Exactly Solvable BCS-Hubbard Model in Arbitrary Dimensions

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We introduce in this Letter an exact solvable BCS-Hubbard model in arbitrary dimensions. The model describes a p-wave BCS superconductor with equal spin pairing moving on a bipartite (cubic, square, etc.) lattice with on-site Hubbard interaction U. We show that the model becomes exactly solvable for arbitrary U when the BCS pairing amplitude Δ equals the hopping amplitude t. The nature of the solution is described in detail in this Letter. The construction of the exact solution is parallel to the exactly solvable Kitaev honeycomb model for $S = 1/2$ quantum spins and can be viewed as a generalization of Kitaev's construction to $S = 1/2$ interacting lattice fermions. The BCS-Hubbard model discussed in this Letter is just an example of a large class of exactly solvable lattice fermion models that can be constructed similarly.

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Introduction.—Exact solutions of quantum interacting-particle models in dimensions > 1 are rare and are important resources for understanding the physics of strongly correlated systems in dimensions > 1 [1–13]. More recently, a major advance in understanding the mathematics of topological order was put forth by the introduction of the exactly solvable Toric Code [14] and honeycomb [15–18] models by Kitaev and their generalizations [19–27]. The exact solvability of the Kitaev honeycomb model is a result of the existence of an infinite number of conserved quantities (for an infinite lattice) in the model. In this Letter, we show that Kitaev's construction can be generalized to a class of $S = 1/2$ lattice fermion models that describe p-wave BCS superconductors with equal spin pairing (ESP) and with on-site Hubbard interaction U. The generalization is based on the observation that the Kitaev honeycomb lattice model can be expressed in terms of a spinless fermion model [17,18] via a Jordan-Wigner transformation. The "generalized" lattice fermion models carry both "quasiparticle" and "solitonic" excitations as in the Kitaev honeycomb model except that the solitonic excitations are in general nontopological in lattice fermion models.

To illustrate, we consider in this Letter a particular BCS-Hubbard model with equal spin pairing [28–30] on cubic (3D) and square (2D) lattices. The more general constructions are discussed at the end of the Letter.

Model.—The Hamiltonian for our BCS-Hubbard model is given by $H = H_0 + H_{\text{int}}$, where

$$H_0 = \sum_{\langle i,j\rangle,\sigma} (t_{ij} c^\dagger_{i\sigma} c_{j\sigma} + \text{H.c.} + \Delta_{ij} c^\dagger_{i\sigma} c^\dagger_{j\sigma} + \text{H.c.})$$

$$H_{\text{int}} = U \sum_l \left(n_{l\uparrow} - \frac{1}{2}\right)\left(n_{l\downarrow} - \frac{1}{2}\right) \qquad (1)$$

where $\langle i,j\rangle$ describes nearest neighbor sites with $i \in A$, $j \in B$ being lattice sites belonging to different sublattices of the cubic or square lattice. $t_{ij} = t_{ji}$ and $\Delta_{ij} = -\Delta_{ji}$ are the hopping matrix and BCS-pairing term between sites i and j, respectively. The last term describes on-site Hubbard interaction U where $l \in A$, B, i.e., all lattices sites where $n_{l\sigma} = c^\dagger_{l\sigma} c_{l\sigma}$. Notice that the BCS-pairing term describes equal spin pairing. We shall consider a pairing term Δ_{ij} which is positive when $i \in A$, $j \in B$, corresponding to a staggering nearest neighbor pairing field on the cubic and square lattices (see Fig. 1). The Hamiltonian Eq. (1) is in general not solvable. In the following we shall show that it becomes exactly solvable when $\Delta_{ij} = t_{ij} = t$ where t is a real number.

