Covariant Structure of Models of Geophysical Fluid Motion

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Geophysical models approximate classical fluid motion in rotating frames. Even accurate approximations can have profound consequences, such as the loss of inertial frames. If geophysical fluid dynamics are not strictly equivalent to Newtonian hydrodynamics observed in a rotating frame, what kind of dynamics are they? We aim to clarify fundamental similarities and differences between relativistic, Newtonian, and geophysical hydrodynamics, using variational and covariant formulations as tools to shed the necessary light. A space-time variational principle for the motion of a perfect fluid is introduced. The geophysical action is interpreted as a synchronous limit of the relativistic action. The relativistic Levi-Civita connection also has a finite synchronous limit, which provides a connection with which to endow geophysical space-time, generalizing Cartan (1923). A covariant mass-momentum budget is obtained using covariance of the action and metric-preserving properties of the connection. Ultimately, geophysical models are found to differ from the standard compressible Euler model only by a specific choice of a metric-Coriolis-geopotential tensor akin to the relativistic space-time metric. Once this choice is made, the same covariant mass-momentum budget applies to Newtonian and all geophysical hydrodynamics, including those models lacking an inertial frame. Hence, it is argued that this mass-momentum budget provides an appropriate, common fundamental principle of *dynamics*. The postulate that Euclidean, inertial frames exist can then be regarded as part of the Newtonian theory of *gravitation*, which some models of geophysical hydrodynamics slightly violate.

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Geophysical fluid dynamics study hydrodynamics in rotating frames. Insight is gained from geophysical models that *approximate* classical fluid motion in rotating frames. However, even accurate approximations can have profound, unexpected consequences. Consider, for instance, inviscid fluid motion under the equatorial β-plane approximation

$$Ds/Dt = 0,$$

$$\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0,$$

$$\frac{Du}{Dt} - \beta y v + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0,$$

$$\frac{Dv}{Dt} + \beta y u + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0,$$

$$\frac{Dw}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial z} = -g,$$

where D/Dt is the material derivative, x, y, and z are Cartesian coordinates, u, v, and w are their material derivatives (velocity components), g is gravity, $\beta = 2\Omega/a$, with a and Ω being the planetary radius and rotation rate, and $p(\rho, s)$, ρ, and s are the fluid pressure, density, and entropy per unit mass. In this system, because the Coriolis parameter βy varies linearly with y (the equator is at $y = 0$ and latitude $\simeq y/a$), no change of reference frame can cancel the Coriolis force. Therefore, inertial frames do not exist. If geophysical fluid dynamics are not strictly equivalent to Newtonian hydrodynamics observed in a rotating frame [1], what kind of dynamics are they?

In order to discuss similarities and differences between physical models, it is useful to characterize their formal structure. To this end, variational principles and covariant formulations are powerful tools. Least action principles have been presented for relativistic [2] and Newtonian [3] flows. In addition to their variational structure, Newtonian hydrodynamics possess a space-time covariant formulation closely parallel to that of relativistic hydrodynamics, based on the Milne-Cartan structure of space-time [4–6]. Geophysical models have been formulated from a least action principle [7–9] and, recently, as a space-time covariant mass-momentum budget involving a connection generalizing the Milne-Cartan structure [1]. However, this mass-momentum budget and space-time connection could not be related to a space-time invariant action, as in relativistic hydrodynamics [6].

This Letter aims to clarify fundamental similarities and differences between relativistic, Newtonian, and geophysical hydrodynamics, especially how their action, space-time structure, and mass-momentum budget relate. For this, a variational principle starting from a space-time devoid of any geometrical structure is devised. The geophysical action is interpreted as a synchronous limit of the relativistic action. Applying the same limit to the Levi-Civita connection of relativistic space-time, a connection is obtained that endows the space-time of geophysical models. Furthermore, covariance of the geophysical action leads to a covariant mass-momentum budget. These results suggests a slight redefinition of the classical distinction between dynamics and gravitation, such that Newtonian and geophysical fluid dynamics share identical dynamics but differ slightly in their specification of gravitation.

