_1^{n-|\beta|} |h_1| e^{i\vartheta'}|^2, \\ p(a_N) &= |a_0^n |h_0| + a_1^n |h_1| e^{i[n\theta_a - \theta_h]}|^2, \end{aligned}$$

respectively, where $\vartheta = (n - |\alpha|)\theta_a + |\alpha|\theta_b - \theta_h$ and $\vartheta' = (n - |\beta|)\theta_a + |\beta|\theta_b - \theta_h + |\beta|\pi$.

Let $p(b_\alpha a_{\bar{\alpha}}) = p(\bar{b}_\beta a_{\bar{\beta}}) = 0$, we have equations of angles

$$\begin{aligned} (n - |\alpha|)\theta_a + |\alpha|\theta_b - \theta_h &= (2m_1 + 1)\pi, \\ (n - |\beta|)\theta_a + |\beta|\theta_b - \theta_h + |\beta|\pi &= (2m_2 + 1)\pi, \end{aligned}$$

with m_1, $m_2 = 0, 1, 2, \ldots$, and of norms

$$b_0^{|\alpha|}a_0^{n-|\alpha|}|h_0| = b_1^{|\alpha|}a_1^{n-|\alpha|}|h_1|,$$
$$b_1^{|\beta|}a_0^{n-|\beta|}|h_0| = b_0^{|\beta|}a_1^{n-|\beta|}|h_1|.$$

The following arguments are split into two cases:

Case 1:$|\beta| < |\alpha|$: we let $m_1 = m_2 = 0$, then, we have $\theta_b = [|\beta|\pi/(|\alpha| - |\beta|)] + \theta_a$, $n\theta_a - \theta_h = (1 - [|\alpha||\beta|/(|\alpha| - |\beta|)])\pi$, and $[(a_0/a_1) = (b_0/b_1)^{(|\alpha|+|\beta|)/(|\alpha|-|\beta|)} = \gamma^{\{|\alpha|+|\beta|/[(|\alpha|+|\beta|)n-2|\alpha||\beta|]\}}$, with $\gamma = [|h1|/(|h0|)]$, and so, the success probability equals

$$p(a_{I_n}) = \frac{\gamma^2|\kappa_0 - \kappa_1|^2}{(1+\gamma^2)(1+\kappa_2)^n} > 0,$$
$$\kappa_0 = e^{i[(|\alpha||\beta|)/(|\alpha|-|\beta|)]\pi},$$
$$\kappa_1 = \gamma^{[(2|\alpha||\beta|)/n(|\alpha|+|\beta|)-2|\alpha||\beta|]},$$
$$\kappa_2 = \gamma^{[2(|\alpha|+|\beta|)/n(|\alpha|+|\beta|)-2|\alpha||\beta|]}.$$

At $\gamma = 1$, on the other hand, the success probability equals

$$p(a_{I_n}) = \frac{1}{2^n}\left[1 - \cos\left(\frac{|\alpha||\beta|\pi}{|\alpha| - |\beta|}\right)\right].$$

Note that $p(a_{I_n})$ is strictly smaller than $(1/2^{n-1})$ because $[|\alpha||\beta|/(|\alpha| - |\beta|)]$ cannot be odd. For the standard Hardy's paradox, i.e., $\alpha = n, \beta = 1$, it reduces to the result in [14] as

$$P_n^S \equiv p(a_{I_n}) = \frac{1}{2^n}\left[1 + \cos\left(\frac{\pi}{n-1}\right)\right], \tag{1}$$

where P_n^S represents the success probability for the standard Hardy's paradox for the n-qubit GHZ state.

Case 2: $|\beta| = |\alpha|$: we let $m_1 = 0$, $m_2 = (|\beta|/2)$ (here, $|\alpha|$ and $|\beta|$ must be even). Note that, in this case, we have an independent θ_h, then, we further let $n\theta_a - \theta_h = 0$ and $b_0 = b_1 = 1/\sqrt{2}$, $a_0/a_1 = \gamma^{(1/n-|\alpha|)}$, with $\gamma = [|h1|/(|h0|)]$, and the success probability equals

$$p(a_{I_n}) = \frac{\gamma^2(1+\kappa_1')^2}{(1+\gamma^2)(1+\kappa_2')^n} > 0,$$
$$\kappa_1' = \gamma^{(|\alpha|/n-|\alpha|)}, \qquad \kappa_2' = \gamma^{2/n-|\alpha|}.$$

The success probability at $\gamma = 1$ equals

$$P_n^G \equiv p(a_{I_n}) = \frac{1}{2^{n-1}},$$

where P_n^G represents the success probability for the generalized Hardy's paradox for the n-qubit GHZ state.

