Discrete Time-Crystalline Order in Cavity and Circuit QED Systems

Zongping Gong,[1] Ryusuke Hamazaki,[1] and Masahito Ueda[1,2]
[1]*Department of Physics, University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan*
[2]*RIKEN Center for Emergent Matter Science (CEMS), Wako, Saitama 351-0198, Japan*

(Received 8 August 2017; revised manuscript received 11 November 2017; published 25 January 2018)

Discrete time crystals are a recently proposed and experimentally observed out-of-equilibrium dynamical phase of Floquet systems, where the stroboscopic dynamics of a local observable repeats itself at an integer multiple of the driving period. We address this issue in a driven-dissipative setup, focusing on the modulated open Dicke model, which can be implemented by cavity or circuit QED systems. In the thermodynamic limit, we employ semiclassical approaches and find rich dynamical phases on top of the discrete time-crystalline order. In a deep quantum regime with few qubits, we find clear signatures of a transient discrete time-crystalline behavior, which is absent in the isolated counterpart. We establish a phenomenology of dissipative discrete time crystals by generalizing the Landau theory of phase transitions to Floquet open systems.

DOI: 10.1103/PhysRevLett.120.040404

Introduction.—Phases and phase transitions of matter are key concepts for understanding complex many-body physics [1,2]. Recent experimental developments in various quantum simulators, such as ultracold atoms [3,4], trapped ions [5,6], and superconducting qubits [7,8], motivate us to seek for quantum many-body systems out of equilibrium [9–11], such as many-body localized phases [12–17] and Floquet topological phases [18–25].

In recent years, much effort has been devoted to periodically driven (Floquet) quantum many-body systems that break the discrete time-translation symmetry (TTS) [26]. In contrast to the continuous TTS breaking [27–29] that has turned out to be impossible at thermal equilibrium [30,31], the discrete TTS breaking has been theoretically proposed [32–36] and experimentally demonstrated [37,38]. Phases with broken discrete TTS feature discrete time-crystalline (DTC) order characterized by periodic oscillations of physical observables with period nT, where T is the Floquet period and $n = 2, 3, \cdots$. The DTC order is expected to be stabilized by many-body interactions against variations of driving parameters. Note that the system is assumed to be in a localized phase [33–37] or to have long-range interactions [38–40]. Otherwise, the DTC order only exists in a prethermalized regime [41,42] since the system will eventually be heated to a featureless infinite-temperature state due to persistent driving [43–45].

While remarkable progress is being made concerning the DTC phase, most studies focus on isolated systems. Indeed, as has been experimentally observed [37,38] and theoretically investigated [46], the DTC order in an open system is usually destroyed by decoherence. On the other hand, it is known that dissipation and decoherence can also serve as resources for quantum tasks such as quantum computation [47] and metrology [48]. From this perspective, it is natural to ask whether the DTC order exists and can even be stabilized in open systems [49]. Such a possibility has actually been pointed out in Ref. [41], but neither a detailed theoretical model nor a concrete experimental implementation is presented.

In this Letter, we propose a concrete open-system setup for realizing the DTC order by using a prototypical dissipative model—a modified open Dicke model [50–52], which describes a collective atom-photon interaction in the presence of interaction modulation and photon loss. This model is relevant to cavity QED systems based on cold atoms [53–56] and circuit QED systems based on superconducting qubits [57–63]. As schematically illustrated in Fig. 1, the DTC order manifests itself through periodic switch-on and switch-off of a sufficiently strong atom-photon coupling. For the cavity QED

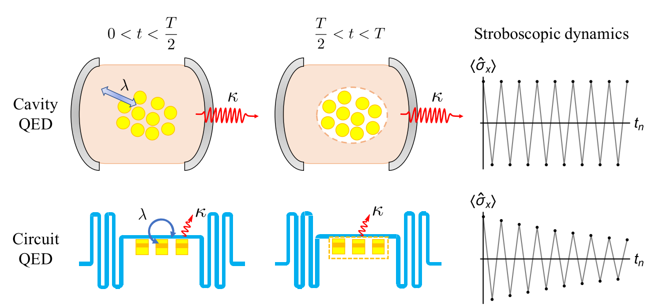

FIG. 1. Cavity and circuit QED setups for realizing the DTC order. In the first (second) half of a Floquet period T, we switch on (off) the coupling λ between photons and (artificial) atoms. For sufficiently large λ, almost persistent DTC order in the stroboscopic dynamics of a local observable is expected for an ensemble of a large number of atoms in an optical cavity, while transient DTC behavior can be observed for few superconducting qubits coupled to a microwave transmission line. Here κ denotes the loss rate of (microwave) photons.

