DOES SPONTANEOUS BREAKDOWN OF SYMMETRY IMPLY ZERO-MASS PARTICLES?*

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There is relatively intense interest at present in exploring more deeply and widely the suggestion[1] that the mathematical methods essential to the understanding of material media which exhibit long-range order (ferromagnets, superconductors, etc.) may also be basic or useful for the theory of elementary particles. The original work had two connected aspects: the generation of fermion masses (by extension of the underlying group, mass differences) by the "spontaneous breakdown of symmetry," and the consequent occurrence of collective boson excitations.

Though the two kinds of consequences appear inextricably linked in the above work, further studies have tended to emphasize one or the other aspect. Illustrative of one kind of study has been the effort to "derive" the mass formula for the octet of fermions in the SU(3) symmetry model of strong interactions,[2,3] ignoring the possible occurrence – in the particular models employed – of boson excitations. By analogy with the results of reference 1 the models in question would, however, appear to offer the embarrassment of nonexistent strongly interacting scalar bosons with zero mass. A second line of investigation dealing largely with this latter problem traces to Goldstone's[4] conjecture that all broken-symmetry models thus far considered are indeed plagued with this difficulty. Goldstone, Salam, and Weinberg[5] gave two proofs of this conjecture whereas Bludman and Klein[6] have refined and generalized one of these proofs.

In the latter work, it was emphasized, though perhaps insufficiently, that the proof given depended both on the class of models and on the prescribed method of calculation. The second proof of Goldstone, Salam, and Weinberg, however, appears to soar above such detailed considerations and may be summarized as follows: Lorentz invariance + continuous internal symmetry group (represented by commutation relations between generators and operator representations as well as by conserved currents) + spontaneous breakdown of symmetry (conditions of long-range order) implies massless bosons. If this proof is correct, it would seem to spell finis to many of the interesting possibilities for the application of the ideas of spontaneous breakdown of symmetry.

It is well known, however, that the theorem cannot obtain if one removes the requirement of Lorentz invariance. For in this case there is the example of the theory of superconductivity, where the presence of the long-range Coulomb interaction results in a collective boson excitation of finite rest mass – the plasmon. Anderson[7] has conjectured that one should be able to construct the relativistic analog of the plasmon phenomenon. Baker, Johnson, and Lee[8] have presented arguments that a new version of quantum electrodynamics due to Johnson, Baker, and Willey,[9] which possesses formal γ_5 gauge symmetry in consequence of vanishing bare electron mass, does not conform to the Goldstone theorem. As it stands presently, however, this work is merely suggestive.

In this note we shall show the following: (1) The formally exact proof of Goldstone, Salam, and Weinberg has a nonrelativistic analog, which, if followed through faithfully, yields, independently of model, a boson excitation of zero rest mass. (2) The flaw in the argument will then be given, and will be seen to nullify as well the covariant proof.

We deal explicitly with the operators and with the invariance property relevant to the theory of superconductivity. Let $\Psi_\alpha(x)$, $\alpha=1,2$, be the electron destruction operator at the point $x=(\mathbf{x},x_0)$, $\Psi_\alpha^\dagger(x)$ its Hermitian conjugate. We shall utilize the densities

$$\varphi_1(x)=\Psi_1^\dagger(x)\Psi_2^\dagger(x)+\Psi_2(x)\Psi_1(x), \tag{1a}$$

$$\varphi_2(x)=-i[\Psi_1^\dagger(x)\Psi_2^\dagger(x)-\Psi_2(x)\Psi_1(x)], \tag{1b}$$

$$\rho(x)=\Psi_1^\dagger(x)\Psi_1(x)+\Psi_2^\dagger(x)\Psi_2(x), \tag{2a}$$

$$\vec{\mathrm{j}}(x)=-\frac{i\hbar}{2m}\sum_{\alpha=1,2}[\Psi_\alpha^\dagger\nabla\Psi_\alpha-(\nabla\Psi_\alpha^\dagger)\Psi_\alpha], \tag{2b}$$

where $\rho(x)$ and $\vec{\mathrm{j}}(x)$ are the particle density and current, respectively. We assume the Hamiltonian to be particle conserving, so that

$$\nabla\cdot\vec{\mathrm{j}}(x)+\dot{\rho}(x)=0. \tag{3}$$

Thus we have $[N,H]=0$, $N=\int d^3x\,\rho(x)$.

