ION GYRO-RADIUS EFFECTS AND STABILIZATION OF PLASMA DISSIPATIVE MODES*

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In the wake of the instabilities found by taking into account the effects of diffusion due to collisions[1,2] in the hydromagnetic equations, effort has been made in order to find possible mechanisms for stabilization of the new interchange modes.[3] A mechanism of this type has been described in a previous paper,[4] where a stability criterion is also given. However, the problem has remained open as to whether by inserting into the theory the ion gyro-radius effects,[5-7] one could get stabilization as obtained for the interchange modes existing in the absence of collisional effects.

Here we choose a class of equilibrium configurations which do not undergo the stabilization previously found,[4] and show that the ion gyro-radius effects result in strongly slowing down the growth rates, but broaden the region where instabilities take place and do not eliminate them. Overstability appears only if no ion drift velocity exists in the equilibrium. All the results obtained in the absence of collisional effects are recovered as special cases.

The present stage of the theory would indicate that possible fusion devices should be based on having strongly sheared magnetic fields, as suggested by our criterion, rather than on having very low pressures, as suggested by the theory without dissipation,[8] in order to be stable against interchange modes.

Here we consider a sheet pinch configuration[1] so that the conclusions can be extended to more complex ones[3-9] by a proper change of variables. All equilibrium quantities are supposed to be x-dependent, the gravitational acceleration $\vec{g}$ acting in the x direction, and the magnetic field $\vec{B}$ not having x components. By equilibrium, we mean all quantities having negligible variation on any time scale less than the mass diffusion time. A set of macroscopic equations derived from the Fokker-Planck equation[10] is used. A high-temperature situation is considered[11] so that $a \gg (\mu^2/\chi)^{1/3}R$, where a is the ion Larmor radius, R is the finite width of the sheet, $\mu \equiv \mu_\perp/(\rho v_H R)$ and $\chi \equiv \eta_\parallel/(Rv_H)$ where $\mu_\perp$ is the transverse viscosity, $\eta_\parallel$ the longitudinal resistivity, ρ the mass density, and v_H the hydromagnetic velocity.

We first assume the ions have a drift velocity $v_d\vec{e}_\perp$ so that $v_d = \Omega^{-1}(p_i'/\rho + g)$, $\vec{B}\cdot\vec{B}' = \rho g - p'$, and $p_i/\rho = p_e/\rho = \text{const}$ are the only equilibrium conditions. Here we use a set of local coordinates $\vec{e}_x$, $\vec{e}_B \equiv \vec{B}/B$, and $\vec{e}_\perp \equiv \vec{e}_B \times \vec{e}_x$, and rationalized Gaussian units with $c = 1$. Moreover, $\vec{B}$ is the magnetic field, p_e and p_i the electron and ion pressures, $p = p_i + p_e$, Ω the ion gyro-frequency, and "prime" means derivative with respect to x. Let $s\vec{\xi}(\vec{r},t) = s\vec{\xi}(x)\exp(st + i\vec{k}\cdot\vec{r})$, where $\vec{k}\cdot\vec{e}_x = 0$, represent a perturbation of the ion velocity around its equilibrium value.

Instabilities take place in a small region ϵR where the material flow is decoupled from the lines of force. The region ϵR is centered around the surface $x = x_0$, where $\vec{k}\cdot\vec{B} = 0$ and $\rho' < 0$. We consider the perturbed linearized form of the macroscopic equations given in reference 10, valid inside ϵR, and expand them in ϵ as shown in reference 4. We further assume $\beta \ll 1$, in particular $\beta = O(\epsilon^\alpha)$ where $\alpha > 0$ and $\beta \equiv 2p/B^2$, $Rp'/p = O(1)$, and $g\rho/p' = O(\beta)$, so that $(\vec{B}\cdot\vec{B}')R/B^2 \sim (\vec{k}\cdot\vec{B}')R^2/B = O(\beta)$ and the quantity $D \equiv g\rho' k^2/(\vec{k}\cdot\vec{B}')^2 = O(1)$. In these conditions our stability criterion[11] $D < \beta f(\eta_\perp/\eta_\parallel)$ is not satisfied as happens for a stellarator configuration. Here f is a finite function of $\eta_\perp/\eta_\parallel$, $\eta_\perp$ being the transverse resistivity.[10] In particular,[11] for $\eta_\perp = 1.98 \times \eta_\parallel$, $f \cong 1.7$. Therefore, if we further expand in β the perturbed equations, we find that in lowest order $\vec{\nabla}\cdot\vec{\xi} = 0$ and