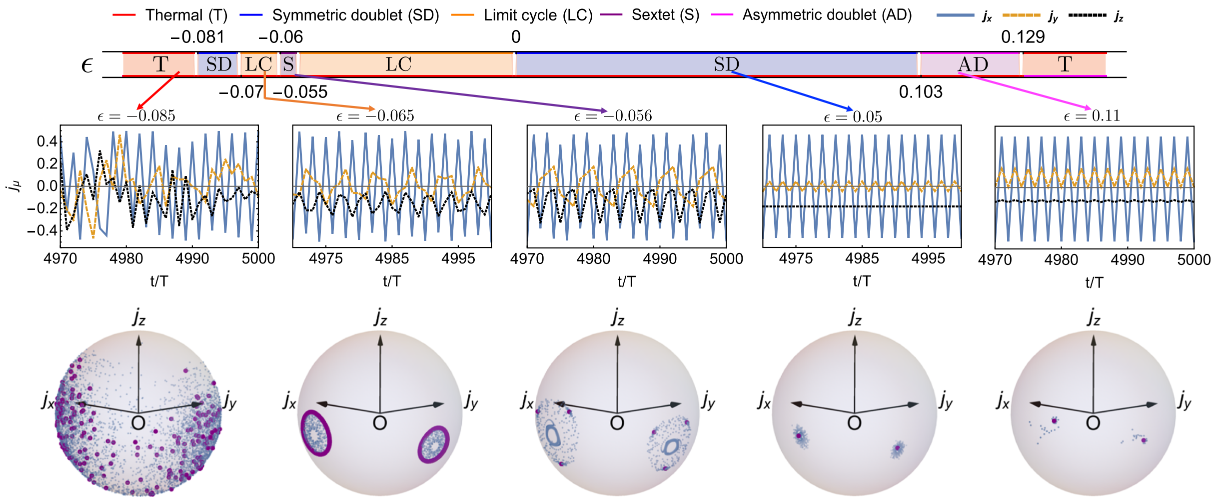

FIG. 2. Dynamical phase diagram (top), typical stroboscopic dynamics (middle), and stroboscopic trajectories (bottom) of the atomic pseudospin for atom-photon coupling $\lambda = 1$ and photon-loss rate $\kappa = 0.05$. Top: As the detuning ϵ [see Eq. (3)] is varied, five different dynamical phases emerge: thermal (T, red), symmetric period doubling (normal DTC order, SD, blue), limit-cycle pair (LC, orange), period sextupling (S, purple), and asymmetric period doubling (AD, magenta). The phase boundaries are marked in white with resolution 10^{-3}. Middle: Typical stroboscopic dynamics of $j_\mu \equiv \langle \hat{J}_\mu \rangle / N$ [$\mu = x$ (solid blue), y (dashed orange), z (dotted black)] for the last 30 periods of the entire 5000-period evolution. Bottom: Full stroboscopic phase-space-point trajectories (light blue) and those of the last 200 periods (purple) projected on the pseudospin Bloch sphere.

case, we consider the thermodynamic limit and find unexpectedly rich dynamical phases as the detuning parameter is varied (see Fig. 2). For the circuit QED case, we examine a deep quantum regime with few qubits to find a clear transient DTC behavior even for two qubits, a minimal setup of superradiance [60]. We also discuss a phenomenological model that demonstrates the exponentially long lifetime of the DTC order. These predictions should be testable in light of the state-of-the-art experimental developments in atomic, molecular, and optical physics.

Modulated open Dicke model.—We consider N identical two-level atoms in a single-mode cavity. Neglecting the atomic motional degrees of freedom, the dynamics of the system can be described by the open Dicke model [64]:

$$\frac{d\hat{\rho}_t}{dt} = \mathcal{L}(\lambda)\hat{\rho}_t = -i[\hat{H}(\lambda), \hat{\rho}_t] + \kappa \mathcal{D}[\hat{a}]\hat{\rho}_t,$$
$$\hat{H}(\lambda) = \omega \hat{a}^\dagger \hat{a} + \omega_0 \hat{J}_z + \frac{2\lambda}{\sqrt{N}}(\hat{a} + \hat{a}^\dagger)\hat{J}_x, \tag{1}$$

where $\mathcal{D}[\hat{a}]\hat{\rho} \equiv \hat{a}\,\hat{\rho}\,\hat{a}^\dagger - \frac{1}{2}\{\hat{a}^\dagger \hat{a}, \hat{\rho}\}$, $\hat{a}$ is the annihilation operator of the photon field, $\hat{J}_\mu \equiv \frac{1}{2}\sum_{j=1}^{N} \hat{\sigma}_j^\mu$ ($\mu = x, y, z$) is the collective atomic pesudospin operator, and ω, ω_0, λ, and κ are the optical frequency, the atomic frequency, the coupling strength, and the photon-loss rate, respectively.