Construction of the exact solution.—To see how the model becomes exactly solvable we introduce Majorana fermion representation [31]

$$c_{i\sigma} = \eta_{i\sigma} + i\beta_{i\sigma}, \qquad c^\dagger_{i\sigma} = \eta_{i\sigma} - i\beta_{i\sigma}$$
$$c_{j\sigma} = \beta_{j\sigma} + i\eta_{j\sigma}, \qquad c^\dagger_{j\sigma} = \beta_{j\sigma} - i\eta_{j\sigma} \qquad (2)$$

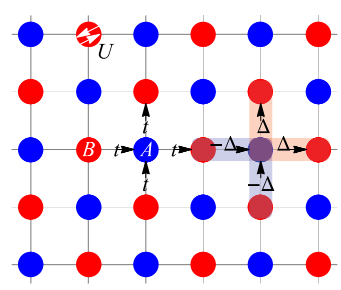

FIG. 1. BCS-Hubbard model on square lattice, naming the blue dot as A sublattice and red dot B sublattice. The hopping potential t is uniform along all the nearest neighbor bond, that is $t_{A\to B} = t_{B\to A}$, while the ESP potential has a staggered form which explicitly is $\Delta_{A\to B} = -\Delta_{B\to A}$. U represents the Hubbard on-site interaction.

for fermions on $i \in A$ and $j \in B$ sublattices, respectively. It is straightforward to show that the Hamiltonian H can be represented in terms of the Majorana fermions η and β where

$$H_0 \to 4i\tilde{t} \sum_{\langle i,j\rangle,\sigma} (-\beta_{i\sigma}\beta_{j\sigma} + \delta\eta_{i\sigma}\eta_{j\sigma})$$
$$H_{\mathrm{int}} \to U \sum_l (2i\eta_{l\uparrow}\beta_{l\uparrow})(2i\eta_{l\downarrow}\beta_{l\downarrow}) \tag{3}$$

where $t = \tilde{t}(1+\delta)$ and $\Delta = \tilde{t}(1-\delta)$. We notice that in the limit $t = \Delta$ (or $\delta = 0$), the kinetic (H_0) term is expressed in terms of Majorana fermions β only and the η terms are absent in H_0. In this limit,

$$\frac{d}{dt}(\eta_{l\uparrow}\eta_{l\downarrow}) = \frac{1}{i\hbar}[\eta_{l\uparrow}\eta_{l\downarrow}, H] = 0,$$

$\forall l$ and $(2i\eta_{l\uparrow}\eta_{l\downarrow}) = D_l$ are constants of motion. Using the identities $(\eta_{l\sigma})^2 = (\beta_{l\sigma})^2 = \frac{1}{4}$, we obtain $(D_l)^2 = \frac{1}{4}$ and $D_l = \pm\frac{1}{2}$.

Substituting D_l into Eq. (3), we obtain in the limit $\delta \to 0$,

$$H \to -4it \sum_{\langle i,j\rangle,\sigma} (\beta_{i\sigma}\beta_{j\sigma}) - \sum_l (UD_l)(2i\beta_{l\uparrow}\beta_{l\downarrow}) \tag{4}$$

where D_l are C numbers. The Hamiltonian Eq. (4) is quadratic and is exactly diagonalizable. The many-body eigenstates of the Hamiltonian are divided into different *solitonic* sectors characterized by different sets of eigenvalues $\{D_l\}$. The ground state of the system is given by the set of $\{D_l\}$ with lowest energy. The construction of the exact solution is parallel to the construction of the exact solution of the spin-1/2 Kitaev honeycomb model [15] when the model is expressed in terms of spinless fermions [17,18]. We show here that the Kitaev construction can be extended to $S = 1/2$ lattice fermions with Hubbard-type interaction rather straightforwardly.

Properties of the exact solution (1)–$U = 0$ limit.—We first study the solution of Hamiltonian Eq. (3) in the limit $U = 0$. In this limit $H \to H_0$ describes a p wave, ESP BCS superconductor with staggered nearest neighbor pairing fields and with chemical potential $\mu = 0$, i.e., half-filled bands.

It is convenient to "re-fermionize" the Majorana fermions by introducing the composite fermions

$$d_{l2} = \eta_{l\uparrow} + i\xi_l\eta_{l\downarrow}, \qquad d^{\dagger}_{l2} = \eta_{l\uparrow} - i\xi_l\eta_{l\downarrow}$$
$$d_{l1} = \beta_{l\uparrow} - i\xi_l\beta_{l\downarrow}, \qquad d^{\dagger}_{l1} = \beta_{l\uparrow} + i\xi_l\beta_{l\downarrow}, \tag{5}$$

where $\xi_l = +(-)1$ for $l \in A(B)-$ sublattices.