Combining the above two cases, the theorem is proved as was claimed. ■

Remark 1.—As an example, given the GHZ state $|\Psi\rangle_{\rm GHZ} = (|00\cdots0\rangle + |11\cdots1\rangle)/\sqrt{2}$ of n qubits, Cereceda [14] found that the maximal success probability for the standard Hardy's paradox is Eq. (1) But, by choosing $|\beta| = |\alpha|$ in the generalized Hardy's paradox, for any $n \geq 3$, we can have a greater success probability (see, also, Fig. 1):

$$P_n^G \equiv p(a_{I_n}) = \frac{1}{2^{n-1}} > P_n^S. \tag{2}$$

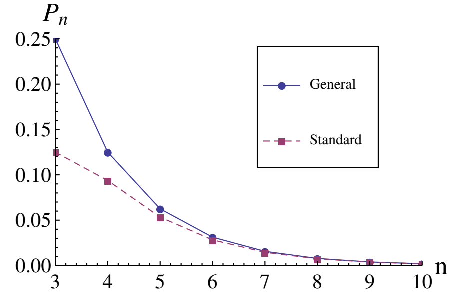

FIG. 1. The success probability P_n versus particle number n ($n \geq 3$). The blue points correspond to P_n^G in the generalized Hardy's paradox (with $|\alpha| = |\beta| =$ even number), and the purple points correspond to P_n^S in the standard Hardy's paradox (with $|\alpha| = n$, $|\beta| = 1$). The relation $P_n^G > P_n^S$ implies that there is always a stronger paradox in comparison to the standard Hardy's paradox for $n \geq 3$. Especially, in the [3;2,2] scenario, we have $P_3^G = 1/4$, which is twice $P_3^S = 1/8$. Thus, it is feasible to observe the stronger quantum paradox in a three-qubit system.

Indeed, for GHZ states with $n \geq 3$, $|\alpha| = |\beta| =$ even number is the best choice for generalized Hardy's paradoxes [37].

Remark 2.—The $[n; |\alpha| = 2, |\beta| = 1]$ scenario resembles the paradox presented in [38], but the former concerns the Bell scenario, while the latter discusses the genuine multipartite nonlocality, which is a subset of the Bell nonlocality; the $[n; 2 < |\alpha| < n, |\beta| = 1]$ scenario is related to the paradox presented in [39], which discussed the hierarchy of multipartite nonlocality. Thus, it is of great interest to further investigate possible connections of the results in [38,39] with the structure of Theorem 1.

Remark 3.—For the paradox of $[n; |\alpha|, |\beta|]$ scenario, one can have its corresponding generalized Hardy's inequality as

$$\begin{aligned}\mathcal{I}[n; |\alpha|, |\beta|; x, y] &= F(n; \alpha, \beta; x, y)p(a_{I_n}) \\ &\quad - x\sum_{\alpha} p(b_\alpha a_{\bar{\alpha}}) - y\sum_{\beta} p(\bar{b}_\beta a_{\bar{\beta}}) \leq 0,\end{aligned} \tag{3}$$

with $x > 0$, $y > 0$. Usually for convenience, one can choose x, y as positive integers, and to make the inequality meaningful (i.e., it can possibly be violated by quantum states), one needs to require $F(\alpha, \beta; x, y) > 0$. By direct computation, one can determine

$$F(n; \alpha, \beta; x, y) = \min_{0 \leq m \leq n}\left[x\binom{m}{|\alpha|} + y\binom{n-m}{|\beta|}\right],$$

TABLE I. Numerical results of threshold visibility $V_{\rm thr}[n;|\alpha|,|\beta|;1,1]$ for violations of inequality $\mathcal{I}[n;|\alpha|,|\beta|;1,1]\leq 0$ by the n-qubit GHZ states. The boxed number represents the lowest visibility for each n. For $n\geq 4$, the generalized Hardy's inequalities can provide lower visibility than the standard one (which corresponds to $|\alpha|=n$, $|\beta|=1$). For $|\alpha|=|\beta|=q$, (q is even, $2q\leq n+1$), one may have the analytical expression $V_{\rm thr}[n;q,q;1,1]=(2(n/q)-[(n/2)/q]-\{[n-(n/2)]/q\})/(2(n/q)+[(n/2)/q]+\{[n-(n/2)]/q\})$. For $q=2$, we have $V_{\rm thr}[n;2,2;1,1]=\{3[(n+1)/2]-1/5[(n+1)/2]-3\}$. It can be proved that, for the case of $x=y=1$, for GHZ states with $n\geq 5$, the relation $|\alpha|=|\beta|=2$ is the best choice for generalized Hardy's inequality [37].