It is known that, in the thermodynamic limit and when λ exceeds $\lambda_c = \frac{1}{2}\sqrt{(\omega_0/\omega)(\omega^2 + \kappa^2/4)}$, the open Dicke model exhibits a phase transition that breaks the $\mathbb{Z}_2$ symmetry characterized by the parity operator $\hat{P} \equiv e^{i\pi(\hat{a}^\dagger \hat{a} + \hat{J}_z + (N/2))}$ [52,54]. For $\lambda > \lambda_c$, we can construct an exact period-doubling Floquet dynamics as follows: Starting from one of the symmetry-broken steady states $\hat{\rho}_{ss}$, in the first-half period, the dynamics is governed by Eq. (1), so $\hat{\rho}_{ss}$ stays unchanged by definition. In the second-half period, we perform the parity operation on the system, so that the other steady state $\hat{\rho}'_{ss} = \hat{P}\hat{\rho}_{ss}\hat{P}$ is obtained at the end of the Floquet period. If we observe the system stroboscopically at $t_n = nT$, we should find $\hat{\rho}_{ss}$ ($\hat{\rho}'_{ss}$) for even (odd) n.

If the period doubling is robust against imperfection such as the deviation of the evolution in the second-half period from the parity operation, we can identify it as a DTC order. A straightforward way to introduce such imperfection is to switch off the atom-photon coupling in the second-half period. That is, we modulate λ in Eq. (1) periodically as

$$\lambda_{t+T} = \lambda_t = \begin{cases} \lambda & 0 \le t < \frac{T}{2}; \\ 0 & \frac{T}{2} \le t < T. \end{cases} \tag{2}$$

In the resonant ($\omega = \omega_0 = \omega_T \equiv 2\pi/T$) and isolated ($\kappa = 0$) case, the state evolution during the second half of the period generates the parity operator up to an unimportant global phase, i.e., $\hat{P} = e^{-i(T/2)\hat{H}(0) + (i\pi/2)N}$. If we introduce a detuning between ω and ω_0 as

$$\omega = (1 - \epsilon)\omega_T, \qquad \omega_0 = (1 + \epsilon)\omega_T, \tag{3}$$

we can control the degree of imperfection by ϵ. Note that there is always a *nonunitary* imperfection due to photon loss even for $\epsilon = 0$. For simplicity, we set $\omega_T = 1$ in the following discussion.

Dynamical phases in the thermodynamic limit.—In the thermodynamic limit $N \to \infty$, the relative fluctuation in a local observable becomes negligible and the semiclassical approach is justified [65–67]. In terms of the scaled variables $x \equiv \langle \hat{a} + \hat{a}^\dagger \rangle / \sqrt{2N\omega}$, $p \equiv i\langle \hat{a}^\dagger - \hat{a} \rangle / \sqrt{2N/\omega}$ and $\mathbf{j} \equiv (j_x, j_y, j_z)$ with $j_\mu \equiv \langle \hat{J}_\mu \rangle / N$ ($\mu = x, y, z$), the semiclassical dynamics governed by Eq. (1) reads [68]

$$\begin{aligned} \frac{d\boldsymbol{j}}{dt} &= (\omega_0 \boldsymbol{e}_z + 2\lambda_t \sqrt{2\omega} x \boldsymbol{e}_x) \times \boldsymbol{j}, \\ \frac{dx}{dt} &= p - \frac{\kappa}{2} x, \\ \frac{dp}{dt} &= -\omega^2 x - \frac{\kappa}{2} p - 2\lambda_t \sqrt{2\omega} j_x. \end{aligned} \tag{4}$$

Note that the $\mathbb{Z}_2$ symmetry is maintained, since Eq. (4) is invariant under the simultaneous sign reversal of x, p, j_x, and j_y. The dissipative phase transition [69] in the open Dicke model now becomes a dynamical phase transition known as the pitchfork bifurcation [66], where the original unique attractor with $x_0 = p_0 = j_{x0} = j_{y0} = 0$ and $j_{z0} = \frac{1}{2}$ becomes unstable and two new stable attractors with $(j_{x\pm}, j_{y\pm}, j_{z\pm}) = \frac{1}{2}(\pm\sqrt{1-\mu^2}, 0, -\mu)$ and $(x_\pm, p_\pm) = \mp[\sqrt{2\omega(1-\mu^2)}/(\omega^2 + \kappa^2/4)](\lambda, \kappa/2)$ ($\mu \equiv \lambda_c^2/\lambda^2$) emerge as the classical limits from $\hat{\rho}_{ss}$ and $\hat{\rho}'_{ss}$, which are nothing but the steady-state solutions to Eq. (1). To be specific, we fix $\lambda = 1$ and $\kappa = 0.05$ in the following calculations and choose the initial state to be the "+" attractor.