The transformation Eqs. (2) and (5) and be understood by introducing the fermions

$$c_{l\rightleftarrows} = \frac{1}{\sqrt{2}}(c_{l\uparrow} \pm ic_{l\downarrow}) \tag{6}$$

which represents fermions with spin pointing in $+(-)\hat{y}-$ directions, respectively. The d fermions are related to $c_{\rightleftarrows}$ by

$$d_{l\rightarrow} = \frac{1}{\sqrt{2}}(c_{l\rightarrow} + c^{\dagger}_{l\leftarrow}),$$
$$d_{l\leftarrow} = \frac{1}{\sqrt{2}i}(c_{l\leftarrow} - c^{\dagger}_{l\rightarrow}) \tag{7a}$$

and

$$d^{(A)}_{l2} = d^{(B)}_{l1} = d_{l\rightarrow},$$
$$d^{(A)}_{l1} = d^{(B)}_{l2} = d_{l\leftarrow}, \tag{7b}$$

where $d^{(A)}$ and $d^{(B)}$ are fermions on $A(B)$ sublattices, respectively. Equation (7) represents a Bogoliubov–de Gennes transformation between the d and $c_{\rightleftarrows}$ fermions.

As will be seen below and in next section, the d fermions will form the *quasiparticles* for our model Hamiltonians. It's interesting to note from Eq. (7a) that

$$\langle c^{\dagger}_{l\leftarrow}c_{l\leftarrow}\rangle + \langle c^{\dagger}_{l\rightarrow}c_{l\rightarrow}\rangle = 1 + i\langle(d^{\dagger}_{l\rightarrow}d^{\dagger}_{l\leftarrow} - d_{l\leftarrow}d_{l\rightarrow})\rangle, \tag{8a}$$

and

$$\langle c^{\dagger}_{l\leftarrow}c_{l\leftarrow}\rangle - \langle c^{\dagger}_{l\rightarrow}c_{l\rightarrow}\rangle = \langle d^{\dagger}_{l\leftarrow}d_{l\leftarrow}\rangle - \langle d^{\dagger}_{l\rightarrow}d_{l\rightarrow}\rangle. \tag{8b}$$

It is interesting to note from Eq. (8a) that the (c)-fermion charges are not directly proportional to the d-fermion occupation number, and can be changed only by exciting a pair of d fermions. This is because the d fermions are an equal superposition of particle and hole states of c fermions. As a result they carry only spin and no charge individually.

In terms of the composite fermions d, H_0 becomes

$$H_0 \to 2i\tilde{t} \sum_{\langle i,j\rangle} [-(d^{\dagger}_{i1}d^{\dagger}_{j1} - d_{j1}d_{i1}) + \delta(d^{\dagger}_{i2}d^{\dagger}_{j2} - d_{j2}d_{i2})]$$
$$= 2i\tilde{t} \sum_{\langle i,j\rangle} [-(d^{\dagger}_{i\leftarrow}d^{\dagger}_{j\rightarrow} - d_{j\rightarrow}d_{i\leftarrow}) + \delta(d^{\dagger}_{i\rightarrow}d^{\dagger}_{j\leftarrow} - d_{j\leftarrow}d_{i\rightarrow})] \tag{9}$$

which can be diagonalized straightforwardly by introducing sublattice Fourier transforms

$$d^{A(B)}_{\mathbf{k}\alpha} = \frac{1}{V} \sum_{i\in A(B)} e^{i\mathbf{k}\cdot\mathbf{r}_i} d_{i\alpha}$$

etc., where $\alpha = 1, 2$ and

$$H_0 = \sum_{\mathbf{k},\alpha=1,2} \psi^{\dagger}_{\mathbf{k}\alpha} h^{\alpha}(\mathbf{k}) \psi_{\mathbf{k}\alpha}, \tag{10a}$$

where $\psi^{\dagger}_{\mathbf{k}\alpha} = (d^{\dagger A}_{\mathbf{k}\alpha}, d^{B}_{-\mathbf{k}\alpha})$ and

$$h^{\alpha}(\mathbf{k}) = \begin{pmatrix} 0 & \Delta_{\alpha}(\mathbf{k}) \\ \Delta^{*}_{\alpha}(\mathbf{k}) & 0 \end{pmatrix} \tag{10b}$$

with $\Delta_1(\mathbf{k}) = 4i\tilde{t}(\sum_{i=1,\ldots,\text{dim}} \sin k_i)$ where "dim" is the dimension of the system and $\Delta_2(\mathbf{k}) = \delta\Delta_1(\mathbf{k})$.