n	3	4	5	6	7	8	9	10
$\lvert\alpha\rvert=n$, $\lvert\beta\rvert=1$	$\boxed{0.681250}$	0.707107	0.737431	0.764501	0.787467	0.806795	0.823130	0.837049
$\lvert\alpha\rvert=2$, $\lvert\beta\rvert=1$	0.682242	0.703526	0.730699	0.755929	0.777878	0.796691	0.812819	0.826718
$\lvert\alpha\rvert=n-1$, $\lvert\beta\rvert=1$	0.682242	$\boxed{0.671442}$	0.702481	0.734966	0.763073	0.786584	0.806221	0.822742
$\lvert\alpha\rvert=2$, $\lvert\beta\rvert=2$	0.714286	0.714286	$\boxed{0.666667}$	$\boxed{0.666667}$	$\boxed{0.647059}$	$\boxed{0.647059}$	$\boxed{0.636364}$	$\boxed{0.636364}$

which is the largest integer that the inequality still holds, and $\binom{m}{k}=[m!/k!(m-k)!]$ is the binomial coefficient. For $x=y=1$, $|\beta|=1$, one has the coefficient as $F(\alpha,\beta;1,1)=n-|\alpha|+1$. For $x=y=1$, the family of the generalized Hardy's inequalities is particularly interesting, one may have that:

(i) The standard n-qubit Hardy's inequality corresponds to $\mathcal{I}[n;|\alpha|=n,|\beta|=1;x=1,y=1]$, which is a family of tight Bell's inequalities; the 22nd Sliwa's inequality [40] corresponds to $\mathcal{I}[n=3;|\alpha|=2,|\beta|=1;x=1,y=1]$, which is a tight Bell's inequality; also, numerical computation shows that the family of n-qubit Bell's inequalities $\mathcal{I}[n;|\alpha|,|\beta|=1;x=1,y=1]$ is tight [41];

(ii) Based on the visibility criterion, for $n\geq 4$, the generalized Hardy's inequalities can resist more white-noise than the standard Hardy's inequality. For a given n-qubit entangled state ρ, we can mix it with the white noise $I_{\rm noise}=\mathbb{1}^{\otimes n}/2^n$, the resultant density matrix is given by $\rho^V=V\rho+(1-V)I_{\rm noise}$. Resistance to noise can be measured through the threshold visibility $V_{\rm thr}$, below which Bell's inequality cannot be violated. A lower threshold visibility means that the quantum state can tolerate a greater amount of noise. Let us consider ρ as the n-qubit GHZ state. In Table I, we compare the threshold visibility of the generalized Hardy's inequalities and that of the standard Hardy's inequality. We find that, for $n\geq 4$, the generalized Hardy's inequalities can provide lower visibilities than the standard one.

Experimental proposal to observe the stronger paradox with three qubits.—A number of experimental tests of the two-qubit Hardys paradox have been carried out since 1993 [5–13]. The maximal success probability for the two-qubit Hardy's paradox is $(5\sqrt{5}-11)/2\simeq 9\%$, which does not occur for the maximally entangled state [3,14]. For the three-qubit standard Hardy's paradox, the success probability is given by $P_3^S=1/8=0.125$, which occurs for the GHZ state. To our knowledge, such an experiment has not yet been demonstrated. The higher the success probability, the more friendly the experimental observation. Here, we present an experimental proposal to observe a stronger paradox in the $[n=3;|\alpha|=2,|\beta|=2]$ scenario, whose success probability is $P_3^G=1/4=0.25$. In the experiment, the resource is prepared as the three-qubit GHZ state $|\Psi\rangle_{\rm GHZ}=(|000\rangle+|111\rangle)/\sqrt{2}$, and three qubits are sent to three observers Alice, Bob, and Charlie separately (see, also, Fig. S1 in [37] for an illustrative setup). Quantum mechanically, the three observers will all perform the same measurements in the $\hat{x}$- and $\hat{y}$-direction, respectively, i.e.,