Variational principle for all hydrodynamics.—Space-time coordinates are noted as x^μ, $\mu = 0, \ldots, 3$. Fluid flow (relativistic or not) can be described by $M \geq 0$ scalars q_X, $X = 1, \ldots, M$ and $N > 0$ currents U^μ_Y, $Y = 1, \ldots, N$, where to each kind of particle or chemical species there corresponds either one scalar or one current. In addition, one scalar or current is associated with entropy. Depending on the situation, it may be preferable to treat all species equally, as currents, or to single out one of them to define a single current, all other species and entropy being described as scalars. In the presence of a single fluid, all currents are colinear. Furthermore, assuming that no process transforms one species into another, all currents are assumed nondivergent, and all scalars to be transported by the currents:

$$\partial_\mu U^\mu_Y = 0, \qquad \partial_\mu (q_X U^\mu_Y) = 0. \tag{1}$$

The dynamics will derive from the action $\mathcal{S}[q_X, U^\mu_Y] = \int l(q_X, U^\mu_Y) d^4x$, where the Lagrangian density $l(q_X, U^\mu_Y)$ is a function of the scalars q_X and currents U^μ_Y to be specified later. The action varies according to

$$\delta\mathcal{S}[U_Y, q_X] = \int \left(\pi^Y_\mu \delta U^\mu_Y + \frac{\partial l}{\partial q_X} \delta q_X \right) d^4x, \tag{2}$$

where $\pi^Y_\mu \equiv \partial l / \partial U^\mu_Y$ are momenta. In the sequel, δq_X and δU^μ_Y have compact support, so that any boundary terms that would appear after integration by parts vanish.

When a Lagrangian description of the flow is adopted [2], the action varies due to variations of the fluid parcel world lines. These variations translate into Eulerian variations of fields q_X, U^Y in Lie form [6]:

$$\begin{aligned} \delta q_X &= \xi^\mu \partial_\mu q_X, \\ \delta U^\alpha_Y &= \xi^\beta \partial_\beta U^\alpha_Y - U^\beta_Y \partial_\beta \xi^\alpha + U^\alpha_Y \partial_\beta \xi^\beta, \end{aligned} \tag{3}$$

where ξ^μ is an arbitrary vector field related to the displacement of fluid parcel world lines. Standard properties of the Lie derivative imply that such variations preserve Eq. (1). Injecting Eq. (3) into $\delta\mathcal{S} = 0$ and integrating by parts yields the Carter-Lichnerowicz equations of motion

$$U^\beta_Y (\partial_\beta \pi^Y_\alpha - \partial_\alpha \pi^Y_\beta) - \frac{\partial l}{\partial q_X} \partial_\alpha q_X = 0. \tag{4}$$

Relativistic and geophysical actions.—At this point, we can specify the Lagrangian density $l(q_X, U^\mu)$ such that relativistic or geophysical hydrodynamics are recovered from Eq. (4). In a relativistic context, the fluid velocity u^μ is normalized so that $g_{\mu\nu} u^\mu u^\nu = -c^2$, with $g_{\mu\nu}$ being the space-time metric and c an arbitrary, constant velocity, and the flow carries scalars n_Y defined as particle or molecule numbers per unit volume, except the scalar $n_1 = S$ associated with entropy, which defines entropy per unit volume. The usual convention $c = 1$ is not used here because the limit $c \to \infty$ will be needed later. All species are described through their associated current defined by $U^\mu_Y = J n_Y u^\mu$, with $J = J_{\rm rel} = (1/c)\sqrt{-\det g_{\mu\nu}}$. Conversely, n_Y can be obtained from U_Y as

$$n_Y \equiv \sqrt{-g_{\mu\nu} U^\mu_Y U^\nu_Y} / cJ. \tag{5}$$

The equation of state of the fluid is described by the total energy per unit volume $E(n_Y)$. $dE = \mu^Y dn_Y$, with μ^Y chemical potentials (except $\mu^1 = T$, which is the temperature). Defining the action as minus the space-time integral of energy [6],

$$\mathcal{S}_{\rm rel} = -\int E\left(\sqrt{-g_{\mu\nu} U^\mu_Y U^\nu_Y} / cJ\right) J d^4x, \tag{6}$$

Eq. (4) yields $n_Y u^\beta (\partial_\beta \pi^Y_\alpha - \partial_\alpha \pi^Y_\beta) = 0$, with momenta $\pi^Y_\mu = c^{-2} \mu^Y u_\mu$.

