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[16]See, for example, P. G. Dawber and R. J. Elliott, Proc. Roy. Soc. (London) A273, 222 (1963).

[17]This has been derived from (4) by summing from $n_i = 1$ to infinity.

[18]A possible system for such an experiment may be carbon in iron.

SOLUTION OF THE CLASSICAL ELECTROMAGNETIC SELF-ENERGY PROBLEM

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The problem of the electromagnetic self-energy in classical physics can be stated as follows. The theory of systems involving n point charges leads to one or more of the following situations: (1) There occur divergent expressions; these can be traced to the interaction of each charge with itself. (2) The theory can be made mathematically well defined by the introduction of cutoffs, form factors, or other ad hoc modifications. (3) It is necessary to carry out a mass renormalization. A recent presentation of electrodynamics which exhibits clearly the last point was given by Bergmann.[1] It is the purpose of this note to propose a formulation of the classical theory of point charges which is free of these undesirable features.[2]

The two basic assumptions of the theory are these:

(A) There exists an action integral I (to be specified below) which is invariant under the full Lorentz group including inversions.

(B) Asymptotically, for $\tau \to \pm\infty$ the particles are free and are in uniform motion.

The essential point that enables a solution of this old problem is the realization that one is dealing with not only one electromagnetic field, but with two such fields. These two fields are both mathematically and physically entirely different; they satisfy the homogeneous and the inhomogeneous equations, respectively. Thus, the field strength tensor F of the total field will, in general, consist of two parts, $F = F_f + F_b$, a "free" and a "bound" field.

I hasten to add that this separation is not unique but can be made unambiguous by one of the following requirements:

(a) Each part should be time-reversal invariant (i.e., the tensor components should at most change in sign). In this case we write $F = \bar{F} + F_+$.

(b) Asymptotically, for $t \to -\infty$, F_b reduces to a (generalized) Coulomb field (i.e., a field which differs from the usual Coulomb field at most by a Lorentz transformation). In that case we write $F = F_{\text{in}} + F_{\text{ret}}$.

(c) Asymptotically, for $t \to +\infty$, F_b reduces to a Coulomb field: $F = F_{\text{out}} + F_{\text{adv}}$.

The fields $\bar{F}$, F_{in}, and F_{out} satisfy the homogeneous equations; the others satisfy the inhomogeneous ones.

Consider first a closed system of only one point charge and an incident radiation field $F_{\rm in}$. Let the particle position be labeled by the four-vector $z(\lambda)$, where λ is some monotonically increasing parameter. Let the fields by represented by the four-vector potentials $\bar{A}$ and A_+. The theory can then be based on the following action integral:

$$I=-m\int(-\dot{z}^\alpha\dot{z}_\alpha)^{1/2}d\lambda+e\int\bar{A}_\alpha(x)d^4x\,\delta_4(x-z(\lambda))\dot{z}^\alpha d\lambda-\tfrac{1}{4}\int\bar{F}_{\alpha\beta}(x)\bar{F}^{\alpha\beta}(x)d^4x-\tfrac{1}{2}\int\bar{F}_{\alpha\beta}(x)F_+{}^{\alpha\beta}(x)d^4x. \quad (1)$$

Heaviside-Lorentz units are used and $c=1$; $\dot{z}^\alpha\equiv dz^\alpha/d\lambda$, and the metric has signature +2. The field strength tensor is defined in terms of the potential,

$$\bar{F}_{\alpha\beta}\equiv\partial_\alpha\bar{A}_\beta-\partial_\beta\bar{A}_\alpha;\quad F_{\alpha\beta}{}^+\equiv\partial_\alpha A_\beta{}^+-\partial_\beta A_\alpha{}^+.$$