$$\hat{a}_1=\hat{a}_2=\hat{a}_3=|+x\rangle\langle +x|,$$
$$\hat{b}_1=\hat{b}_2=\hat{b}_3=|+y\rangle\langle +y|,$$

with $\bar{\hat{b}}_j=\mathbb{1}-\hat{b}_j=|-y\rangle\langle -y|$, ($j=1,2,3$), $\mathbb{1}$ is the 2×2 unit matrix, and $|+x\rangle=(1/\sqrt{2})(|0\rangle+|1\rangle)$, $|\pm y\rangle=(1/\sqrt{2})(|0\rangle\pm i|1\rangle)$.

First, one needs to experimentally verify the zero-probability conditions, i.e.,

$$\begin{aligned}p(\hat{b}_1\hat{b}_2\hat{a}_3)&=p(\hat{b}_1\hat{a}_2\hat{b}_3)=p(\hat{a}_1\hat{b}_2\hat{b}_3)\\&=p(\bar{\hat{b}}_1\bar{\hat{b}}_2\hat{a}_3)=p(\bar{\hat{b}}_1\hat{a}_2\bar{\hat{b}}_3)=p(\hat{a}_1\bar{\hat{b}}_2\bar{\hat{b}}_3)=0,\end{aligned}\tag{4}$$

with $p(\hat{b}_1\hat{b}_2\hat{a}_3)={\rm tr}[\rho(\hat{b}_1\otimes\hat{b}_2\otimes\hat{a}_3)]$, etc., and ρ stands for the GHZ state. Equations (4) are automatically satisfied in quantum theory. Second, one will experimentally measure the success probability, i.e., the last one in Theorem 1, whose theoretical quantum prediction is given by

$$p(\hat{a}_1\hat{a}_2\hat{a}_3)={\rm tr}[\rho(\hat{a}_1\otimes\hat{a}_2\otimes\hat{a}_3)]=\frac{1}{4}.\tag{5}$$

Taking into account experimental errors due to environment noise such that the six probabilities in (4) are not exactly zeros by measurements, let us denote the conditions as $p(\hat{b}_1\hat{b}_2\hat{a}_3)=p(\hat{b}_1\hat{a}_2\hat{b}_3)=p(\hat{a}_1\hat{b}_2\hat{b}_3)=p(\bar{\hat{b}}_1\bar{\hat{b}}_2\hat{a}_3)=p(\bar{\hat{b}}_1\hat{a}_2\bar{\hat{b}}_3)=p(\hat{a}_1\bar{\hat{b}}_2\bar{\hat{b}}_3)=\epsilon$. With the aid of the inequality $\mathcal{I}[3;2,2;1,1]=a_1a_2a_3-b_1b_2a_3-b_1a_2b_3-a_1b_2b_3-\bar{b}_1\bar{b}_2a_3-\bar{b}_1a_2\bar{b}_3-a_1\bar{b}_2\bar{b}_3\leq 0$, if one can observe the violation, then he must have $1/4-6\epsilon>0$. Thus, the maximal tolerance of measurement error is $\epsilon<1/24\approx 0.041$.

Conclusions and discussion.—While Hardy's paradox and Hardy's inequality have been generalized to arbitrary n qubits by Cereceda, we have found that Cereceda's way of extension is not the unique one. In this Letter, we have presented the most general framework for the n-particle Hardy's paradox and Hardy's inequality. For $n \geq 3$ the generalized paradox may possess higher success probability and, thus, is stronger than the standard Hardy's paradox. For GHZ states with $n \geq 3$, $|\alpha| = |\beta| =$ even number is the best choice for generalized Hardy's paradoxes. For $n \geq 4$, the generalized Hardy's inequalities resist more noise than the standard Hardy's inequality (one can also adopt the generalized Hardy's inequality to prove Gisin's theorem, which we shall discuss elsewhere). Particularly in consideration of Table I and [37], our result shows that for GHZ states with $n \geq 5$, the relation $|\alpha| = |\beta| = 2$ is the best choice for generalized Hardy's inequality $\mathcal{I}[n; |\alpha|, |\beta|; x = 1, y = 1] \leq 0$. Moreover, in the three-qubit system, we have also designed a feasible experiment proposal to observe the stronger quantum paradox. In our opinion, the results here advance the study of Bell's nonlocality both with and without inequality. We anticipate experimental work in this direction in the near future.

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