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PLASMA EQUATION OF STATE*

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Frieman and Book[1] have derived a convergent kinetic equation for a plasma by combining asymptotic [i.e., asymptotic in $\epsilon \equiv (4\pi n \lambda_d^3)^{-1} = (e^2/\theta)\lambda_d^{-1}$] solutions of the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy separately valid in the three regions $r < e^2/\theta$ (Boltzmann), $d \equiv e^2/\theta < r < \lambda_d$ (Fokker-Planck), and $r > \lambda_d$ (Debye). Three-particle correlations are systematically ignored since they are shown to contribute only in higher order in each of the three regions.

In the same article, the equilibrium case is also considered, but the treatment appears to suffer from the disadvantage that the resultant expression for the two-particle correlation function leads to a divergent expression for the internal energy, and no equation of state can be derived. We wish to point out that this difficulty is only apparent, resulting from a minor misinterpretation of the asymptotic solution of the differential equation for the two-particle correlation function.

For a spatially homogeneous plasma in equilibrium, we may write

$$g_{\mathrm{eq}}(\vec{r}, \vec{v}, \vec{v}') = F_{\mathrm{eq}}(\vec{v}) F_{\mathrm{eq}}(\vec{v}') \Psi(r). \tag{1}$$

If three-particle correlations are ignored, Ψ must satisfy[1]

$$\frac{d^2\Psi}{dr^2} + \frac{2}{r}\frac{d\Psi}{dr} - \frac{d}{r^2}\frac{d\Psi}{dr} - \lambda_d^{-2}\Psi = 0, \tag{2}$$

along with the boundary conditions

$$\Psi(0) = -1, \quad \Psi(\infty) = 0. \tag{3}$$

Clearly, for $r \sim \lambda_d$ the third term in (2) is negligible; for $r \sim d$ the fourth term is negligible; for $r \sim n^{-1/3}$ the third and fourth terms may be neglected. Thus

$$\begin{aligned} \Psi &\approx -1 + e^{-d/r} &&= \Psi_{\mathrm{I}}, \quad r \lesssim d, \\ &\approx -(d/r) &&= \Psi_{\mathrm{II}}, \quad r \approx n^{-1/3}, \\ &\approx -(d/r)\exp(-r/\lambda_d) &&= \Psi_{\mathrm{III}}, \quad r \gtrsim \lambda_d, \end{aligned} \tag{4}$$

where the constants of integration have been chosen so that the solutions match where their regions of validity overlap. The prescription of reference 1,

$$\Psi_{\mathrm{approx}} = \Psi_{\mathrm{I}} + \Psi_{\mathrm{III}} - \Psi_{\mathrm{II}}, \tag{5}$$

yields a solution to (2) asymptotic in ϵ for all r. However, this property is of no value in computing thermodynamic functions. There, the pertinent quantity is the contribution to the internal energy per particle,

$$U(\epsilon) = \tfrac{1}{2} n \int_0^\infty (e^2/r)\Psi(r) 4\pi r^2 dr; \tag{6}$$

and if (5) is taken for Ψ, this integral diverges. Nevertheless, there is no difficulty in obtaining an asymptotic expression for $U(\epsilon)$ from the solution (4). A particularly simple technique is to construct a function $\tilde{\Psi}$ such that

$$\tilde{\Psi}/\Psi = 1 + O(\epsilon) \text{ (uniformly in } x). \tag{7}$$

It is easy to see that one such function is

$$\tilde{\Psi} = -(1 - e^{-d/r}) \exp(-x/\lambda_d), \tag{8}$$

which possesses the added convenience that (6) may be evaluated in closed form. The result is[2]

$$U(\epsilon) = -\theta\{1 - 2\epsilon K_2(2\epsilon^{1/2})\}, \tag{9}$$

from which we obtain the equation of state

$$p/n = \theta\{1 - \tfrac{1}{6}\epsilon - \tfrac{1}{12}\epsilon^2 \ln\epsilon + O(\epsilon^2)\}. \tag{10}$$

We have shown that the equation of state for an electron gas in a positive background follows straightforwardly from the truncated Liouville hierarchy. The results obtained agree (to the extent to which three-particle correlations may be neglected) with the results of Abe,[3] Friedman,[4] and Guernsey[5] obtained by other methods. We are presently considering the effects of higher order correlations.

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HEAT CAPACITY OF THE EXCHANGE BATH IN SOLID ^{3}He

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In solid ^{3}He the atomic wave functions overlap because of the finite mass of the ^{3}He atoms ("zero-point motion"). The requirement for antisymmetry of the total wave function under exchange of any two atoms then leads to a nuclear spin-dependent energy usually discussed in terms of the two-body exchange interaction J defined by the phenomenological exchange Hamiltonian

$$\mathcal{H}_{\mathrm{ex}} \equiv \tfrac{1}{2}\hbar J \sum_{i,j}{}' \vec{\mathrm{I}}^{i} \cdot \vec{\mathrm{I}}^{j} \tag{1}$$

(where $\sum'$ runs only over the z nearest neighbors of each atom). Values of J in both bcc and hcp (3.5 Å $< a <$ 3.65 Å), (3.38 Å $< a <$ 3.63 Å) ^{3}He have recently been published as a result of T_1 and T_2 measurements and of a consistent theoretical interpretation.[1,2] In this density range, $J/2\pi \leq 2.7$ Mc/sec. Several workers have attempted to measure the heat capacity associated with this measured exchange interaction in solid ^{3}He, but have thus far been able only to set upper limits on the heat capacity measured calorimetrically[3,4] and a lower limit in a spin-echo experiment.[1] This Letter presents the first actual measurements of the exchange heat capacity C_{ex}. In terms of the Zeeman heat capacity C_Z, we find the exchange heat capacity

$$C_{\mathrm{ex}}^{\mathrm{obs}} = 350 C_Z,$$

for a sample and magnetic field for which the value calculated from the measured T_1 and T_2, by means of the relation

$$C_{\mathrm{ex}} = \tfrac{3}{32} NZ(\hbar J)^2/kT^2, \tag{2}$$

gives

$$C_{\mathrm{ex}} = 0.084 C_Z,$$

a discrepancy of a factor 4000. [We also confirm in detail the exchange-lattice relaxation mechanism (Raman effect) which was reported previously,[1] but which has remained a subject of controversy.]

In Fig. 1, the Zeeman, exchange, and lattice reservoirs are coupled as shown. The present experiment was done at a field such that the Zeeman-exchange cross-relaxation time $T_1 = T_{\mathrm{ZE}} = 0.15$ sec, while the bath temperature T_{L} was low enough that the diffusion-induced Zeeman-to-lattice relaxation time $T_{\mathrm{ZL}} \to \infty$. For the measurement of the exchange-bath heat capacity, we worked at $T_{\mathrm{L}} \approx 0.45$°K, so that the exchange