We solve the nonlinear differential equation (4) up to 5000 periods by using the Runge-Kutta method for different ϵ and map out the full dynamical phase diagram in the top row of Fig. 2 [70]. We find the normal DTC phase and the thermal phase, where the former respects the $\mathbb{Z}_2$ symmetry in which j_x, j_y, x, p reverse their signs after one period, and the latter shows irregular trajectories that cover some areas of the pseudospin sphere [or in the quadrature (x-p) plane]. Furthermore, we find symmetric limit-cycle pairs, where the steady orbit forms two closed loops in the phase space, period sextupling, and asymmetric period doubling, with j_x, j_y, x, p taking on two different values that are not symmetric against inversion. In fact, we find even richer dynamical phases for other κ, such as higher-order period multipling and asymmetric limit-cycle pairs [68]. These phases can unambiguously be diagnosed by a measure of synchronization [71–73] and can systematically be understood by employing bifurcation theory [74–79].

We note that the dynamics of a generalized *time-independent* open Dicke model, which has an additional Stark-shift term $(U/N)\hat{J}_z \hat{a}^\dagger \hat{a}$ in $H(\lambda)$ in Eq. (1), has thoroughly been studied in Ref. [66] on the basis of the semiclassical analysis. While there are only single- (normal) and double-attractor (superradiant) phases for $U = 0$, limit-cycle and multiple-attractor phases emerge for $U \neq 0$. In contrast, in this Letter, the richness of dynamical phases arises from the *time dependence* of λ with $U = 0$. Another distinction is that in Ref. [66] the steady state picks up one of the attractors or the unique limit cycle, whereas in the present Letter the steady state goes around different fixed points or limit cycles in a stroboscopic manner.

Transient DTC behavior in the deep quantum regime.—Let us move on to the few-atom regime [$N \sim O(1)$], which is the case for circuit QED systems. We consider the modulated open Dicke model with $N = 2$. We demonstrate that the interplay between strong coupling and dissipation causes a DTC behavior for unexpectedly long periods even in this deep quantum regime. By unexpectedly long we mean that the DTC transient lasts much longer than the decay time $\kappa^{-1} \sim 3T$.

We employ the exact diagonalization approach to solving the Floquet-Lindblad dynamics governed by Eqs. (1) and (2) under a truncation at 16 photons. Figure 3(a) shows the obtained stroboscopic dynamics of the scaled angular momenta j_μ and quadratures x, p (inset) in the strong-coupling regime, where $\kappa = 0.05$, $\epsilon = 0.1$ and $\lambda = 1$. The initial state is chosen to be $|\Rightarrow\rangle \otimes |0\rangle$, where $|\Rightarrow\rangle \equiv \otimes_{j=1}^N |\rightarrow\rangle$ is the eigenstate of $\hat{J}_x$ with eigenvalue $N/2 (N = 2)$ and $|0\rangle$ is the photon vacuum. We clearly see that j_x and x start oscillating with a period of $2T$ after $t \sim 5T$, which persists even at $t \sim 50T$. This result shows that our strong-coupling

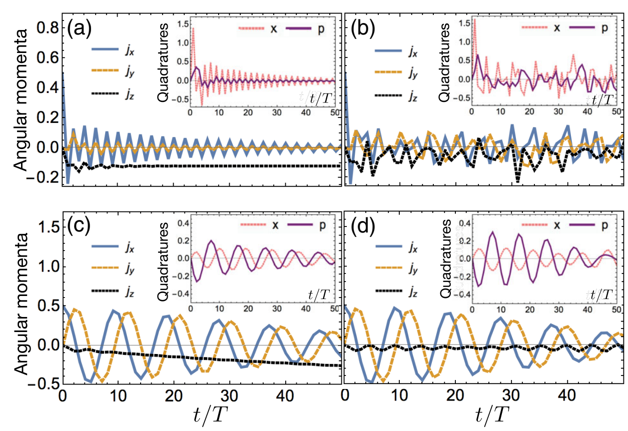

FIG. 3. (a) Stroboscopic dissipative dynamics of the scaled angular momenta of j_x (solid), j_y (dashed), and j_z (dotted) in the two-qubit Dicke model with $\kappa = 0.05$, $\epsilon = 0.1$, and $\lambda = 1$ (strong coupling). The inset shows quadratures, x (dotted) and p (solid). (b) Stroboscopic dynamics for isolated systems ($\kappa = 0$, $\epsilon = 0.1$, and $\lambda = 1$) in the strong-coupling regime. (c) Stroboscopic dissipative dynamics in the weak-coupling regime ($\kappa = 0.05$, $\epsilon = 0.1$, and $\lambda = 0.1$). (d) Stroboscopic unitary dynamics in the weak-coupling regime ($\kappa = 0$, $\epsilon = 0.1$, and $\lambda = 0.1$). Only (a) shows a DTC transient. The initial state is always $|\Rightarrow\rangle \otimes |0\rangle$, where $|\Rightarrow\rangle$ is the eigenstate of $\hat{J}_x$ with eigenvalue $N/2 (N = 2)$ and $|0\rangle$ is the photon vacuum.