Let us now turn to nonrelativistic motion. In this context, it would make the most sense to use time as the coordinate x^0, but for the sake of covariance we introduce absolute time as a scalar $t(x^\mu)$ in an arbitrary coordinate system and define $t_\mu \equiv \partial_\mu t$. When $dx^0 = dt$, we will say that the coordinate system is classical. In this particular case, $t_\mu = \delta^0_\mu$. In addition to $t(x^\mu)$, a metric-Coriolis-geopotential tensor $H_{\mu\nu}$ is introduced whose role is to define free-fall trajectories as those that minimize the action:

$$\mathcal{S}_{\rm fall}[x^\mu(\lambda)] = \frac{1}{2} \int H_{\mu\nu} u^\mu u^\nu dt, \qquad u^\mu \equiv \frac{dx^\mu}{dt}, \tag{7}$$

where $x^\mu(\lambda)$ is a world line such that $t_\mu dx^\mu \neq 0$. Notice that $u^\mu t_\mu = 1$ by design; hence, $u^0 = 1$ in classical coordinates. For example, with

$$d\sigma^2 \equiv H_{\mu\nu} dx^\mu dx^\nu = -2V dt^2 + dx^2 + dy^2 + dz^2, \tag{8}$$

the Euler-Lagrange equations $(d/dt)(H_{\mu\nu} u^\nu) = \frac{1}{2} u^\alpha u^\beta \partial_\mu H_{\alpha\beta}$ yield the Newtonian free-fall equations in a Cartesian, inertial frame (t, x, y, z) in the presence of a gravitational field $V(x, y, z, t)$ [1].

For geophysical applications, $d\sigma^2$ is reexpressed in a rotating frame and approximated [9]. The resulting model admits a global, Euclidean inertial frame if it is possible, by a change of coordinates, to transform $d\sigma^2$ back to Eq. (8).

Sometimes such a transformation is impossible because approximations introduce three-dimensional curvature [10] or destroy global inertial frames [11]. A classical model exemplifying the latter is the equatorial β-plane approximation

$$d\sigma^2 = -2gzdt^2 + dx^2 + dy^2 + dz^2 - \beta y^2 dxdt. \quad (9)$$

The cross term $-\beta y^2 dxdt$ is responsible for the Coriolis force and the lack of an inertial frame. Therefore, for the sake of generality it is important to look beyond Eq. (8) and consider a general $H_{\mu\nu}$.

The volume form Jd^4x is given the covariant definition

$$J = J_{\text{geo}} \equiv \sqrt{\lim_{\varepsilon\to 0} \det\left(\varepsilon^{-1} t_\mu t_\nu + \varepsilon H_{\mu\nu}\right)}, \quad (10)$$

which, in classical coordinates, yields $J = \sqrt{\det H_{ij}}$, with $i,\ j = 1,\ 2,\ 3$. Indeed H_{ij} is then the nonsingular 3×3 spatial metric tensor.

The fluid and its motion are described by N transported scalars q_X and a single current representing the mass flux $U^\mu = J\rho u^\mu$, where ρ is the total mass per unit volume. The scalars are mixing ratios (in kg/kg), except for q_1, which is specific entropy. Energetics are specified through the specific internal energy $e(v, q_Y)$, such that $de = -pdv + \chi^Y dq_Y$, where $v = 1/\rho$ is the specific volume, p is the pressure, and χ^Y is the chemical potential per unit mass (with the exception of $\chi^1 = T$).

From U^μ one recovers $\rho = t_\mu U^\mu/J$, $u^\mu = U^\mu/\rho J$. The action is then defined as

$$\mathcal{S}_{\text{geo}} = \int \left[\frac{1}{2} H_{\mu\nu} \frac{U^\mu U^\nu}{t_\alpha U^\alpha} - J\epsilon\left(\frac{t_\beta U^\beta}{J}, q_X\right)\right] d^4x, \quad (11)$$

where $\epsilon(\rho, q_X) = \rho e$. $\delta\mathcal{S}_{\text{geo}} = 0$ generates the equations of motion

$$u^\beta(\partial_\beta \pi_\alpha - \partial_\alpha \pi_\beta) + \chi^X \partial_\alpha q_X = 0,$$
$$\text{where} \quad \pi^\mu = H_{\mu\nu} u^\nu - t_\mu\left(e + \frac{p}{\rho} + \frac{1}{2} H_{\alpha\beta} u^\alpha u^\beta\right).$$

As only three of the four above equations are independent, it is natural in classical coordinates to keep $\alpha = i = 1, 2, 3$:

$$\partial_t \pi_i + u^j(\partial_j \pi_i - \partial_i \pi_j) - \partial_i \pi_0 + \chi^X \partial_i q_X = 0, \quad (12)$$

where $-\pi_0 = K + e + p/\rho$, with $K = \frac{1}{2} H_{\alpha\beta} u^\alpha u^\beta - H_{0\nu} u^\nu$.

