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ROLE OF COMPRESSIBILITY IN THE MAGNETOHYDRODYNAMICAL STABILITY OF THE DIFFUSE PINCH DISCHARGE

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In the approximation of infinite conductivity and zero Larmor radius, the stability of a cylindrical diffuse pinch discharge against displacements of the form $\vec{\xi}\exp i[\omega t - (m/r)\theta - k_z z]$ has been widely studied using the energy principle of Bernstein et al.[1] It is well known that minimization of the energy integral ξ_θ, ξ_z in all cases, except the doubly special one of modes satisfying $\vec{k}\cdot\vec{B}=0$ in a shearless magnetic field $[(d/dr)(rB_z/B_\theta)=0]$, leads to the conditions

$$\nabla\cdot\vec{\xi}=0, \tag{1}$$

$$\xi_{\perp k}=\frac{2ik_z B_\theta \xi_r}{rkk_{\parallel}B}, \tag{2}$$

where $\vec{k}\equiv(m/r)\vec{i}_\theta+k_z\vec{i}_z$ and where here and below the subscripts $\parallel$ and $\perp$ denote vector components in the directions of $\vec{B}$ and $(\vec{B}\times\vec{r})$ and the subscripts k and $\perp k$ denote components in the directions of $\vec{k}$ and $(\vec{r}\times\vec{k})$.

Since the minimization is carried out without applying any constraint, the conditions (1) and (2) will satisfy the plasma equations only in the special case of marginal stability. In fact (1) and (2) can be obtained directly by putting $\omega=0$ in the θ and z components of the equation of motion,

$$-\omega^2\rho\vec{\xi}=(1/4\pi)[(\nabla\times\delta\vec{B})\times\vec{B}+(\nabla\times\vec{B})\times\delta\vec{B})]-\nabla\delta p, \tag{3}$$

after substitution from the other plasma equations

$$\delta\vec{B}=\nabla\times(\vec{\xi}\times\vec{B}), \tag{4}$$

$$\delta p=-\xi_r(dp/dr)-\gamma p\nabla\cdot\vec{\xi}. \tag{5}$$

That $\nabla\cdot\vec{\xi}=0$ is zero at marginal stability was first pointed out by Shafranov.[3] Because of this result and the mathematical simplification it leads to, workers have generally used the incompressibility approximation when calculating growth rates.

In the case of a shearless magnetic field, if $k_{\parallel}=0$ the minimization of the energy integral yields

$$\nabla\cdot\vec{\xi}=\frac{2B_\theta^{\,2}\xi_r}{4\pi r[(B^2/4\pi)+\gamma p]} \tag{6}$$

and the discontinuity in $\nabla\cdot\vec{\xi}$ as $k_{\parallel}$ passes from a small nonzero value to zero leads to a discontinuity in the stability condition.[4,5] For small nonzero values of $k_{\parallel}$, stability requires approximately

$$\frac{dp}{dr}>0, \tag{7}$$

whereas for $k_{\parallel}=0$, the condition is

$$\frac{dp}{dr}\geqslant-\frac{2\gamma pB_\theta^{\,2}}{4\pi r[(B^2/4\pi)+\gamma p)]}. \tag{8}$$

The pressure gradients observed experimentally for moderately constricted discharges[5] are close to the equality in (8) and not to (7).

A corresponding discontinuity occurs when the energy principle is applied to the stability of a magnetized plasma supported by gravity,[6,7] but in Newcomb's derivation of the growth rates[6] with compressibility included, the discontinuity disappears. In the work reported here, an analysis similar to that of Newcomb is applied to the cylindrical pinch discharge.

By taking the components of (3) parallel to $\vec{k}$ and $(\vec{r}\times\vec{k})$, substituting from (4) and (5), ξ_k and $\xi_{\perp k}$ can be eliminated yielding

$$\nabla\cdot\vec{\xi}=\omega^2\rho_0\left\{\frac{2k_z k_\perp B_\theta B\xi_r}{4\pi r k^2}-\frac{[(k_\parallel{}^2B^2/4\pi)-\omega^2\rho]}{k^2 r}\frac{\partial}{\partial r}(r\xi_r)\right\}$$

$$\times\left[\frac{\gamma p k_\parallel{}^2B^2}{4\pi}-\omega^2\rho\left(\frac{B^2}{4\pi}+\gamma p\right)+\frac{\omega^4\rho^2}{k^2}\right]^{-1}. \quad (9)$$

Putting $\omega=0$ gives either Eq. (1) or (6) depending on whether $k_\parallel$ is nonzero or zero. In addition, for $k_\parallel\neq 0$, as conditions depart from marginal stability $\nabla\cdot\vec{\xi}$ becomes comparable with the interchange value [Eq. (6)] when $|\omega^2|\sim k_\parallel{}^2\gamma p/\rho$.