The action integral is a functional of three four-vectors, z, $\bar{A}$, and A_+. They are to be varied independently. Keeping the boundaries and end points fixed, Hamilton's principle yields the Euler-Lagrange equations in the usual way,

$$ma^\mu=e\bar{F}^{\mu\nu}v_\nu, \quad (2)$$

$$\partial_\alpha(\bar{F}^{\alpha\mu}+F_+{}^{\alpha\mu})=-j^\mu, \quad (3)$$

$$\partial_\alpha\bar{F}^{\alpha\mu}=0. \quad (4)$$

Here,

$$j^\mu(x)\equiv e\int\delta_4(x-z(\tau))v^\mu(\tau)d\tau,$$

$$v^\mu\equiv dz^\mu/d\tau,\quad a^\mu\equiv dv^\mu/d\tau, \quad (5)$$

and the proper time is defined by

$$d\tau=(-\dot{z}^\alpha\dot{z}_\alpha)^{1/2}d\lambda. \quad (6)$$

Equation (2) is the equation of motion of the charged particle; Eqs. (3) and (4) characterize $\bar{F}$ and F_+ as solutions of the homogeneous and the inhomogeneous Maxwell-Lorentz equation, respectively. The total field is $F=\bar{F}+F_+$.

Equations (3) and (5) give F_+ as some linear combinations of retarded and advanced Liénard-Weichert field strengths. But since we require time-reversal invariance in the sense of (a) above, F_+ is uniquely

$$F_+{}^{\mu\nu}=\tfrac{1}{2}(F_{\rm ret}{}^{\mu\nu}+F_{\rm adv}{}^{\mu\nu}). \quad (7)$$

The linear combination

$$F_-{}^{\mu\nu}\equiv\tfrac{1}{2}(F_{\rm ret}{}^{\mu\nu}-F_{\rm adv}{}^{\mu\nu}) \quad (8)$$

is also time-reversal invariant, but satisfies the homogeneous equation. Since assumption (B) implies

$$\lim_{\tau\to-\infty}(F_{\rm ret}{}^{\mu\nu}-F_{\rm C}{}^{\mu\nu})=0,$$

$$\lim_{\tau\to+\infty}(F_{\rm adv}{}^{\mu\nu}-F_{\rm C}{}^{\mu\nu})=0, \quad (9)$$

where $F_{\rm C}$ is the Coulomb field, the above separations (b) and (c) hold,

$$\bar{F}+F_+=F_{\rm in}+F_{\rm ret}=F_{\rm out}+F_{\rm adv}.$$

From this follows the identification

$$\bar{F}=\tfrac{1}{2}(F_{\rm in}+F_{\rm out})=F_{\rm in}+F_-=F_{\rm out}-F_-. \quad (10)$$

One can show[2] that

$$eF_-{}^{\mu\alpha}v_\alpha=\Gamma^\mu, \quad (11)$$

where

$$\Gamma^\mu=(e^2/6\pi)(da^\mu/d\tau-a^\alpha a_\alpha v^\mu)$$

is the Abraham four-vector of radiation reaction. Therefore, the equation of motion (2) is the Lorentz-Dirac equation[3]

$$ma^\mu=eF_{\rm in}{}^{\mu\nu}v_\nu+\Gamma^\mu. \quad (12)$$

Noether's theorem assures that the Lorentz invariance of the action integral I implies ten conservation laws. In particular, momentum conservation results in the form

$$dp^\mu+d\bar{P}^\mu+dP_D{}^\mu=0. \quad (13)$$

Here $p^\mu=mv^\mu$ is the particle momentum,

$$\bar{P}^\mu{}_{(\tau)}=\int_{\sigma(\tau)}\bar{\Theta}^{\mu\alpha}d^3\sigma_\alpha,$$

$$\bar{\Theta}^{\mu\nu}\equiv\bar{F}^{\mu\alpha}\bar{F}_\alpha{}^\nu+\tfrac{1}{4}\eta^{\mu\nu}\bar{F}_{\alpha\beta}\bar{F}^{\alpha\beta},$$

is the free-field momentum, and

$$P_D{}^{\mu}(\tau) = \int_{\sigma(\tau)} \Theta_D{}^{\mu\alpha} d^3\sigma_\alpha,$$