$$s^2\rho\tilde{\xi}_x'' = i(\vec{k}\cdot\vec{B})\tilde{B}_x'' - k^2 g\tilde{\rho}, \qquad (1)$$

$$s\tilde{B}_x = i(\vec{k}\cdot\vec{B})s\tilde{\xi}_x + \eta_\parallel \tilde{B}_x'' - i(\vec{k}\cdot\vec{v}_d)\tilde{B}_x - ik\frac{B}{\Omega p}\left[i(\vec{k}\cdot\vec{B})\tilde{B}_B - \frac{\vec{k}\cdot\vec{J}}{k}\tilde{B}_x\right], \qquad (2)$$

$$B\tilde{B}_B = -\tilde{p} = \tilde{\xi}_x p'(1 + i\vec{k}\cdot\vec{v}_d/s)^{-1}, \qquad (3)$$

$$\tilde{\rho} = \frac{\rho}{p}\tilde{p}. \qquad (4)$$

Here, $\tilde{\xi}_x \equiv \vec{\xi}\cdot\vec{e}_x$, $\tilde{B}_x \equiv \vec{B}_p\cdot\vec{e}_x$, $\tilde{B}_B \equiv \vec{B}_p\cdot\vec{e}_B$, $k^2 = k_\perp^2 + k_B^2$, $\vec{B}_p$ is the perturbed magnetic field, $\tilde{p}$ and $\tilde{\rho}$ the perturbed pressure and density, and $\vec{J}$ the equilibrium current. The ordering leading to Eqs. (1) through (4) is chosen with the criterion of obtaining the fastest modes and including simultaneously the effects of ion Larmor radius and resistivity. Therefore, $s \sim \vec{k}\cdot\vec{v}_d \sim \eta_{\parallel}(\epsilon R)^{-2} \sim \epsilon Rk(g\rho'/\rho)^{1/2}$, $kR = O(1)$, $k \approx k_{\perp}$, $\vec{k}\cdot\vec{B} \approx \vec{k}\cdot\vec{B}'(x - x_0)$. Clearly, $v_d = p'/(\Omega\rho) = av_{\rm th}p'/(2p)$, where a is the ion gyro-radius and $v_{\rm th}$ the ion thermal velocity.

After using the equilibrium conditions, Eqs. (1) through (4) reduce to

$$s(s+ikv_d)\rho\tilde{\xi}_x'' = i(\vec{k}\cdot\vec{B}')(x-x_0)\tilde{B}_x''(1+ikv_d/s) - k^2g\rho'\tilde{\xi}_x, \quad (5)$$

$$\tilde{B}_x = i\frac{\vec{k}\cdot\vec{B}'}{1+ikv_d/s}(x-x_0)\tilde{\xi}_x + \frac{\eta_{\parallel}}{s-ikv_d}\tilde{B}_x''. \quad (6)$$