A further interesting result is the component of Eq. (3) parallel to $\vec{B}$ which reduces to

$$\omega^2\rho\xi_\parallel=ik_\parallel\gamma p\nabla\cdot\vec{\xi}, \quad (10)$$

showing that motion paralled to $\vec{B}$ is directly dependent on $\nabla\cdot\vec{\xi}$ when $\omega\neq 0$.

The results (9) and (10) are general. Turning to the particular case of a moderately constricted discharge ($B_\theta\lesssim B_z$) with little magnetic shear, if the plasma extends to a conducting wall the unstable modes will satisfy $k_\parallel\ll k$. In addition it is assumed that $|\omega^2|\ll k^2B_\theta{}^2/4\pi\rho$ and $|(1/r\xi_r)\times(d/dr)(r\xi_r)|\ll k$. The latter is a reasonable approximation for $m>1$ but is only a rough approximation for $m=1$. With these approximations, equation (9) takes the simple form

$$\nabla\cdot\xi=\left(-\frac{\omega^2\rho B_\theta{}^2\xi_r}{2\pi r}\right)\left[\gamma p\frac{k_\parallel{}^2B^2}{4\pi}-\omega^2\rho\left(\frac{B^2}{4\pi}+\gamma p\right)\right]^{-1}. \quad (11)$$

The other equations required are the curl of Eq. (3) in the direction of $\vec{k}$ and the component of (4) parallel to $\vec{B}$; these reduce to

$$\omega^2\rho k^2\xi_r=\frac{2k_z{}^2B\delta B_\parallel}{4\pi r}+O\left(\frac{kk_\parallel B\delta B}{4\pi r}\right), \quad (12)$$

$$\delta B_\parallel=\frac{4\pi\xi r}{B}\left(\frac{dp}{dr}+\frac{B_\theta{}^2}{2\pi r}\right)-B\nabla\cdot\vec{\xi}-ik_\parallel B\xi_\parallel. \quad (13)$$

These equations show clearly the two main destabilizing mechanisms. ω^2 is made negative by a negative value of $\delta B_\parallel$ and, in (13), assuming $dp/dr+B_\theta{}^2/2\pi\simeq[(-B^2/4\pi B_z)(dB_z/dr)]>0$, the two terms which can lead to a negative $\delta B_\parallel$ are those containing $\nabla\cdot\vec{\xi}$ and $\xi_\parallel$. The former involves the interchange mechanism, i.e., the expansion of plasma as it moves out into a region of lower $(B^2/8\pi+p)$. The latter gives the destabilizing effect of plasma motion along a tube of force from the region where the tube has moved in and been compressed towards the region where it has moved out and expanded. From (10) it is seen that the $\xi_\parallel$ effect will dominate for $|\omega^2|<k_\parallel{}^2\gamma p/\rho$ and $\nabla\cdot\vec{\xi}$ for $|\omega^2|>k_\parallel{}^2\gamma p/\rho$.

Solving Eqs. (10) to (13) yields a quadratic in ω^2. If dp/dr is appreciably different from the equality in (8), there is one large and one small root; these are given by

$$\omega_1{}^2\simeq\frac{2B_\theta{}^2}{r\pi B^2}\left[\frac{dp}{dr}+\frac{\gamma pB_\theta{}^2}{2\rho r(B^2/4\pi+\gamma p)}-O\left(\frac{k_\parallel B^2}{4\pi rk}\right)\right], \quad (14)$$

$$\omega_2{}^2=\frac{\gamma pk_\parallel{}^2}{\rho}\left(\frac{B^2/4\pi}{B^2/4\pi+\gamma p}\right)\left[\frac{dp}{dr}-O\left(\frac{k_\parallel B^2}{4\pi rk}\right)\right]$$

$$\times\left[\frac{dp}{dr}+\frac{\gamma pB_\theta{}^2}{2\pi r(B^2/4\pi+\gamma p)}\right]^{-1}. \quad (15)$$

($\beta=8\pi p/B^2$ is assumed to be of the order unity.) For $dp/dr<-\gamma pB_\theta{}^2/2\pi r(B^2/4\pi+\gamma p)$ the large root (ω_1) is unstable and the small root stable. For $dp/dr>-\gamma pB_\theta{}^2/2\pi r(B^2/4\pi+\gamma p)$ the large root is stable and the small root is stable or unstable depending approximately whether dp/dr is greater or less than 0.