modulated open Dicke model features a DTC transient even in the deep quantum regime before reaching the stationary state. For the sake of comparison, we show in Fig. 3(b) the stroboscopic dynamics for an isolated Dicke model ($N = 2$, $\kappa = 0$, $\epsilon = 0.1$, $\lambda = 1$) starting from the same initial state. We can see that the expectation value of each observable randomly fluctuates and does not have temporal order in contrast to its dissipative counterpart.

We note that no DTC transient emerges in the weak-coupling regime. Figure 3(c) shows the Floquet dynamics for an open ($\kappa = 0.05$) Dicke model with $\epsilon = 0.1$ and $\lambda = 0.1$. The low-frequency oscillation has a period around T/ϵ which is susceptible to detuning ϵ. This is similar to the observation that the DTC order is fragile in noninteracting spin systems [36,37]. A similar dynamics is found in a weakly coupled isolated Dicke system ($\kappa = 0$, $\epsilon = 0.1$, and $\lambda = 0.1$) as shown in Fig. 3(d). Thus, neither photon loss nor strong coupling alone gives rise to the DTC transient.

Floquet-Lindblad-Landau theory.—With all the obtained numerical results in mind, we now establish a general phenomenology for such an open-system DTC. As illustrated in Fig. 4(a), the eigenvalues of the Floquet-Lindblad superoperator $\mathcal{U}_F \equiv \mathcal{T} e^{\int_0^T dt \mathcal{L}(\lambda_t)}$ generally locate inside the unit circle in the complex plane, except for the steady state which always locates at 1. Even if the initial state is a complex mixture of many eigenmodes, the state will eventually be described by fewer modes due to an exponential decay during time evolution. A semiclassical picture of this process is the convergence to attractors. When the state is described as a mixture of two eigenmodes, it can exhibit oscillatory DTC behavior with the double period if the distinguishably long-lived mode other than the steady state has a negative eigenvalue close to -1 [68].

An important question is how the lifetime of this DTC mode scales with N. A natural expectation is that it becomes *exponentially* long with increasing N, since the underlying dissipative phase transition features an exponentially small damping gap [80]. However, it is highly nontrivial to find whether this is the case even in a Floquet open system. It turns out to be difficult to handle this problem numerically in the modulated open Dicke model. This difficulty emphasizes the importance of scalable circuit-QED-based quantum simulation with up to tens of qubits [81]. Nevertheless, we can gain qualitative insights by considering a numerically tractable effective theory for the photon field:

$$\frac{d\hat{\rho}_t}{dt} = -i[\hat{H}_L(\Omega_2(t), \Omega_4(t)), \hat{\rho}_t] + \kappa \mathcal{D}[\hat{a}]\hat{\rho}_t,$$

$$\hat{H}_L(\Omega_2, \Omega_4) = \omega \hat{a}^\dagger \hat{a} - \frac{\Omega_2}{4}(\hat{a}^\dagger + \hat{a})^2 + \frac{\Omega_4}{32N}(\hat{a}^\dagger + \hat{a})^4. \quad (5)$$

These equations can be derived from the open Dicke model (1) by adiabatically eliminating the atomic degrees of freedom under specific conditions [68]. Remarkably, Eq. (5) can be regarded as the Floquet-Lindblad generalization of the scalar-field Landau theory in $0 + 1$ dimension, and it is thus expected to capture the general qualitative features of a wide class of Floquet open systems in addition to the Dicke model. In Fig. 4(b), we show the lifetime of the DTC (longest-lived) and that of the second longest-lived mode (except for the steady state) for a specific protocol $\Omega_4(t) = \Omega_2(t) = \Omega_2(t + T)$, where $\Omega_2(t) = 1.5\omega$ for $0 \le t < \pi/\omega$ and $\Omega_2(t) = 0$ for $\pi/\omega \le t < T = (2 - \varepsilon)\pi/\omega$ with $\kappa = 0.05\omega$. We do find an exponential scaling of the lifetime of the DTC order with respect to N and the saturation of the lifetime of the second longest-lived mode. Note that the lifetime of a one-dimensional many-body localized DTC obeys the same exponential scaling in the system size [82], although the mechanism of DTC order is different [33–36].