In an inertial, Cartesian frame, Eq. (11) coincides with the well-established action for a perfect fluid [3,12] and Eq. (12) is Crocco's theorem. More generally, in a classical, rotating frame, with curvilinear Eulerian coordinates, Eq. (11) coincides with the action considered in Ref. [9] and Eq. (12) reduces to their Eq. (18).

Having derived relativistic and geophysical hydrodynamics from a common least action principle, we now wish to relate their two different actions. For this, consider a family of space-time metrics $g_{\mu\nu}(c)$. If $d\tau = c^{-1}\sqrt{-g_{\mu\nu} dx^\mu dx^\nu} \to dt$ as $c \to \infty$ for some scalar field $t(x^\mu)$, time intervals measured by clocks along world lines with identical end points become independent of the world line; i.e., synchronicity is recovered. This property is satisfied by

$$g_{\mu\nu} \equiv -c^2 t_\mu t_\nu + H_{\mu\nu}. \quad (13)$$

Notice that Eq. (13) is also obeyed to $O(c^{-2})$ in the weak-gravity, slow velocity limit [13]. However, in that limit, $g_{\mu\nu}(c)$ obeys Einstein's equations, which constrain $H_{\mu\nu}$, while $g_{\mu\nu}$ and $H_{\mu\nu}$ in Eq. (13) are *not* constrained by gravitational equations.

Since $d\tau = \sqrt{1 - c^{-2} H_{\mu\nu} u^\mu u^\nu}\, dt$, the least action principle $\delta \int d\tau = 0$ defining space-time geodesics tends to that defining the classical free-fall trajectories:

$$-c^2 \delta \int d\tau = \delta \int \frac{1}{2} H_{\mu\nu} u^\mu u^\nu dt + O(c^{-2}).$$

Indeed, the $O(c^2)$ term $c^2 \int dt$ has vanishing variations because the end points are kept fixed. Similarly, $J_{\text{rel}} \to J_{\text{geo}}$ as $c \to \infty$.

Now define the total energy in Eq. (6) as $E = c^2 m^Y n_Y + \epsilon$, with m^Y being the mass per particle or molecule ($m^1 = 0$). Then $dE = \mu^Y dn_Y$, with $\mu^Y = m^Y(c^2 + \chi^Y)$ (with the exception of $\mu^1 = T$). As $c \to \infty$, the $O(c^2)$ term of S_{rel} is again a boundary term $m^Y \int \partial_\alpha (tU_Y^\alpha) d^4x$. Variations of the relativistic action therefore reduce to $\delta\mathcal{S}_{\text{rel}} = \delta\mathcal{S}_{\text{geo}} + O(c^{-2})$.

Mass-momentum budget.—Thus far, relativistic and geophysical hydrodynamics have been cast as Eq. (4) with properly defined actions, and the geophysical action has been interpreted as a synchronous limit of the relativistic action. Besides Eq. (4), another fundamental formulation of relativistic hydrodynamics is in the form of a mass-momentum budget,

$$D_\mu T^{\mu\nu} = 0,$$

where $T^{\mu\nu}$ is the mass-momentum stress tensor and D_μ is the covariant derivative associated with the Levi-Civita connection $\Gamma^\mu_{\alpha\beta} \equiv \frac{1}{2} g^{\mu\nu}(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta})$. We finally use Noether's theorem to derive a similar budget for geophysical and Newtonian hydrodynamics. Indeed, action (11) is invariant under arbitrary changes of space-time coordinates, i.e., if all its arguments, including the background fields J, t_μ, $H_{\mu\nu}$, have variations in Lie form:

$$\delta J = \partial_\gamma J + J\partial_\gamma \xi^\gamma,$$
$$\delta t_\mu = \xi^\alpha \partial_\alpha t_\mu + t_\alpha \partial_\mu \xi^\alpha,$$
$$\delta H_{\mu\nu} = \xi^\beta \partial_\beta H_{\mu\nu} + H_{\alpha\nu}\partial_\mu \xi^\alpha + H_{\mu\alpha}\partial_\nu \xi^\alpha.$$