Diagonalizing the Hamiltonian, we obtain

$$H_0 = \sum_{\mathbf{k},\alpha=1,2} E_{\alpha}(\mathbf{k})(\gamma^{\alpha\dagger}_{\mathbf{k}+}\gamma^{\alpha}_{\mathbf{k}+} + \gamma^{\alpha\dagger}_{\mathbf{k}-}\gamma^{\alpha}_{\mathbf{k}-}), \tag{11}$$

where $E_{\alpha}(\mathbf{k}) = |\Delta_{\alpha}(\mathbf{k})|$,

$$\gamma^{\alpha}_{\mathbf{k}+}(\gamma^{\alpha\dagger}_{-\mathbf{k}-}) = \frac{1}{\sqrt{2}}(d^{A}_{\mathbf{k}\alpha} - (+)id^{B\dagger}_{-\mathbf{k}\alpha}),$$

with ground state energy $E_G = -\sum_{\mathbf{k}\alpha} E_{\alpha}(\mathbf{k})$, the ground state being defined by the usual BCS requirement $\gamma^{\alpha}_{\mathbf{k},\pm}|G\rangle = 0$ for all $\mathbf{k}$ and α.

We notice that $E_1(\mathbf{k}) \neq E_2(\mathbf{k})$, reflecting that the fermion pairing breaks spin-rotation symmetry, as can be seen directly from Eq. (9). The spectrum has a Fermi surface denoted by $E_{1(2)}(\mathbf{k}) = 0$ which describes a rather unusual *gapless* BCS superconductor [32].

It is also easy to show that for $\delta \neq 0$,

$$\langle d^{\dagger}_{l\rightarrow} d_{l\rightarrow}\rangle = \langle d^{\dagger}_{l\leftarrow} d_{l\leftarrow}\rangle = \frac{1}{2} \tag{12}$$

for l in both sublattices, indicating that the ground state is nonmagnetic.

Properties of the exact solution (2)–$U \neq 0$, $\delta = 0$.—In terms of the composite fermions d, the Hubbard interaction term can be expressed as

$$U\sum_{l}(2i\eta_{l\uparrow}\beta_{l\uparrow})(2i\eta_{l\downarrow}\beta_{l\downarrow}) = U\sum_{l}\left(n^{(d)}_{l\rightarrow} - \frac{1}{2}\right)\left(n^{(d)}_{l\leftarrow} - \frac{1}{2}\right) \tag{13}$$

where $n^{(d)}_{l\rightleftarrows} = d^{\dagger}_{l\rightleftarrows} d_{l\rightleftarrows}$. In the limit $\delta = 0$, the Hamiltonian in terms of composite fermions becomes

$$H \rightarrow -2i\tilde{t}\sum_{\langle i,j\rangle}(d^{\dagger}_{i1}d^{\dagger}_{j1} - d_{j1}d_{i1}) + U\sum_{l}(D_l)\left(n^{(d)}_{l1} - \frac{1}{2}\right) \tag{14}$$

[see Eq. (7b) for the relation between 1(2) and $\rightleftarrows$] and $D_l = n^{(d)}_{l2} - \frac{1}{2} = \pm\frac{1}{2}$ are conserved quantities which can be determined (in the ground state) by minimizing the energy of the system [see discussions after Eq. (3)]. We have performed the calculation numerically and find that D_l have uniform value $UD_l = -(|U|/2)$ on the ground state. In particular, the U and $-U$ ground states are related by flipping D_l to $-D_l$ with the solution for d_1 fermions remaining unchanged.

It may be surprising that although the d fermions carry zero c-fermion charge, nevertheless they are affected by the presence of Hubbard interaction as indicated in Eqs. (13) and (14). To clarify this we construct the on-site states with zero, one, and two occupied d fermions, respectively. Using Eq. (7), it is easy to show that

$$\begin{aligned} |0_d\rangle &= \frac{1}{\sqrt{2}}(1 + c^{\dagger}_{\leftarrow}c^{\dagger}_{\rightarrow})|0\rangle, \\ |s_d\rangle &= c^{\dagger}_{s}|0\rangle; \\ |\rightleftarrows_d\rangle &= \frac{1}{\sqrt{2}}(1 - c^{\dagger}_{\leftarrow}c^{\dagger}_{\rightarrow})|0\rangle, \end{aligned} \tag{15}$$