Summary and outlook.—We have proposed a simple scheme for realizing DTC order in cavity and circuit QED systems via switching on and off of the atom-photon coupling. In particular, we focus on the modulated open Dicke model both in the thermodynamic limit and in the deep quantum regime. In the former case, we find rich dynamical phases. In the latter case, we show that the interplay between dissipation and strong coupling gives rise to a clear transient DTC behavior. We have demonstrated an exponentially long lifetime of the DTC order in the Floquet-Lindblad-Landau theory. These predictions have direct experimental relevance [68].

Our model can readily be generalized by taking into account the atomic motional degrees of freedom [83], interactions between atoms [84], local decoherence, and spontaneous emission [85–87]. In particular, our study raises an intriguing question of whether an intrinsically nonunitary DTC can possess absolute stability [82] against arbitrary *nonunitary* perturbation. Further studies along this

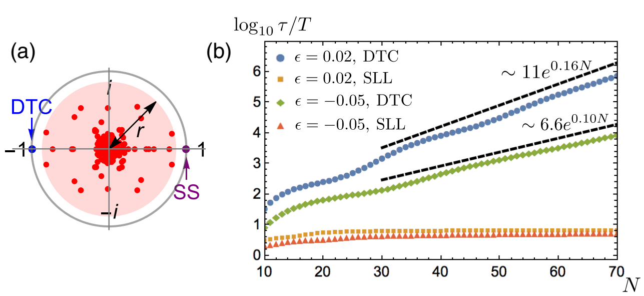

FIG. 4. (a) Typical Floquet-Lindblad spectrum of an open-system DTC. The DTC mode and the steady state (SS) locate at $-1 + \delta$ and 1, respectively, with $\delta \sim O(e^{-cN})$. The other modes locate on a disk (shaded) with radius $r < 1$ for $\forall N$, so their lifetime is bounded by a constant $-T/\ln r$. (b) Finite-size scaling for the lifetime $\tau = -T/\ln(1 - \delta)$ of the DTC and the second longest-lived (SLL) modes in the Floquet-Lindblad-Landau model (5) for $\epsilon = 0.02$ and -0.05.

line should give valuable hints for realizing a persistent DTC in the presence of realistic uncontrollable dissipation and decoherence. Another direction of research is to understand the Floquet-Lindblad spectra of other dynamical phases shown in Fig. 1. We have already made some progress on the asymmetric DTC behavior [68].

This work was supported by KAKENHI Grant No. JP26287088 from the Japan Society for the Promotion of Science, a Grant-in-Aid for Scientic Research on Innovative Areas "Topological Materials Science" (KAKENHI Grant No. JP15H05855), and the Photon Frontier Network Program from MEXT of Japan, and the Mitsubishi Foundation. Z. G. was supported by MEXT. R. H. was supported by the Japan Society for the Promotion of Science through Program for Leading Graduate Schools (ALPS) and JSPS fellowship (JSPS KAKENHI Grant No. JP17J03189).