Now, since the variations are in Lie form, one may replace ∂_μ with the covariant derivative ∇_μ associated with any torsion-free connection $\gamma^\alpha_{\mu\nu} = \gamma^\alpha_{\nu\mu}$. As shown below, $\Gamma^{\mu\nu}$ has a finite limit as $c \to \infty$ in Eq. (13). This provides a natural connection with which to endow geophysical space-time. Since the least action principle implies that the variation $\delta\mathcal{S}$ due to q_X, U_Y vanishes, what remains after integration by parts is

$$-\nabla_\alpha\left(2\frac{\partial l}{\partial H_{\alpha\beta}}H_{\gamma\beta} + t_\gamma\frac{\partial l}{\partial t_\alpha} + J\frac{\partial l}{\partial J}\delta^\alpha_\gamma\right) + \frac{\partial l}{\partial J}\nabla_\gamma J + \frac{\partial l}{\partial H_{\alpha\beta}}\nabla_\gamma H_{\alpha\beta} + \frac{\partial l}{\partial t_\alpha}\nabla_\gamma t_\alpha = 0, \qquad (14)$$

where J is formally considered an independent argument of l. Using identities (17) and (18) derived below, Eq. (14) simplifies into mass-momentum budget (20).

Assuming Eq. (13), some algebra yields

$$g^{\mu\nu} = G^{\mu\nu} - (c^2 + 2V)^{-1} e^\mu e^\nu, \qquad (15)$$

where $V(x^\alpha)$ is such that $G_{\mu\nu} \equiv H_{\mu\nu} + 2V t_\mu t_\nu$ is singular, $G^{\mu\nu}$ is the pseudoinverse of $G_{\mu\nu}$ such that $G^{\mu\nu} t_\mu = 0$, and e^μ is the null vector of $G_{\mu\nu}$ such that $e^\nu t_\nu = 1$. As a result, $H_{\mu\nu}G^{\nu\sigma} = G_{\mu\nu}G^{\nu\sigma} = \delta^\sigma_\mu - e^\sigma t_\mu$. Equation (15) shows that as $c \to \infty$, $g^{\mu\nu} \to G^{\mu\nu}$, a rank-3 tensor that defines a spatial metric on the three-dimensional slices $t(x^\mu) = cst$. More algebra shows that $\Gamma^\mu_{\alpha\beta} \to \gamma^\mu_{\alpha\beta}$, with

$$\gamma^\mu_{\alpha\beta} \equiv \frac{1}{2}G^{\mu\nu}(\partial_\alpha H_{\nu\beta} + \partial_\beta H_{\alpha\nu} - \partial_\nu H_{\alpha\beta}) + e^\mu \partial_{\alpha\beta} t. \qquad (16)$$

The covariant derivative D_μ associated with $\Gamma^\mu_{\alpha\beta}$ preserves the metric $D_\gamma g_{\alpha\beta} = 0$, $D_\gamma g^{\alpha\beta} = 0$, $D_\gamma\sqrt{\det - g_{\mu\nu}} = 0$. Furthermore, $D_\alpha(g^{\gamma\delta} g_{\gamma\beta}) = 0$ and $G^{\gamma\delta}H_{\gamma\beta} = \delta^\delta_\beta - t_\beta e^\delta$. Letting $c \to \infty$ in those identities yields

$$\nabla_\mu t_\alpha = 0, \qquad \nabla_\mu G^{\alpha\beta} = 0, \qquad \nabla_\mu J = 0, \qquad (17)$$

$$G^{\gamma\delta}\nabla_\alpha H_{\gamma\beta} + t_\beta \nabla_\alpha e^\delta = 0. \qquad (18)$$

