DYNAMICAL INSTABILITY OF GASEOUS MASSES APPROACHING THE SCHWARZSCHILD LIMIT IN GENERAL RELATIVITY*

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It is a well-known result of general relativity that a mass M, under conditions of hydrostatic equilibrium, cannot have a (coordinate) radius R which is less than a certain lower limit. This is the Schwarzschild limit and is given by[1]

$$R > 1.125(2GM/c^2) = 1.125R_0, \tag{1}$$

where G is the constant of gravitation, c is the velocity of light, and R_0 is the "gravitational radius" appropriate to the mass M.

The existence of the Schwarzschild limit has been the subject of much recent discussion in the context of the astronomical discoveries pertaining to the "quasistellar" radio sources.[2] However, quite apart from astronomical implications, the question of the stability of gaseous masses as they approach the Schwarzschild limit is one of definite physical interest. An examination of this question, in the framework of general relativity, shows that gaseous masses become dynamically unstable (with respect to radial oscillations) well before they reach the Schwarzschild limit, a fact which must have astronomical implications.

A rigorous discussion of the stability of gaseous masses with respect to purely radial oscillations can be carried out on the basis of Einstein's field equations for a metric of the form

$$ds^2 = e^\nu (dx^0)^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) - e^\lambda dr^2, \tag{2}$$

where ν and λ are allowed to be functions of the world time x^0 and the coordinate radius r. The field equations appropriate to the metric (2) are known and can be found in standard textbooks.[3] If in those field equations, we suppose that none of the quantities depend on x^0, we obtain the following equations which govern hydrostatic equilibrium in general relativity:

$$(re^{-\lambda_0})' = 1 - \kappa r^2 \epsilon_0, \tag{3}$$

$$re^{-\lambda_0}\nu_0' = 1 - e^{-\lambda_0} + \kappa r^2 p_0, \tag{4}$$

and

$$\nu_0' = -2p_0'/(p_0 + \epsilon_0), \tag{5}$$

where $\kappa = 8\pi G/c^4$. In Eqs. (3)-(5), the subscript "0" distinguishes that the quantity refers to the equilibrium static state, the primes denote differentiations with respect to r, and p_0 and ϵ_0 are the equilibrium values of the pressure and the energy density (the latter including not only the energy ρc^2 due to the mass density ρ but also all the other forms of energy that may be present).

Next considering the general time-dependent field equations appropriate for the metric (2) and supposing that the various quantities in the time-dependent state differ from their equilibrium values (distinguished by the subscript "0") by quantities of the first order of smallness, we obtain the following linearized set of equations:

$$(re^{-\lambda_0}\delta\lambda)' = \kappa r^2 \delta\epsilon, \tag{6}$$

$$re^{-\lambda_0}(\delta\nu' - \nu_0'\delta\lambda) = e^{-\lambda_0}\delta\lambda + \kappa r^2 \delta p, \tag{7}$$

$$e^{-\lambda_0}\delta\dot{\lambda} = -\kappa(p_0 + \epsilon_0) r\dot{r}, \tag{8}$$

and

$$e^{\lambda_0 - \nu_0}(p_0 + \epsilon_0)\ddot{r} + \delta p' + \tfrac{1}{2}(p_0 + \epsilon_0)\delta\nu' + \tfrac{1}{2}(\delta p + \delta\epsilon)\nu_0' = 0, \tag{9}$$

where $\delta\lambda$, $\delta\nu$, δp, and $\delta\epsilon$ are the first-order changes in the respective quantities caused by the perturbation, and dots denote differentiations with respect to x^0.

Letting $\xi(r, x^0)$ denote the "Lagrangian displacement" with respect to the "world time" x^0, we find that Eqs. (6)-(8), when suitably combined with the equations governing equilibrium, give

$$\delta\epsilon = -[r^2(p_0 + \epsilon_0)\xi]'/r^2 = -\xi(p_0 + \epsilon_0)' - (p_0 + \epsilon_0)(r^2\xi)'/r^2, \tag{10}$$

and

$$(p_0 + \epsilon_0)\delta\nu' = (\lambda_0' + \nu_0')[\delta p - (p_0 + \epsilon_0)(\nu_0' + 1/r)\xi]. \tag{11}$$