[1] P. M. Chaikin and T. C. Lubensky, *Principles of Condensed Matter Physics* (Cambridge University Press, Cambridge, England, 2000).
[2] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, England, 2011).
[3] I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. **80**, 885 (2008).
[4] I. Bloch, J. Dalibard, and S. Nascimbène, Nat. Phys. **8**, 267 (2012).
[5] D. Leibfried, R. Blatt, C. Monroe, and D. J. Wineland, Rev. Mod. Phys. **75**, 281 (2003).
[6] R. Blatt and C. F. Roos, Nat. Phys. **8**, 277 (2012).
[7] A. A. Houck, H. E. Türeci, and J. Koch, Nat. Phys. **8**, 292 (2012).
[8] M. H. Devoret and R. J. Schoelkopf, Science **339**, 1169 (2013).
[9] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, Rev. Mod. Phys. **83**, 863 (2011).
[10] J. Eisert, M. Friesdorf, and C. Gogolin, Nat. Phys. **11**, 124 (2015).
[11] R. Moessner and S. L. Sondhi, Nat. Phys. **13**, 424 (2017).
[12] A. Pal and D. A. Huse, Phys. Rev. B **82**, 174411 (2010).
[13] R. Vosk and E. Altman, Annu. Rev. Condens. Matter Phys. **6**, 383 (2015).
[14] R. Nandkishore and D. A. Huse, Annu. Rev. Condens. Matter Phys. **6**, 15 (2015).
[15] P. Ponte, Z. Papić, F. Huveneers, and D. A. Abanin, Phys. Rev. Lett. **114**, 140401 (2015).
[16] A. Lazarides, A. Das, and R. Moessner, Phys. Rev. Lett. **115**, 030402 (2015).
[17] J.-y. Choi, S. Hild, J. Zeiher, P. Schauß, A. Rubio-Abadal, T. Yefsah, V. Khemani, D. A. Huse, I. Bloch, and C. Gross, Science **352**, 1547 (2016).
[18] T. Kitagawa, E. Berg, M. Rudner, and E. Demler, Phys. Rev. B **82**, 235114 (2010).
[19] L. Jiang, T. Kitagawa, J. Alicea, A. R. Akhmerov, D. Pekker, G. Refael, J. I. Cirac, E. Demler, M. D. Lukin, and P. Zoller, Phys. Rev. Lett. **106**, 220402 (2011).
[20] M. S. Rudner, N. H. Lindner, E. Berg, and M. Levin, Phys. Rev. X **3**, 031005 (2013).
[21] P. Titum, E. Berg, M. S. Rudner, G. Refael, and N. H. Lindner, Phys. Rev. X **6**, 021013 (2016).
[22] C. W. von Keyserlingk and S. L. Sondhi, Phys. Rev. B **93**, 245145 (2016).
[23] D. V. Else and C. Nayak, Phys. Rev. B **93**, 201103(R) (2016).
[24] A. C. Potter, T. Morimoto, and A. Vishwanath, Phys. Rev. X **6**, 041001 (2016).
[25] H. C. Po, L. Fidkowski, T. Morimoto, A. C. Potter, and A. Vishwanath, Phys. Rev. X **6**, 041070 (2016).
[26] K. Sacha and J. Zakrzewski, Rep. Prog. Phys. **81**, 016401 (2018).
[27] F. Wilczek, Phys. Rev. Lett. **109**, 160401 (2012).
[28] A. Shapere and F. Wilczek, Phys. Rev. Lett. **109**, 160402 (2012).
[29] T. Li, Z.-X. Gong, Z.-Q. Yin, H. T. Quan, X. Yin, P. Zhang, L.-M. Duan, and X. Zhang, Phys. Rev. Lett. **109**, 163001 (2012).
[30] P. Bruno, Phys. Rev. Lett. **111**, 070402 (2013).
[31] H. Watanabe and M. Oshikawa, Phys. Rev. Lett. **114**, 251603 (2015).
[32] K. Sacha, Phys. Rev. A **91**, 033617 (2015).
[33] V. Khemani, A. Lazarides, R. Moessner, and S. L. Sondhi, Phys. Rev. Lett. **116**, 250401 (2016).
[34] D. V. Else, B. Bauer, and C. Nayak, Phys. Rev. Lett. **117**, 090402 (2016).
[35] C. W. von Keyserlingk and S. L. Sondhi, Phys. Rev. B **93**, 245146 (2016).
[36] N. Y. Yao, A. C. Potter, I.-D. Potirniche, and A. Vishwanath, Phys. Rev. Lett. **118**, 030401 (2017).
[37] J. Zhang, P. W. Hess, A. Kyprianidis, P. Becker, A. Lee, J. Smith, G. Pagano, I.-D. Potirniche, A. C. Potter, A. Vishwanath *et al.*, Nature (London) **543**, 217 (2017).
[38] S. Choi, J. Choi, R. Landig, G. Kucsko, H. Zhou, J. Isoya, F. Jelezko, S. Onoda, H. Sumiya, V. Khemani *et al.*, Nature (London) **543**, 221 (2017).
[39] W. W. Ho, S. Choi, M. D. Lukin, and D. A. Abanin, Phys. Rev. Lett. **119**, 010602 (2017).
[40] A. Russomanno, F. Iemini, M. Dalmonte, and R. Fazio, Phys. Rev. B **95**, 214307 (2017).
[41] D. V. Else, B. Bauer, and C. Nayak, Phys. Rev. X **7**, 011026 (2017).
[42] T.-S. Zeng and D. N. Sheng, Phys. Rev. B **96**, 094202 (2017).
[43] L. D'Alessio and M. Rigol, Phys. Rev. X **4**, 041048 (2014).
[44] A. Lazarides, A. Das, and R. Moessner, Phys. Rev. E **90**, 012110 (2014).
[45] H. Kim, T. N. Ikeda, and D. A. Huse, Phys. Rev. E **90**, 052105 (2014).
[46] A. Lazarides and R. Moessner, Phys. Rev. B **95**, 195135 (2017).
[47] F. Verstraete, M. M. Wolf, and J. I. Cirac, Nat. Phys. **5**, 633 (2009).
[48] M. Beau and A. del Campo, Phys. Rev. Lett. **119**, 010403 (2017).
[49] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, London, 2002).
[50] R. H. Dicke, Phys. Rev. **93**, 99 (1954).