Contracting Eq. (14) with $G^{\gamma\delta}$ and using Eqs. (17) and (18) yields

$$\nabla_\alpha T^{\alpha\delta} = e^\delta \nabla_\alpha (t_\beta T^{\alpha\beta}),$$
$$\text{where } T^{\mu\nu} \equiv \frac{2}{J}\frac{\partial l}{\partial H_{\mu\nu}} + \frac{\partial l}{\partial J}G^{\mu\nu} \qquad (19)$$

defines the mass-momentum stress tensor. Finally, the expression of l implied by Eq. (11) yields the desired mass-momentum budget:

$$T^{\mu\nu} = \rho u^\mu u^\nu + pG^{\mu\nu}, \qquad \nabla_\mu T^{\mu\nu} = 0, \qquad (20)$$

where $t_\beta T^{\alpha\beta} = \rho u^\alpha$, $J\nabla_\alpha(t_\beta T^{\alpha\beta}) = \nabla_\alpha U^\alpha = 0$ has been used.

Summary and discussion.—A least action principle valid for all hydrodynamics has been identified. Relativistic hydrodynamics derive from action (6), while geophysical models, possibly lacking inertial frames, derive from action (11), interpreted as a synchronous limit of Eq. (6). Relativistic dynamics involve a space-time metric which defines a Levi-Civita connection. Similarly, geophysical models involve a metric-Coriolis-geopotential tensor $H_{\mu\nu}$ which defines a connection (16). This new family generalizes the Newton-Cartan connection [4–6], obtained in the special case (8). A covariant mass-momentum budget (20) for geophysical models has been derived from the covariance of the action. In terms of applications, Eq. (12) is at the heart of recent energy-conserving numerical schemes [14,15]. Following Ref. [16], discretizing $\delta\mathcal{S} = 0$ itself may also be possible. These results call for some final remarks.

Geometry of geophysical space-time.—Unlike in Ref. [17], covariance is not restricted to transforms between classical coordinates. This full covariance allows the use of Noether's theorem to derive the mass-momentum budget (20), from which gravitational, Coriolis, and centrifugal forces are apparently absent. Indeed, Eq. (20) treats them not as forces but as a geometric property of space-time. The corresponding source terms are included (hidden) in the covariant derivative.

Mass-momentum stress tensor.—In the relativistic case, Eq. (14) holds with $g_{\mu\nu}$ in lieu of $H_{\mu\nu}$ and without t_μ, so that preservation of the metric by D_μ is enough to yield $D_\mu T^{\mu\nu} = 0$ for an arbitrary Lagrangian. In the geophysical case, the covariance of the action and the metric-preserving properties (17) and (18) of the connection alone are not enough, and the Lagrangian cannot be arbitrary for Eq. (20) to hold: it must be such that $t_\beta \partial l/\partial H_{\alpha\beta}$ be nondivergent. This requirement is related to Eq. (11) being the limit of Eq. (6) as $c \to \infty$. For this limit to exist, the $O(c^2)$ term must be a pure boundary term, which keeps the $O(c^2)$ term of $E(n_Y)$ linear in n_Y, implying in turn that the $c \to \infty$ limit of $E(n_Y)$ is of the classical form $K - P$, with $JK = H_{\alpha\beta}U^\alpha U^\beta/(t_\mu U^\mu)$ being the kinetic energy and P some function of $t_\mu U^\mu/J$, q_Y.

Dynamics vs gravitation.—Usually, the laws of motion of a point mass or a fluid subject to gravity are conceptually split into a fundamental principle of dynamics (FPD) and a theory of gravity. In a classical context, the standard FPD states that mass times acceleration equals force in inertial frames, while the law of gravity specifies the force. In a relativistic context, action (6) governs dynamics, while Einstein's equations specify $g_{\mu\nu}$.

Using the present results, the relativistic distinction between dynamics and gravitation can be transposed to a classical context, building upon and extending Ref. [4]. Indeed, since Eq. (11) is the synchronous limit of Eq. (6), it makes sense to regard Eqs. (16) and (20) as the general form of the classical FPD, to be complemented by a theory of gravity which fully specifies $H_{\mu\nu}$. For Newtonian gravity, it would include the law for the gravitational potential *and* the postulate that inertial frames exist. Drawing the boundary between dynamics and gravitation this way differs from the standard way, where the postulate about inertial frames is part of the FPD.

From this point of view, both geophysical and Newtonian hydrodynamics are governed by the classical FPD [Eqs. (16) and (20)], but in space-times obeying different laws of gravity, namely, geophysical hydrodynamics being allowed to slightly deviate from the Newtonian laws of gravity.

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