where $|0\rangle$ denotes vacuum for the c fermions, $|0_d\rangle$ denotes vacuum for the d fermions, $|s_d\rangle$ denotes a state occupied by a single d fermion with spin $s = \rightleftarrows$, and $|\rightleftarrows_d\rangle$ denotes a state occupied by two d fermions. Notice that the c-fermion number is equal to 1 in all four states. However, there exists doubly occupied c-fermion state components in states $|0_d\rangle$ and $|\rightleftarrows_d\rangle$, and they are both affected by the Hubbard interaction U.

Fourier transforming, the quasiparticle Hamiltonian Eq. (14) in the ground state sector $UD_l = -(|U|/2)$ can be rewritten as $H = \sum_{\mathbf{k}}\psi^{\dagger}_{\mathbf{k}}h^{(1)}(\mathbf{k})\psi_{\mathbf{k}}$, where $\psi^{\dagger}_{\mathbf{k}\alpha} = (d^{\dagger A}_{\mathbf{k}1}, d^{B}_{-\mathbf{k}1})$ and

$$h^{(1)}(\mathbf{k}) = \begin{pmatrix} -\frac{|U|}{2} & \Delta_1(\mathbf{k}) \\ \Delta^{*}_1(\mathbf{k}) & \frac{|U|}{2} \end{pmatrix}. \tag{16}$$

The quasiparticle energy spectrum for the d_1 fermions is given by

$$H = \sum_{\mathbf{k}} E_1(\mathbf{k})(\gamma^{(1)\dagger}_{\mathbf{k}+}\gamma^{(1)}_{\mathbf{k}+} + \gamma^{(1)\dagger}_{\mathbf{k}-}\gamma^{(1)}_{\mathbf{k}-}), \tag{17a}$$

where

$$E_1(\mathbf{k}) = \sqrt{|\Delta_1(\mathbf{k})|^2 + \left(\frac{U}{2}\right)^2} \tag{17b}$$

and

$$\gamma^{(1)}_{\mathbf{k}+}(\gamma^{(1)\dagger}_{-\mathbf{k}-}) = u_{\mathbf{k}+(-)}d^{A}_{\mathbf{k}\leftarrow} - (+)iv_{\mathbf{k}+(-)}d^{B\dagger}_{-\mathbf{k}\rightarrow} \tag{17c}$$

where

$$u(v)_{\mathbf{k}+} = \sqrt{\frac{1}{2}\left(1 - (+)\frac{|U|}{2E_1(\mathbf{k})}\right)}$$

and $u_{\mathbf{k}-} = v_{\mathbf{k}+}$; $v_{\mathbf{k}-} = u_{\mathbf{k}+}$. The quasiparticles are chargeless and carry spin 1/2 along the $\hat{y}$ direction.

It is also straightforward to show that

$$\begin{aligned} \langle d^{\dagger}_{l1}d_{l1}\rangle &= \frac{1}{2V}\sum_{\mathbf{k}}\left(1 + \frac{|U|}{2E_1(\mathbf{k})}\right) \\ \langle d^{\dagger}_{l\rightarrow}d^{\dagger}_{l\leftarrow}\rangle &= \langle d_{l\leftarrow}d_{l\rightarrow}\rangle = 0 \end{aligned} \tag{18}$$

in the ground state for both sublattices $l \in A(B)$ and

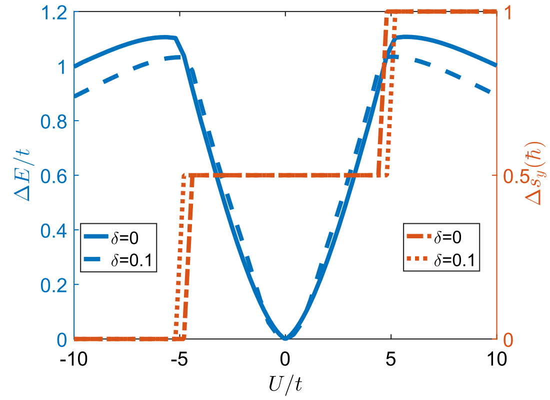

FIG. 2. The excitation energy (ΔE left axis) and spin along the $\hat{y}$ direction (Δs_y right axis) as a function of U for $\delta = 0$ and 0.1. The blue-solid and blue-dashed lines represent the excitation energies for $\delta = 0$, 0.1, and the orange-dotted and orange-dash-dotted lines represent the spins carried by the excitations for $\delta = 0$, 0.1, respectively.

