ment has found agreement with the predictions of QED, in both slope and normalization, at values of q_m^2 about an order of magnitude larger than that previously attained in the pair-production process.

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THEORY OF RADIATIVE CORRECTIONS FOR NON-ABELIAN GAUGE FIELDS

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In a series of unpublished researches Feynman has investigated the problems which arise when one attempts to quantize a field possessing a non-Abelian infinite-dimensional invariance group (e.g., Yang-Mills field; gravitational field). In particular, he has posed the question: Is it possible, in perturbation theory, to calculate the radiative corrections for such a field while simultaneously (1) carrying out a consistent renormalization scheme, (2) maintaining manifest covariance, and (3) securing unitarity of the S matrix?

Feynman found that he could do these things in lowest order, i.e., for diagrams involving only single closed loops, by removing the cyclic products of retarded (or advanced) Green's functions from the loops, thereby breaking them open, and then rearranging the result into sets of associated tree diagrams having their external lines on the mass shell. He next restricted the summations involving external lines to the "transverse" or "physical" quantum states, and proved that the sum of all the thus restricted tree diagrams of given order is group invariant. This theorem is important, because for the fields in question "manifest covariance" means "group invariance" (or "gauge invariance") and not merely Lorentz invariance. Indeed, in the case of the gravitational field the theorem can be extended to arbitrary Riemannian manifolds and not merely to quasi-Minkowskian space-time.

The author has published an alternative version[1] of Feynman's results which removes some of the complexity of the theory and introduces perhaps enough formal elegance to justify its appearing in print. In particular, it shows how the "spurious particles" which Feynman was compelled to introduce, and which have been regarded by some as the least attractive element of his theory, arise in a perfectly natural way. They appear when the effort is made to cast the theory—which is known to be covariant because of the tree theorem but has its covariance obscured through the explicit use of the transverse states—into a manifestly covariant form. Manifest covariance is essential for correct renormalization, but covariant propagators necessarily propagate nonphysical as well as physical quanta around the closed loops. To compensate for the unwanted quanta one must associate with each closed loop another closed loop, with a different set of covariant propagators and vertices, which may be regarded as respectively describing the propagation of a set of fictitious particles and the interaction of these particles with real quanta. For the Yang-Mills field the spurious particles have spin zero; for gravity they have spin one. In principle, they are needed even in electrodynamics, but in that special case the associated vertices vanish owing to the Abelian character of the gauge group, and the spurious particles consequently never interact with anything.

In dealing with the problem of higher order radiative corrections, in which overlapping loops must be considered, Feynman was less successful in formulating a consistent set of calculational rules. The technique of removing cyclic products of Green's functions and collecting the resulting tree diagrams into complete sets no longer works, and the complexity of the difficulties has frustrated further progress in this direction. It is the purpose of this note to indicate briefly how, by taking a slightly different approach, the difficulties can be overcome. A more complete account of the method will appear elsewhere.

The chief formal tool for the purpose consists of the introduction of a "background" field satisfying the classical dynamical equations. By manipulating this field one may convert all physical questions into questions about the vacuum-to-vacuum amplitude[2]

$$\langle 0,\infty | 0,-\infty\rangle \equiv e^{i\Gamma}. \tag{1}$$

This technique replaces the more familiar one employing external sources and avoids the well-known difficulties which arise in using sources with gauge fields.

In functional integral form Eq. (1) becomes

$$e^{i\Gamma} = N\int \exp\left[i\left(S+\frac{1}{2!}S_{,ij}\varphi^i\varphi^j + \frac{1}{3!}S_{,ijk}\varphi^i\varphi^j\varphi^k+\cdots\right)\right]\delta\varphi, \tag{2}$$

where the classical action S and its functional derivatives are regarded as functionals of the background field. In operator language this is equivalent to expressing the total field operator in the form

$$\tilde{\varphi}^i = \varphi^i + \tilde{\varphi}^i, \tag{3}$$

where φ^i is the background field. (The notation is the same as in reference 1,[3] except that φ^i is here real and satisfies the classical equations $S_{,i}=0$, while the symbol G^{ij} will be used to denote the bare propagator; also, the boldface symbol of that reference is replaced by a tilde.)