As shown numerically in reference 3 and analytically in reference 11, the two fastest modes, solutions of Eqs. (5) and (6), have the same growth rate and have either $\tilde{\xi}_x$ even ($\tilde{B}_x$ odd) or $\tilde{\xi}_x$ odd ($\tilde{B}_x$ even). As long as D is finite they are not localized and their eigenvalue s is determined by connecting[1] their solution with the solution of the corresponding hydromagnetic equations, valid where $(x-x_0)/R = O(1)$, missing the inertial and resistive terms. However, for D small the mode with $\tilde{\xi}_x$ even becomes localized in ϵR and the solution can be obtained by further expansion[11] in D of Eqs. (5) and (6). Then they reduce to

$$s(s+ikv_d)\rho\tilde{\xi}_x'' = (\vec{k}\cdot\vec{B}')^2\frac{s-ikv_d}{\eta_{\parallel}}(x-x_0)^2\tilde{\xi}_x + k^2g\rho'\tilde{\xi}_x. \quad (7)$$

There is a set of eigensolutions of Eq. (7) of the form

$$\tilde{\xi}_x = \exp[-\sigma(x-x_0)^2/2]H_n(x-x_0).$$

The one corresponding to the largest eigenvalue occurs for $n=0$ and leads to the dispersion relation

$$s(s^2+k^2v_d^{\ 2}) = \frac{\eta_{\parallel}}{\rho}\left(\frac{k^2g\rho'}{\vec{k}\cdot\vec{B}'}\right)^2 \equiv \chi(D\vec{k}\cdot\vec{B}')^2v_HR/\rho. \quad (8)$$

When D is not small we obtain, generalizing a procedure given in reference 1,

$$s(s^2+k^2v_d^{\ 2}) = \chi(\Delta\vec{k}\cdot\vec{B}')^2v_HR/\rho, \quad (9)$$

where $\Delta \equiv \frac{1}{2}[1-(1-4D)^{1/2}]$ and $D \le \frac{1}{4}$. Equation (8) shows that when $v_d = 0$, $s = (\chi v_HR/\rho)^{1/3}(D\vec{k}\cdot\vec{B}')^{2/3} \propto \eta_{\parallel}^{\ 1/3}k^{2/3}g^{2/3}$, and when the ion gyro-radius effects are relatively large,

$$s \approx \frac{\eta_{\parallel}}{a^2}g^2\frac{\rho k^2}{(\vec{k}\cdot\vec{B}')^2} = \frac{\chi}{v_d^{\ 2}}\left(\frac{D}{k}\vec{k}\cdot\vec{B}'\right)^2 v_H\frac{R}{\rho}. \quad (10)$$

The modes remain purely growing but strongly slowed down. In particular the growth rates do not depend on the magnitude of wavelength, so that the fastest purely resistive modes[3] are the most slowed down. Moreover,

$$\epsilon R \equiv [\mathrm{Re}(\sigma)]^{-1/2} = \frac{a}{2}\left(\frac{p_i'}{g\rho'R}\right)^{1/2}\left(\frac{p_i'R}{p}\right)^{1/2} \quad (11)$$

for D small, and $\epsilon R = v_d(\rho/\Delta)^{1/2}k/(\vec{k}\cdot\vec{B}')$ for $D \le \frac{1}{4}$. Since $p'/(g\rho'R) \gg 1$, $\epsilon R \gg a$, as required for the validity of the present treatment.

If there is no drift velocity but an electric field $E\vec{e}_x = Bp_i'\vec{e}_x/(\Omega\rho)$ at equilibrium, we obtain, instead of Eqs. (1), (2), and (3),

$$s^2(s-ikv_{\perp})\rho\tilde{\xi}_x'' = i(\vec{k}\cdot\vec{B})\tilde{B}_x'' - k^2g\tilde{\rho}, \quad (12)$$

$$s\tilde{B}_x = i(\vec{k}\cdot\vec{B})s\tilde{\xi}_x + \eta_{\parallel}\tilde{B}_x'' - ik\frac{B}{\Omega\rho}\left[i(\vec{k}\cdot\vec{B})B_B - \frac{\vec{k}\cdot\vec{J}}{k}\tilde{B}_x\right], \quad (13)$$