These expressions can now be inserted in Eq. (9). And if we further suppose that all the quantities have a dependence on x^0 of the form $e^{i\sigma x^0}$, Eq. (9) then gives

$$\sigma^2(p_0 + \epsilon_0)e^{\lambda_0 - \nu_0}\xi = \delta p' + (\tfrac{1}{2}\lambda_0' + \nu_0')\delta p + \tfrac{1}{2}\nu_0'\delta\epsilon - \tfrac{1}{2}(\lambda_0' + \nu_0')(p_0 + \epsilon_0)(\nu_0' + 1/r)\xi, \tag{12}$$

where $\delta\epsilon$ is given by Eq. (10). If the oscillations are now considered to take place adiabatically, then[4] [see Eq. (10)]

$$\delta p = -\xi p_0' - \gamma p_0 (r^2\xi)'/r^2, \qquad (13)$$

where γ denotes the ratio of the specific heats. (It should be noted that γ will, in general, depend on the local values of p_0 and ϵ_0.)

Equations (12) and (13), further supplemented by the boundary conditions

$$\xi = 0 \text{ at } r = 0 \text{ and } \delta p = 0 \text{ at } r = R, \qquad (14)$$

represent a characteristic value problem for σ^2. It can be readily shown that this problem is self-adjoint and that the following equation provides a variational base for determining σ^2:

$$\sigma^2 \int_0^R e^{3\lambda_0/2}(p_0+\epsilon_0)r^2\xi^2 dr = \int_0^R e^{\nu_0+\lambda_0/2}[4rp_0'\xi^2 + \gamma p_0(r\xi' + 2\xi)^2]dr$$

$$+\kappa \int_0^R e^{\nu_0 + 3\lambda_0/2} p_0(p_0+\epsilon_0)r^2\xi^2 dr - \int_0^R dr(e^{\nu_0+\lambda_0/2})\frac{(p_0')^2 r^2\xi^2}{(p_0+\epsilon_0)}. \qquad (15)$$

In the Newtonian limit Eq. (15) reduces to the known formula[5]

$$c^2\sigma^2 \int_0^R \rho r^2\xi^2 dr = \int_0^R [4rp_0'\xi^2 + \gamma p_0(r\xi' + 2\xi)^2]dr. \qquad (16)$$

With the simplest "trial function" $\xi = r$, Eq. (15) gives

$$\sigma^2 \int_0^R e^{3\lambda_0/2}(p_0+\epsilon_0)r^4 dr = \int_0^R e^{\nu_0+\lambda_0/2}\ (4r^3 p_0' + 9\gamma p_0 r^2)dr$$

$$+\kappa \int_0^R e^{\nu_0+3\lambda_0/2} p_0(p_0+\epsilon_0)r^4 dr - \int_0^R dr e^{\nu_0+\lambda_0/2} r^4 (p_0')^2/(p_0+\epsilon_0). \qquad (17)$$

Since Eq. (15) expresses a minimal (and not merely an extremal) principle for determining the lowest characteristic value of σ^2, it is clear that a sufficient condition for the occurrence of dynamical instability is that the right-hand side of Eq. (17) be negative. By applying this sufficient condition to a homogeneous sphere (as an illustrative example), we shall show that instability with respect to radial pulsations of the kind we are considering does occur before the Schwarzschild limit is reached.

For a homogeneous sphere of uniform energy density ϵ_0, the equations governing equilibrium, namely, Eqs. (3)-(5), can be explicitly solved. Thus, writing

$$r_0 = (3c^4/8\pi G\epsilon_0)^{1/2}, \quad y = (1 - r^2/r_0^2)^{1/2},$$

and

$$y_1 = (1 - R^2/r_0^2)^{1/2}, \qquad (18)$$

we have

$$e^{-\lambda_0} = y^2, \quad e^{\nu_0} = y_1^2/(1 + p_0/\epsilon_0),$$

and

$$p_0/\epsilon_0 = (y - y_1)/(3y_1 - y). \qquad (19)$$

We may parenthetically note here that according to the solution (19), the condition that p_0 be everywhere positive requires

$$3y_1 > 1 \text{ or } R < (8/9)^{1/2} r_0 = 0.9428 r_0; \qquad (20)$$

and this last condition, when expressed in terms of M $(= 4\pi R^3\epsilon_0/3c^2)$, instead of ϵ_0, is equivalent to the Schwarzschild limit on R.

The various integrals in Eq. (17) can be explicitly evaluated for the solution (19) and an assumed constant γ. And it is found that for an assigned value of R/r_0, the configuration is stable

Table I. The critical values of the ratio of the specific heats which limit the stability of the compressible homogeneous sphere in general relativity: R_{max} (given in the unit r_0) is the maximum radius, for a given ϵ_0, compatible with stability if $\gamma < \gamma_c$; and R_* (given in the unit R_0) is the minimum radius, for a given mass, similarly compatible with stability.