When no infinite-dimensional invariance group is present, Eq. (2) immediately yields the graphical expansion depicted in Fig. 1. The solid lines denote propagators G^{ij} and the vertices denote bare vertex functions $S_{,ij}$, $S_{,ijkl}$, etc. The simple circle denotes the familiar logarithm of a Fredholm determinant. Feynman's investigations may be summed up by the statement that when an infinite-dimensional invariance group is present, the circle diagram must be replaced by two circles, one of which represents the propagation (by a covariant propagator) of longitudinal and scalar as well as transverse quanta, while the other represents the propagation of the compensatory spurious particles. The value to be assigned to the two circles taken together is

$$\ln\left[\frac{\det(G^{ij})}{\det(G_0{}^{ij})}\,\frac{\det(G_0{}^{\alpha\beta})^2}{\det(G^{\alpha\beta})^2}\,\frac{\det(G_0{}^{-ij})}{\det(G^{-ij})}\,\frac{\det(G^{-\alpha\beta})^2}{\det(G_0{}^{-\alpha\beta})^2}\right], \tag{4}$$

an expression which must be discussed in some detail.

$G^{\alpha\beta}$ is the propagator for the spurious particles and is a Green's function for the operator

$$F_{\alpha\beta} \equiv R^i{}_\alpha \gamma_{ij} R^j{}_\beta, \tag{5}$$

where the functions $R^i{}_\alpha$ are the quantities which enter into the infinitesimal group transformation

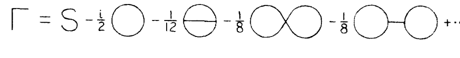

FIG. 1. Graphical representation of vacuum processes when no infinite dimensional invariance group is present.

law for the field φ^i:

$$\delta\varphi^i = R^i{}_\alpha \delta\xi^\alpha \quad (\delta\xi^\alpha = \text{group parameter}), \tag{6}$$

and where γ_{ij} is a symmetric continuous matrix which may be used, together with $R^i{}_\alpha$, to fix supplementary conditions on small disturbances (see reference 1) but which for present purposes need merely be required to satisfy the group transformation law

$$\delta\gamma_{ij} \equiv \gamma_{ij,k} R^k{}_\alpha \delta\xi^\alpha$$
$$= -(\gamma_{kj} R^k{}_{\alpha,\, i} + \gamma_{ik} R^k{}_{\alpha,\, i})\delta\xi^\alpha, \tag{7}$$

so as to assure manifest covariance of the theory. (It is usually chosen to be a tensor delta function.)

When the group is Abelian, the first functional derivative of $R^i{}_\alpha$ with respect to φ^j vanishes, and in all cases, owing to the linearity of the group transformation laws involved, the second derivatives vanish. The $R^i{}_\alpha$ satisfy the identity

$$R^i{}_{\alpha,j} R^j{}_\beta - R^i{}_{\beta,j} R^j{}_\alpha = R^i{}_\gamma c^\gamma{}_{\alpha\beta}, \tag{8}$$

where the $c^\gamma{}_{\alpha\beta}$ are the structure constants of the group.

Since $S_{,ij}$ is now singular (it satisfies $S_{,ij}R^j{}_\alpha = 0$), one must replace it by a nonsingular operator in order to obtain a well-defined propagation law for the field quanta. A suitable choice is

$$F_{ij} \equiv S_{,ij} + \gamma_{ik} R^k{}_\alpha \gamma^{\alpha\beta} R^l{}_\beta \gamma_{lj}, \tag{9}$$

where $\gamma^{\alpha\beta}$ is a nonsingular symmetric continuous matrix having the group transformation law

$$\delta\gamma^{\alpha\beta} \equiv \gamma^{\alpha\beta}{}_{,i} R^i{}_\gamma \delta\xi^\gamma = (c^\alpha{}_{\gamma\delta}\gamma^{\delta\beta} + c^\beta{}_{\gamma\delta}\gamma^{\alpha\delta})\delta\xi^\gamma. \tag{10}$$

(It too is usually chosen to be a tensor delta function.) G^{ij} is the propagator for F_{ij}.