$$m_y = \frac{1}{2}(\langle d_{l1}^\dagger d_{l1}\rangle - \langle d_{l2}^\dagger d_{l2}\rangle) = \frac{1}{4V}\sum_{\mathbf{k}}\left(\frac{|U|}{2E_1(\mathbf{k})} + \mathrm{sgn}(U)\right) \quad (19)$$

is the staggered magnetization carried by the ground state [recall that $d_1 = d_{\leftarrow(\rightarrow)}$ in sublattices $A(B)$, respectively]. We see that the ground state is spin polarized for any $U \neq 0$. The spins are fully polarized in the $U \to \infty$ limit and the spin polarization approaches zero when $U \to -\infty$. The singular behavior of magnetization at $U \to 0$ reflects the singular nature of our Hamiltonian in the $\delta \to 0$ limit where all d_2 quasiparticles are localized.

Besides quasiparticle excitations d_1, we may create *solitonic* excitations by flipping D_l's from the ground state. The energy of a single soliton excitation E_{sol} is obtained by calculating the "ground state" energy of Hamiltonian Eq. (14) with a singly flipped D_l. We have performed this calculation numerically in a square lattice for various values of U/t and the results are shown in Fig. 2. We note that the excitation energy is proportional to U^2 at small U but is proportional to $\tilde{t}^2/|U|$ for $|U| \gg \tilde{t}$. Physically, the soliton excitation is created by adding (or subtracting) a localized d_2 fermion with a dressed cloud of d_1 fermions. The charge and spin carried by the soliton is calculated using Eq. (8a) and we find that the soliton is chargeless and carries spin 1 for $U \geq 5\tilde{t}$ and spin 0 for $U \leq -5\tilde{t}$. There exists also a small region around $|U| \leq 5\tilde{t}$ where the spin is $1/2$ [see Fig. 2]. Physically, the soliton is a bound state between the d_2 particle and d_1 hole when $U/5\tilde{t}$ is large and positive and is a bound state between d_2 hole and d_1 hole when $U/5\tilde{t}$ is large and negative. The $d_{1(2)}$ fermions are unbounded when $|U| \leq 5\tilde{t}$. Consequently, we expect that the soliton is a boson (spin = 0, 1) when $|U| \geq 5\tilde{t}$ and is a spin-1/2 fermion when $|U| \leq 5\tilde{t}$.

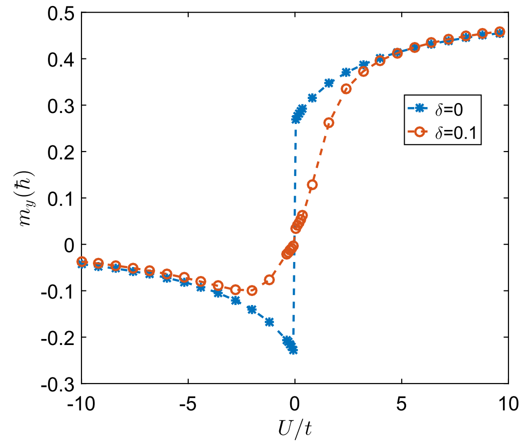

FIG. 3. The staggered magnetization in the $\hat{y}$ direction m_y as a function of U for different δ's. The blue-star and orange-circle lines are for $\delta = 0$, 0.1 respectively.

To study the $\delta \to 0$, $U \to 0$ region more carefully we show in Fig. 3 the ground state staggered magnetization magnitude as a function of $U/\tilde{t}$ in our square lattice model with $\delta = 0.0$ and 0.1 computed using a mean-field approximation