$\sin^{-1}(R/r_0)$	γ_c	R_{max}/r_0	R_*/R_0
0	1.3333	0	∞
20°	1.3809	0.3420	8.549
30°	1.4536	0.5000	4.000
40°	1.5925	0.6428	2.420
45°	1.7062	0.7071	2.000
50°	1.8771	0.7660	1.704
55°	2.1489	0.8192	1.490
60°	2.6266	0.8660	1.333
65°	3.6158	0.9063	1.217
70.529°	∞	0.9428	1.125

only if γ exceeds a certain lower limit γ_c. Values of γ_c determined in accordance with Eqs. (17) and (19) are listed in Table I. From the results of Table I, we may conclude, for example, that if $\gamma < 1.7062$, the configurations are unstable for $0.9428 > R/r_0 > 0.7071$.

The critical values of γ_c limiting the stability of configurations of a given ϵ_0, when expressed for a given mass, require that the radius exceeds a certain lower limit R_* if the configuration is to be stable. We have, in fact, the relation [see the limits set in Eqs. (1) and (20)]

$$R_*/R_0 = (r_0/R_{max})^2. \qquad (21)$$

Thus, in the example considered above, $\gamma < 1.7062$, configurations with radii less than twice their gravitational radius are unstable.

Since the maximum permissible value of γ, for a perfect gas, is 5/3, it follows from the results of Table I that dynamical instability will certainly intervene before the configuration has contracted to 2.12 times its gravitational radius. However, it must be remembered in this connection that, for configurations as massive as the quasistellar radio sources are contemplated to be, the ratio of the specific heats can exceed 4/3 only by a very small amount; and when this is the case, the instability with respect to radial pulsations will arise, already, when the configuration is many times its gravitational radius. Thus, when $\gamma \to 4/3$, one obtains from Eqs. (17) and (19) the asymptotic relation

$$R_*/R_0 \to 5/[14(\gamma - 4/3)] \quad (\gamma \to 4/3). \qquad (22)$$

If $M = 10^8$ solar masses (the mass currently assigned to quasistellar radio sources), it can be estimated that $\gamma - 4/3 \simeq 7\times10^{-5}$; and the lower limit to R set by Eq. (22), for dynamical stability, is $4.4\times10^4 R_0 = 0.16$ light year; and this radius is of the same order as the radii estimated for the objects.

The explicit results that have been derived from a consideration of the homogeneous compressible model serve to illustrate the nature of the phenomena we may expect. But the principal conclusion—that, for a ratio of the specific heats only slightly in excess of 4/3, dynamical instability will intervene long before the mass contracts to anywhere near the Schwarzschild limit—is not likely to be affected by the consideration of more realistic models: Indeed, the estimates of R_* given by Eq. (22) are likely to be underestimates. In any event, it is clear that the instability we have considered is entirely relativistic in its origin. An unambiguous demonstration, that the instability is manifested in nature when the conditions for its occurrence required by Eq. (15) are fulfilled, will provide a unique confirmation for general relativity.

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[1]See H. A. Buchdahl, Phys. Rev. 116, 1027 (1959).

[2]For a general account of these discoveries, see J. L. Greenstein, Sci. Am. 209, No. 6, 54 (1963).

[3]See, for example, L. D. Landau and E. M. Lifshitz, The Classical Theory of Fields (Pergamon Press, New York, 1962), p. 325.

[4]The use of this equation in the present context requires some explanation. Equation (10) (which follows directly from Einstein's field equation) can be interpreted to mean that the Eulerian change $\Delta\epsilon$ in ϵ_0 is given by $\Delta\epsilon/(p_0+\epsilon_0) = -\Delta(r^2\xi)/r^2\Delta r = -\Delta V/V$, where V is the "specific volume." And since the formulation of thermodynamic behavior is affected by the presence of motions only in the second order [see W. Pauli, Theory of Relativity (Pergamon Press, New York, 1958), p. 134], we can, in a linear theory, write $\Delta p/p_0 = -\gamma\Delta V/V$ (and regard this relation as defining γ); Eq. (13) for the corresponding Lagrangian change then follows.

[5]See P. Ledoux and T. Walraven, Handbuch der Physik (Springer-Verlag, Berlin, Germany, 1958), Vol. 51, p. 645.