G^{-ij} and $G^{-\alpha\beta}$ are the retarded Green's functions for the operators F_{ij} and $F_{\alpha\beta}$. (Advanced Green's functions would do just as well.) Their presence in expression (4) corresponds to the removal of the cyclic products which arise when external lines are inserted into the circle diagrams. The subscript zero on half of the G's provides a necessary zero point for the logarithm and indicates that these G's are to be evaluated with vanishing background field in flat space-time.

Evidently the operators $F_{\alpha\beta}$ and F_{ij}, and hence the Green's functions $G^{\alpha\beta}$, G^{ij}, $G^{-\alpha\beta}$, and G^{-ij}, are not uniquely determined, since they depend on arbitrary choices for the continuous matrices γ_{ij} and $\gamma^{\alpha\beta}$, which are restricted only by the easily fulfilled conditions (7) and (10).[4] It is not difficult to show that expression (4) is nevertheless invariant under changes in the γ's. The exponent 2 associated with the Green's functions $G^{\alpha\beta}$ and $G^{-\alpha\beta}$ proves to be essential to the demonstration, and in fact no other combination of the G's would be similarly invariant.

This suggests that an easy way to discover the correct expressions for the higher order radiative corrections is to add to the diagrams of Fig. 1 other topologically similar diagrams, involving the spurious particles in all possible ways, each with an arbitrary coefficient, and then to adjust the coefficients so that the total expression becomes invariant under changes in the γ's. This procedure indeed works, and a sample result of it is shown in Fig. 2, which includes the radiative corrections of second order as well as the first. In the process of verifying the invariance of the sum of these diagrams one is easily convinced that the procedure gives unique results to all orders and that there is no ambiguity about the choice of coefficients. Moreover, all diagrams of a given order must be considered together; they are not separately invariant. Finally, although the verification that this procedure yields a unitary S matrix is very tedious and has been carried out explicitly only up to the second order, there can be little doubt that unitarity holds in general.

The solid lines in Fig. 2 represent the propagator G^{ij}, and the dashed lines represent $G^{\alpha\beta}$. The vertices at which the solid lines meet are the functional derivatives of S, and the vertices at which a solid line meets a dashed line have the value $R^j{}_\alpha \gamma_{jk} R^k{}_{\beta,i}$, which vanishes when the group is Abelian.[5] Since these vertices are not symmetric in α and β, flags have been affixed to them to show their orientation. Note that not all algebraically distinguishable orientations appear.

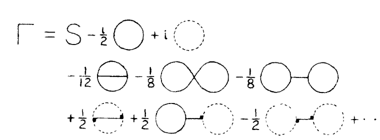

FIG. 2. Graphical representation of vacuum processes when an infinite dimensional invariance group is present. The dashed lines represent the spurious particles, and the flags indicate the orientation of the asymmetric vertices.

In order to secure unitarity of the S matrix it turns out that the maximum possible number of cyclic products of retarded (or advanced) Green's functions must be removed from the diagrams. Moreover, the removal must be carried out in a way which gives equal weight to both dashed and solid lines and is symmetric with respect to the flags. (Remember that $G^{-\alpha\beta}$ is not symmetric.) In simple field theories, in which the field components commute with one another at the same space-time point and the field couplings involve no derivatives, removal of the cyclic products is superfluous; they drop out automatically, and standard theorems assure the unitarity of the resultant S matrix. The fact that they must be forcibly removed in the present case is related to the oft-discussed difficulty of finding a consistent factor ordering for the operator dynamical equations of non-Abelian gauge fields. On the other hand, the fact that the simple removal procedure suffices for the attainment of consistency outside the operator framework testifies to the essential awkwardness of the operator viewpoint. Indeed, it is difficult to translate either the removal procedure or the use of the spurious particles back into operator language. Whether or not it is desirable ultimately to derive the present calculational rules from a tradiational operator formalism is perhaps a matter of taste.