Through the intermediate region, as dp/dr decreases, the unstable small root changes continuously into the unstable large root and the stable large root becomes the stable small root. The only place where ω is zero is where ω_2 passes through zero. (In fact, the numerator in the expression for ω_2 is proportional to the energy principle integrand obtained after minimization for ξ_θ, ξ_z.)

Thus for pressure gradients in the range

$$0>\frac{dp}{dr}>-\frac{\gamma pB_\theta{}^2}{2\pi r(B^2/4\pi+\gamma p)}, \quad (16)$$

the discharge will be unstable to a mode which has small growth rate and small $\nabla\cdot\vec{\xi}$. The instability is driven predominantly by motion parallel to $\vec{B}$. The growth rate is zero for $k_{\parallel}=0$ and will be largest for a small positive $k_{\parallel}$. For negative pressure gradients beyond this range, the growth rate increases more rapidly with $|dp/dr|$, and the values of ω^2, $\nabla\cdot\vec{\xi}$ and $\delta\vec{B}$ are approximately the same as for a pure interchange ($k_{\parallel}=0$). Motion parallel to $\vec{B}$ is small and unimportant. Newcomb's term[6] "quasi-interchange" is appropriate for this mode.

In the range given by (16), since ω is small, finite Larmor radius effects will be important. A simple calculation involving only the Hall effect shows that the ω_2 mode will be stabilized when the ion Larmor radius ρ_i satisfies

$$\rho_i^2\left(\frac{1}{p}\frac{dp}{dr}\right)\left(\frac{1}{B_z}\frac{dB_z}{dr}\right)\geq\frac{4k_{\parallel}^2}{k^2}, \tag{17}$$

provided $j_{\parallel}$ and $j_{\perp}$ are of the same order of magnitude. If (17) is satisfied, the pressure gradient for marginal stability will shift towards the value given by the equality in (8). This can explain why, in experiments, diffuse pinch discharges are observed to support pressure gradients of this magnitude [Eq. (8)] in the central core.[5]

The observed fluctuations, which are small for $\vec{B}$ (~10%) but large for density ($\lesssim$50%) are consistent with a large-amplitude interchange-type instability (ω_1) convecting the plasma and maintaining this gradient. (From theory $\delta B_{\parallel}/B \simeq \beta\delta p/2p$.) In addition the reduction of electrical resistance in toroidal discharges at the magic number conditions[8] given by

$$2\pi r B_z/B_\theta = L/n_1, \tag{18}$$

where L is the torus circumference and n an integer, can be explained. When (18) is satisfied, the only permissible wavelengths ($2\pi/k_z = L/n_2$) are likely to be the pure interchanges ($k_{\parallel}=0$) because of the narrow range of k_z for which a moderately constricted discharge in unstable. Because no bending of the field lines occurs when $k_{\parallel}=0$, the discharge can support a moderate radial electron temperature gradient. Since the volume of a tube of force is proportional to $1/B_z$, the plasma compression in the convection will generate a gradient

$$\frac{1}{Te}\frac{dTe}{dr}=\frac{2}{3B_z}\frac{dB_z}{dr},$$

which in fact is that required to maintain the shearless magnetic field.[9] When (18) is not satisfied, $k_{\parallel}\neq 0$ and there is bending of the field lines. Because of the large amplitude of the instability, a line of force near the center will be connected with plasma near the wall after a distance $\pi/k_{\parallel}$, so that no significant electron temperature gradient can be supported. In addition, the plasma is more unstable for $k_{\parallel}\neq 0$ and for both reasons a higher discharge resistance should result. Lastly the wave velocities with which the fluctuations are observed to propagate in the discharge core[10] agree in both sign and order of magnitude with those predicted for the quasi-interchange mode due to finite Larmor radius effects.[11]

Thus the retention of compressibility removes the discontinuity in the stability condition at $k_{\parallel}=0$ for field curvature modes as well as for gravitational modes, and it leads to stability predictions which are in closer agreement with experiment.

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