$$\begin{aligned}\left(n_{l\rightarrow}^{(d)} - \frac{1}{2}\right)\left(n_{l\leftarrow}^{(d)} - \frac{1}{2}\right) &\to \left(n_{l\rightarrow}^{(d)} - \frac{1}{2}\right)\left\langle\left(n_{l\leftarrow}^{(d)} - \frac{1}{2}\right)\right\rangle \\ &+ \left\langle\left(n_{l\rightarrow}^{(d)} - \frac{1}{2}\right)\right\rangle\left(n_{l\leftarrow}^{(d)} - \frac{1}{2}\right) \\ &- \left\langle\left(n_{l\rightarrow}^{(d)} - \frac{1}{2}\right)\right\rangle\left\langle\left(n_{l\leftarrow}^{(d)} - \frac{1}{2}\right)\right\rangle \\ &= -i(c_{l,\uparrow}^\dagger c_{l,\downarrow}\mathrm{Im}(\langle c_{l,\downarrow}^\dagger c_{l,\uparrow}\rangle) + \xi_l c_{l,\uparrow}^\dagger c_{l,\downarrow}^\dagger \mathrm{Im}(\langle c_{l,\downarrow} c_{l,\uparrow}\rangle)) \\ &+ \mathrm{H.c.},\end{aligned} \quad (20)$$

where we have used Eqs. (6) and (7) in deriving the last equality. The ground state expectation values $\langle(n_{l\rightleftarrows}^{(d)} - \frac{1}{2})\rangle$ are determined self-consistently from the mean field theory. The mean field result becomes exact in the limit $U = 0$ and $U/\delta \to \infty$ and the $\delta \neq 0$ mean-field calculation provides an extrapolation between the two exact limits. We find that the singular behavior of staggered magnetization m_y at $\delta = 0$ as given by Eq. (19) is smoothed out for $\delta = 0.1$. The excitation energies and spins carried by the solitions for $\delta = 0.1$ are also calculated in the mean field theory for different values of U and are shown in Fig. 2 for comparison. We see that both the excitation energies and spins carried by the solition are similar for $\delta = 0.0$ and 0.1.

Summary and discussions.—Summarizing, we introduce in this Letter an exact solvable BCS-Hubbard model in arbitrary dimensions. The construction of the exact solution is parallel to the exactly solvable Kitaev honeycomb model for $S = 1/2$ quantum spins and can be viewed as a generalization of Kitaev's construction to $S = 1/2$

interacting lattice fermions. In fact, any Hamiltonian which when represented in terms of Majorana fermions, has the form

$$H = 4i\sum_{\langle i,j\rangle,\sigma}(t_{ij}\beta_{i\sigma}\beta_{j\sigma}) + U\sum_{l}(2i\eta_{l\uparrow}\beta_{l\uparrow})(2i\eta_{l\downarrow}\beta_{l\downarrow}), \quad (21)$$

is exactly solvable following our discussion on the construction on exact solution, independent of dimension and lattice structure. The nearest neighbor hopping (t) + pairing (Δ) model on square (and cubic) lattices we consider in this Letter is just an example of a large class of exactly solvable lattice fermion models that can be written in the form Eq. (21).

Physically, the presence of the ESP pairing term Δ breaks spin rotation symmetry making one of the two quasiparticle bands completely flat in the limit $\delta \to 0$. The quasiparticles in the flat band are localized making the resulting Hamiltonian exactly solvable. The same happens in the Kitaev honeycomb model. We note that the quasiparticles are nonperturbative objects that are related to the original c-fermion states by a *local* Bogoliubov–de Gennes transformation Eq. (7) in our model. As a result the quasiparticle and solitonic excitations both carry nontrivial charge and spin quantum numbers as discussed in the main text.

We comment also that our construction of an exactly solvable model suggests a new mean field decoupling channel of Hubbard interaction Eq. (20) which can be applied to any interacting fermion model when expressed in terms of Majoranan fermions. The decoupling scheme breaks spin-rotation symmetry and becomes exact when one of the quasiparticle band becomes flat.

Experimentally, the model described by Hamiltonian Eq. (1) requires a staggered ESP potential as well as the equality between hopping term t and ESP potential Δ. One plausible way to realize these is by optical Feshbach resonance [33] in cold atom systems through which one can have a spatial control of parameters characterizing the system. However, this so far has been very difficult to achieve. We note, however, that the exactly solvable model described by Hamiltonian Eq. (21) can be obtained from other tight-binding models, our BCS-Hubbard model with a staggered ESP potential model is just one example among many. We are currently exploring the most general exactly solvable BCS-Hubbard type models that can be transformed to Hamiltonian Eq. (21) through Bogoliubov–de Gennes transformations.

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