In proving the invariance of Fig. 2 under variations of the γ's one must make use of the identities

$$R^i{}_{\alpha,i}=0, \quad c^\beta{}_{\alpha\beta}=0 \tag{11}$$

$$S_{,ij\cdots k}R^k{}_\alpha = -S_{,kj\cdots}R^k{}_{\alpha,i} - S_{,ik\cdots}R^k{}_{\alpha,j} - \cdots, \tag{12}$$

$$R^i{}_\alpha G^{\alpha\beta} = G^{ij}\gamma_{jk}R^k{}_\alpha\gamma^{\alpha\beta}. \tag{13}$$

Equations (11) hold in the case of the Yang-Mills field because the generating Lie group is necessarily compact, and in the case of the gravitational field because space-time has no metric-independent preferred directions.[6] Equations (13) hold only when the background field equations $S_{,i}=0$ are satisfied. This means that φ^i can be varied only subject to $S_{,i}=0$. However, this does not prevent one from obtaining correct scattering amplitudes by differentiating with respect to the φ's. Thus, for example, the complete amplitude for n interacting quanta, including all radiative corrections, is obtained by attaching initial- and final-state wave functions to the expression

$$\left(\frac{\delta}{\delta\Delta\varphi_0{}^{i_1}}\cdots\frac{\delta}{\delta\Delta\varphi_0{}^{i_n}}\Gamma[\varphi+\Delta\varphi]\right)_{\Delta\varphi_0=0}, \tag{14}$$

where $\Delta\varphi_0{}^i$ satisfies $S_{,ij}\Delta\varphi_0{}^j=0$ and where[7]

$$\Delta\varphi^i = \Delta\varphi_0{}^i + G^{ij}(2!^{-1}S_{,jkl}\Delta\varphi^k\Delta\varphi^l + 3!^{-1}S_{,jklm}\Delta\varphi^k\Delta\varphi^l\Delta\varphi^m + \cdots). \tag{15}$$

No matter what choice we make for the γ's in defining G^{ij}, it can be shown[1] that $\varphi^i+\Delta\varphi^i$ will satisfy the classical dynamical equations if φ^i does. Hence all the invariance properties which hold for $\Gamma[\varphi]$ hold also for $\Gamma[\varphi+\Delta\varphi]$.

The author wishes to acknowledge his indebtedness to Professor Feynman for discussions of his unpublished research, which have been of the greatest help in bringing a long and difficult personal project to a successful conclusion. With the formal difficulties connected with the quantization of non-Abelian gauge fields now solved in a manifestly covariant way, one can begin to consider the deeper questions which go beyond the iterative schemes on which the formal methods are based. (For example: Does gravity provide its own cutoff?)

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[1]B. S. DeWitt, 1963 Les Houches Lectures (Gordon and Breach, New York, 1964).

[2]The vacua $|0,-\infty\rangle$ and $|0,\infty\rangle$, corresponding to remote past and future, respectively, are "relative vacua" (i.e., relative to the background field) and hence are not identical.

[3]Functional differentiation with respect to field variables is denoted by a comma followed by one or more indices. The indices themselves do double duty as discrete labels for the field components and as continuous labels over the points of space-time. The summation convention for repeated indices is thus to be understood as including integrations over space-time.

[4]Even these conditions are not needed except to assure the manifest covariance of renormalization pro-

cedures.

[5]Vertices at which three or more dashed lines meet do not occur because $G^{\alpha\beta}$ depends only on the background field φ^i and not on the state of the spurious particles. Vertices such as $R^k{}_{\alpha,i}\gamma_{kl}R^l{}_{\beta,j}$, at which more than one solid line meets a dashed line, occur only when external lines are inserted; they do not occur for the vacuum diagrams themselves since they do not cancel with anything when the γ's are varied.

[6]Both $R^i{}_{\alpha,i}$ and $c^\beta{}_{\alpha\beta}$, when written explicitly in a given case, involve meaningless divergent quantities. Their transformation laws, however, are well defined. In the case of the general coordinate transformation group they are covariant vector densities.

[7]The iterative solution of Eq. (15) yields all tree diagrams.