$$\Theta_D{}^{\mu\nu} \equiv \bar{F}^{\mu\alpha} F_{+\alpha}{}^{\nu} + F_+{}^{\mu\alpha} \bar{F}_\alpha{}^{\nu} + \tfrac{1}{2}\eta^{\mu\nu} \bar{F}_{\alpha\beta} F_+{}^{\alpha\beta}$$

is the momentum associated with the "interaction" between the free field $\bar{F}$ and the bound field F_+. It is to be noted that the gauge-dependent canonical energy tensors $\bar{T}^{\mu\nu}$ and $T_D{}^{\mu\nu}$, which arise from the variation of the boundary, combine with the gauge-dependent term due to the second integral in (1), and give symmetrical energy tensors $\bar{\Theta}^{\mu\nu}$ and $\Theta_D{}^{\mu\nu}$ which are gauge invariant.

Since $\bar{F}$, $F_{\rm in}$, $F_{\rm out}$, and F_- are free fields, there also arise the conservation laws

$$d\bar{P}^{\mu} = dP_{\rm in}{}^{\mu} = dP_{\rm out}{}^{\mu} = dP_-{}^{\mu} = 0. \quad (14)$$

If the system is not closed, the action integral (1) can easily be amended by a suitable term in the external field. The conservation laws (13) will then no longer hold.

The action integral for the closed n-particle system differs from the sum of the one-particle integrals only by the "Coulomb" interaction between the charges. In obvious notation,

$$I = -\sum_{i=1}^{n}\left[m_i \int (-\dot{z}_i{}^{\alpha}\dot{z}_{i\alpha})^{1/2} d\lambda_i - e_i \int \bar{A}_\alpha(x)\delta_4(x-z_i)\dot{z}_i{}^{\alpha} d\lambda_i\right] - \tfrac{1}{4}\int \bar{F}_{\alpha\beta}\bar{F}^{\alpha\beta} d^4x - \tfrac{1}{2}\int \bar{F}_{\alpha\beta} F_+{}^{\alpha\beta}(x) d^4x$$
$$+ \frac{1}{2}\sum_{\substack{i=1\\(i\neq j)}}^{n}\sum_{j=1}^{n} e_i e_j \int \dot{z}_i{}^{\alpha}\dot{z}_{j\alpha}\,\delta\big((z_i - z_j)^2\big) d\lambda_i d\lambda_j. \quad (15)$$

The Fokker-Wheeler-Feynman theory of action at a distance with complete absorption is therefore a special case of this theory, obtained by putting all fields equal to zero.

The field equations arising from this action integral are identical with (2) and (3), except that the current density is now given by

$$j^{\mu}(x) = \sum_{i=1}^{n} j_i{}^{\mu}(x) \equiv \sum_{i=1}^{n} e_i \int \delta_4(x - z_i) v_i{}^{\mu} d\tau_i$$

instead of (5). The equations of motion are the set of coupled equations,

$$m_k a_k{}^{\mu} = e_k \bar{F}_k{}^{\mu\alpha} v_\alpha{}^{k} + \sum_{\substack{i=1\\(i\neq k)}}^{n} F_{i,\rm ret}{}^{\mu\alpha} v_\alpha{}^{k},$$

where

$$\bar{F}_k \equiv F_{\rm in} + F_{k-}.$$

Because of (11) these are exactly the Lorentz-Dirac equations for the n-particle system.

The interaction of F_+ can never occur with its own source particle, but occurs via $F_{\rm ret} = F_- + F_+$ with all the other charges. The only self-interaction occurs via F_-; this is just the radiation reaction. As a consequence, there are no self-energies and no divergences, and there is no self-stress problem. The point charge is stable because it is elementary: The theory does not allow it to consist of parts which could repel each other.

Despite the essential appearance of two kinds of fields throughout the theory, a test charge will see only one field, viz. the retarded field, of the charges.

Finally, it should be noted that the basic assumption (B) played an important role in the derivation of (9) and the identification (10). It is also important for singling out the physically meaningful solutions of the equations of motions. Runaway solutions can thereby not occur.[4]

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