From Eqs. (1) and (2), we deduce that

$$[N,\varphi_2(x)]=-2i\varphi_1(x), \tag{4a}$$

$$[N,\varphi_1(x)]=2i\varphi_2(x). \tag{4b}$$

Thus the $\varphi_i(x)$, $i=1,2$, constitute a representation of the gauge group generated by N. This representation plays a fundamental role in the theory of superconductivity: We define the system to be superconducting if the ground state $|0\rangle$ of a sample in contact with a number reservoir has the property (Ω = volume of sample),

$$\lim_{\Omega\to\infty}\langle 0|\varphi_1(x)|0\rangle = \lim_{\Omega\to\infty}\langle 0|\varphi_1(0)|0\rangle = \varphi, \qquad (5)$$

where φ is a finite number.[10]

Our pseudoproof will be based on the spectral representations of the commutators,

$$\langle 0|[\vec{j}(x),\varphi_2(0)]|0\rangle = (2\pi)^{-3}\int d^3p\,dp_0\,\vec{g}_2(\vec{p},p_0)\exp[i(\vec{p}\cdot\vec{x}-p_0x_0)], \qquad (6a)$$

$$\langle 0|[\rho(x),\varphi_2(0)]|0\rangle = (2\pi)^{-3}\int d^2p\,dp_0\,h_2(\vec{p},p_0)\exp[i(\vec{p}\cdot\vec{x}-p_0x_0)], \qquad (6b)$$

where

$$\vec{g}_2(\vec{p},p_0) = (2\pi)^3\sum_n\{\delta(p_0-\omega_n(p))\langle 0|\vec{j}(0)|p,\omega_n(p)\rangle \times\langle p,\omega_n(p)|\varphi_2(0)|0\rangle - \delta(p_0+\omega_n(p))\langle 0|\varphi_2(0)|p,\omega_n(p)\rangle \times\langle p,\omega_n(p)|\vec{j}(0)|0\rangle\}, \qquad (7a)$$

$$h_2(\vec{p},p_0) = (2\pi)^3\sum_n\{\delta(p_0-\omega_n(p))\langle 0|\rho(0)|p,\omega_n(p)\rangle \times\langle p,\omega_n(p)|\varphi_2(0)|0\rangle - \delta(p_0+\omega_n(p))\langle 0|\varphi_2(0)|p,\omega_n(p)\rangle \times\langle p,\omega_n(p)|\rho(0)|0\rangle\}. \qquad (7b)$$

The sum on n is with respect to the various excitation branches, $\omega_n(p)$ equalling the excitation energy of the nth branch for momentum p. The application of the continuity equation to (6) and (7) informs us immedicately that

$$\vec{p}\cdot\vec{g}_2(\vec{p},p_0) - p_0h_2(\vec{p},p_0) = 0. \qquad (8)$$

The result we are after can be obtained by studying the limit of (8) as $p\to 0$, p_0 fixed. In this limit one can discard with confidence the first term.[11] We thereupon conclude that

$$\lim_{\vec{p}\to 0} h_2(\vec{p},p_0) = C_2\delta(p_0), \qquad (9)$$

where from (7b)

$$C_2 = \lim_{\vec{p}\to 0}\sum\{\langle 0|\rho(0)|p,\omega_n(p)\rangle\langle p,\omega_n(p)|\varphi_2(0)|0\rangle - \langle 0|\varphi_2(0)|p,\omega_n(p)\rangle\langle p,\omega_n(p)|\rho(0)|0\rangle\}, \qquad (10)$$

the summation being over those n for which $\omega_n(p)\to 0$. Thus C_2 could be zero either because no branch of the excitation spectrum extends to zero energy or because, even so, the product of matrix elements goes to zero. Though the considerations to this point evidently lack complete rigor, we emphasize here our belief that they are correct but for one subtlety to which attention will be drawn below.

It remains for us to try to decide if C_2 is nonvanishing. The existing argument[5] applies (4a), (5), and (9) to (6b).

This yields apparently

$$\begin{aligned}\langle 0|[N,\varphi_2(0)]|0\rangle &= -2i\langle 0|\varphi_1(0)|0\rangle = -2i\varphi,\\ &= \int d^3p\,dp_0\,\delta^3(\vec{p},p_0),\\ &= \int dp_0\,h_2(0,p_0) = C_2. \end{aligned}\qquad (11)$$

Thus $C_2\neq 0$ and we have "proved" that the excitation spectrum has at least one branch with the property $\omega_n(p)\to 0$ as $p\to 0$, the nonrelativistic analog of the result of Goldstone, Salam, and Weinberg. Since we know, however, that a real superconductor, including the Coulomb interaction among the electrons, has no such property, we are impelled to seek at least one flaw in the argument.