[51] K. Rzażewski, K. Wódkiewicz, and W. Żakowicz, Phys. Rev. Lett. **35**, 432 (1975).
[52] C. Emary and T. Brandes, Phys. Rev. E **67**, 066203 (2003).
[53] K. Baumann, C. Guerlin, F. Brennecke, and T. Esslinger, Nature (London) **464**, 1301 (2010).
[54] K. Baumann, R. Mottl, F. Brennecke, and T. Esslinger, Phys. Rev. Lett. **107**, 140402 (2011).
[55] H. Ritsch, P. Domokos, F. Brennecke, and T. Esslinger, Rev. Mod. Phys. **85**, 553 (2013).
[56] M. P. Baden, K. J. Arnold, A. L. Grimsmo, S. Parkins, and M. D. Barrett, Phys. Rev. Lett. **113**, 020408 (2014).
[57] A. Blais, R.-S. Huang, A. Wallraff, S. M. Girvin, and R. J. Schoelkopf, Phys. Rev. A **69**, 062320 (2004).
[58] P. Nataf and C. Ciuti, Nat. Commun. **1**, 1 (2010).
[59] O. Viehmann, J. von Delft, and F. Marquardt, Phys. Rev. Lett. **107**, 113602 (2011).
[60] J. A. Mlynek, A. A. Abdumalikov, C. Eichler, and A. Wallraff, Nat. Commun. **5**, 5186 (2014).
[61] T. Jaako, Z.-L. Xiang, J. J. Garcia-Ripoll, and P. Rabl, Phys. Rev. A **94**, 033850 (2016).
[62] M. Bamba, K. Inomata, and Y. Nakamura, Phys. Rev. Lett. **117**, 173601 (2016).
[63] F. Yoshihara, T. Fuse, S. Ashhab, K. Kakuyanagi, S. Saito, and K. Semba, Nat. Phys. **13**, 44 (2017).
[64] F. Dimer, B. Estienne, A. S. Parkins, and H. J. Carmichael, Phys. Rev. A **75**, 013804 (2007).
[65] V. M. Bastidas, C. Emary, B. Regler, and T. Brandes, Phys. Rev. Lett. **108**, 043003 (2012).
[66] M. J. Bhaseen, J. Mayoh, B. D. Simons, and J. Keeling, Phys. Rev. A **85**, 013817 (2012).
[67] R. Chitra and O. Zilberberg, Phys. Rev. A **92**, 023815 (2015).
[68] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.120.040404 for the detailed derivations, further numerical results, and the details of the experimental proposal.
[69] E. M. Kessler, G. Giedke, A. Imamoglu, S. F. Yelin, M. D. Lukin, and J. I. Cirac, Phys. Rev. A **86**, 012116 (2012).
[70] We have a finite resolution 0.001, which is the length of the white stripes that separate adjacent phases. We do not rule out the possibility to find new phases in these narrow stripes.
[71] T. E. Lee and H. R. Sadeghpour, Phys. Rev. Lett. **111**, 234101 (2013).
[72] M. R. Hush, W. Li, S. Genway, I. Lesanovsky, and A. D. Armour, Phys. Rev. A **91**, 061401(R) (2015).
[73] N. Lörch, S. E. Nigg, A. Nunnenkamp, R. P. Tiwari, and C. Bruder, Phys. Rev. Lett. **118**, 243602 (2017).
[74] R. May, Nature (London) **261**, 459 (1976).
[75] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields* (Springer, New York, 1983).
[76] Y. A. Kuznetsov, *Elements of Applied Bifurcation Theory* (Springer, New York, 1998).
[77] X. Liu and D. Xiao, Chaos, Solitons and Fractals **32**, 80 (2007).
[78] F. Flicker, arXiv:1707.09333.
[79] F. Flicker, arXiv:1707.09371.
[80] W. Casteels, R. Fazio, and C. Ciuti, Phys. Rev. A **95**, 012128 (2017).
[81] M. Fitzpatrick, N. M. Sundaresan, A. C. Y. Li, J. Koch, and A. A. Houck, Phys. Rev. X **7**, 011016 (2017).
[82] C. W. von Keyserlingk, V. Khemani, and S. L. Sondhi, Phys. Rev. B **94**, 085112 (2016).
[83] P. Domokos and H. Ritsch, Phys. Rev. Lett. **89**, 253003 (2002).
[84] J. Larson, B. Damski, G. Morigi, and M. Lewenstein, Phys. Rev. Lett. **100**, 050401 (2008).
[85] E. G. DallaTorre, S. Diehl, M. D. Lukin, S. Sachdev, and P. Strack, Phys. Rev. A **87**, 023831 (2013).
[86] E. G. DallaTorre, Y. Shchadilova, E. Y. Wilner, M. D. Lukin, and E. Demler, Phys. Rev. A **94**, 061802(R) (2016).
[87] P. Kirton and J. Keeling, Phys. Rev. Lett. **118**, 123602 (2017).