$$B\tilde{B}_B = -p = \xi_xp', \quad (14)$$

$$\tilde{\rho} = \frac{\rho}{p}\tilde{p}, \quad (15)$$

where $v_{\perp}$ is an effective velocity $v_{\perp} \equiv E/B$, so that $v_{\perp} = v_d$, but $v_{\perp} \neq v_d$. By following the same procedure leading to Eq. (9) we now obtain

$$s(s-ikv_{\perp})(s-i2kv_{\perp}) = \chi(\Delta\vec{k}\cdot\vec{B}')^2v_HR/\rho. \quad (16)$$

The results differ from the previous case only by a Doppler shift $kv_{\perp}$. We see in particular that the resistive modes become overstable with frequency of oscillation

$$\omega = kv_{\perp} \quad (17)$$

and having the same growth rate as expressed by Eq. (9).

When $D = O(\beta)$, then the system can no longer be treated as incompressible.[11] In this case the destabilizing effect of the electric field along the field lines,[12] $\eta_{\parallel}\tilde{J}_B$, is counteracted by the transverse electric field $\eta_{\perp}\tilde{J}_{\perp}$ now directly entering the stability problem (here $\tilde{J}_B$ and $\tilde{J}_i$ are components of the perturbed current). Therefore, there are no evident reasons for expecting that the ion gyro-radius effects should neutralize this stabilizing influence.

Finally, if $D \approx 1$ (see reference 1, Appendix E), the effects of resistivity become unimportant and purely hydromagnetic localized interchange modes with $k \sim (\epsilon R)^{-1}$ can exist. The equations of which they are solutions are, in lowest order,

$$s^2\rho(\tilde{\xi}_x'' - k^2\tilde{\xi}_x) = i(\vec{k}\cdot\vec{B})(\tilde{B}_x'' - k^2\tilde{B}_x) + k^2\rho' g\tilde{\xi}_x(s + ikv_d)^{-1}, \quad (18)$$

$$(1 - ikv_d/s)\tilde{B}_x = i(\vec{k}\cdot\vec{B})(1 - ikv_d/s)(1 + ikv_d/s)^{-1}\tilde{\xi}_x. \quad (19)$$

In Eq. (19) the Hall effect contribution clearly cancels out justifying the hypothesis implicitly expressed in references 5, 6, and 7, that it is negligible. Equations (18) and (19) reduce to

$$s_H^2\rho(\tilde{\xi}_x'' - k^2\tilde{\xi}_x) = -(\vec{k}\cdot\vec{B}')^2(x - x_0)^2 \times[\tilde{\xi}_x'' - k^2\tilde{\xi}_x + 2\tilde{\xi}_x'(x - x_0)^{-1}] + k^2 g\rho'\tilde{\xi}_x \quad (20)$$

where

$$s(s + ikv_d) = s_H^2, \quad (21)$$

as obtained in reference 7 with a different method.

By taking the inner product of Eq. (20) by ξ_x^* over the space variable, one sees that s_H^2 is real and, since the Suydam criterion $D > \frac{1}{4}$ is satisfied,[13] positive. Therefore, stability occurs if $kv_d > 2s_H$, as first obtained in reference 5.

If no drift velocity but an electric field exists at equilibrium, instead of Eq. (21)

$$s(s - ikv_\perp) = s_H^2 \quad (22)$$

results, so that they differ[6] by the Doppler shift kv_d, but lead to the same stability condition.

When collisional effects are not important, stabilization[5] can result since, due to the different velocity with which ions and electrons move across the magnetic field, a charge separation is built up. This is out of phase with the charge separation due to the interchange (e.g., gravitational) force. When collisional effects are important, the destabilizing force results from the difference between the gravitational force and the force restraining the material from crossing the magnetic field line.[1] The charge separation which occurs appears to be no longer out of phase with the one due to the different gyration radii of ions and electrons.

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