That defect resides in the last line of (11). Though the limit (9) may be well defined, the function $h_2(\vec{p},p_0)$ cannot as presently constituted, be continuous at the limit. In (11) we want the function evaluated at the point $\vec{p}=0$. We then ask for the possible contribution of states for which $\omega_n(0)=0$. There is the ground state $|0\rangle$, but this certainly does not contribute since we have supposed that $\langle 0|\varphi_2(0)|0\rangle = 0$. Indeed, if this were the only state involved, we would set $h_2(0,p_0)=0$ and arrive from (11) at the contradiction $\varphi=0$.

To avoid the contradiction, we must suppose that there exists at least one additional state of zero energy and momentum, which we label $|0'\rangle$. The occurrence of such "spurious" states, which are not the limiting states of any branch of the excitation spectrum, is, in fact, a phenomenon well known in approximate calculations where the representation chosen does not diagonalize the generator of some symmetry group.[12,13] Here we wish to emphasize that the existence of such a state or states is an exact consequence of our fundamental assumption. Indeed, it follows directly from (4a) and (5) that

$$-2i\varphi = \langle 0|N|0'\rangle\langle 0'|\varphi_2(0)|0\rangle - \langle 0|\varphi_2(0)|0'\rangle\langle 0'|N|0\rangle, \qquad (12)$$

where we have assumed (as is the case here) but the single extra state. Since N can connect only states of the same energy and momentum, contradiction is avoided by invoking the existence of the state $|0'\rangle$.[14]

To summarize, we have reached the conclusion that the spectral representation (6b) is incomplete. The matter can be set straight if we write (considering momentarily again a finite volume)

$$\rho(x) = (N/\Omega) + \rho'(x), \qquad (13)$$

where $\rho'(x)$ comprises the remaining nonzero-momentum Fourier components of $\rho(x)$,

$$\int \rho'(\vec{x}, x_0) d^3x = 0. \qquad (14)$$

The spectral representations (6) and (7) are then correct with the replacement of $\rho(x)$ by $\rho'(x)$ and with the understanding that spurious states are excluded. The conclusions (8), (9), and (10) are unaffected, but we can then decide nothing *a priori* about value of C_2, since the last step in the demonstration, (11), is now trivially satisfied. On the other hand, the contribution of the first term of (13) teaches us, by means of (12), of the existence of the spurion.

The revised considerations of this note apply equally well to Lorentz-invariant field theories, the special role of Lorentz invariance consisting only in rendering the spectral representations more succinct and familiar. As given in reference 5, these are correct representations only for the primed quantities defined by equations analogous to (13).

In conclusion, we wish therefore to emphasize two points: (1) There exists no general proof, independent of model and method of calculation, which establishes the existence of zero-mass particles in field theories with spontaneous breakdown of symmetry. (2) There are nevertheless classes of such field theories wherein zero-mass particles do occur *in consequence of* the broken symmetry. We may now anticipate with greater confidence, however, that the theory of superconductivity, as a definite example where such is not the case, does not stand isolated because of its nonrelativistic aspect. Some possible relativistic counterparts are being studied and will be presented in a subsequent communication.

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[10]In (5) we have, without loss of generality, made a definite choice of an arbitrary phase. In this connection see R. Haag, Nuovo Cimento 25, 287 (1962).

[11]This statement is correct, for example, both in the theory of superconductivity and for a corresponding expression in the theory of a boson superfluid, insofar as one can trust existing calculations, these theories serving as the nonrelativistic prototypes for the nonexistence and existence, respectively, of zero-mass particles. Increased rigor for the statement and for some of the cavalier reasoning which succeeds it can be supplied by means of suitable sum rules.

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[14]The argument of this paragraph should be considered distinct from the main assertions of this paper contained in the preceding and following paragraphs. These have nothing to do with our choice of a number-nonconserving representation, but with the limiting behavior of a spectral function and an unjustified interchange of limits. In a number-conserving representation, we would simply omit the discussion of the spurion, perhaps reinforcing our point that no profound physical conclusions can be reached from the commutation relations Eqs. (4).