Topological Vortex and Knotted Dissipative Optical 3D Solitons Generated by 2D Vortex Solitons

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We predict a new class of three-dimensional (3D) topological dissipative optical one-component solitons in homogeneous laser media with fast saturable absorption. Their skeletons formed by vortex lines where the field vanishes are tangles, i.e., N_c knotted or unknotted, linked or unlinked closed lines and M unclosed lines that thread all the closed lines and end at the infinitely far soliton periphery. They are generated by embedding two-dimensional laser solitons or their complexes in 3D space after their rotation around an unclosed, infinite vortex line with topological charge M_0 (N_c, M, and M_0 are integers). With such structure propagation, the "hula-hoop" solitons form; their stability is confirmed numerically. For the solitons found, all vortex lines have unit topological charge: the number of closed lines $N_c = 1$ and 2 (unknots, trefoils, and Solomon knots links); unclosed vortex lines are unknotted and unlinked, their number $M = 1$, 2, and 3.

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Vortices and knots, widely known in everyday life, have now become objects of deep research in topology [1] with important applications to hydrodynamics, physics, chemistry, biology, etc., from elementary particles to Early Universe cosmology [2–7]. In classical fields knots correspond to closed vortex lines, where the field vanishes, without intersections not equivalent topologically to a circle. In linear homogeneous media, a wave's diffraction and dispersion diffuse field structures and decrease their contrast; eventually, vortices and knots [8] are practically indistinguishable after a certain evolution period. Solitons—stable nonlinearly localized structures of fields of a different nature—are not common in our day-to-day life, and they were discovered comparatively recently. However, now solitons are considered as one of the most important nonlinear phenomena in a wide variety of academic and applied sciences [9]. The combination of these two lines of investigation—(i) topologic structures like vortices and knots and (ii) solitons—with the formation of topological, e.g., knotted solitons, was pioneered by Faddeev [10,11]. It is a complicated and challenging problem, and the topological solitons' study is proceeding vigorously only recently [12]. Knotted solitons were predicted for homogeneous multicomponent systems that serve as models of Bose-Einstein condensates, plasmas, liquid crystals, and superconductivity [13–17]. Vortex solitons with axially symmetric intensity distribution (without knotting) were found numerically in the framework of the complex cubic-quintic Ginzburg-Landau equation [18,19]; however, the parameters used in these simulations are far from those in real optical schemes. Stable localized vortices, including knotted ones, are possible in media with periodic [20–22] or local [23–26] inhomogeneities. In the latter case, the structures can be referred to as nonlinear defect modes, contrary to solitons.

Although there are impressive experiments with mechanical creation of knots in fluids [27], topological structures' research is readier in optics using modern lasers and optical materials. Isolated optical vortex loops were generated in free space with laser beams controlled by holograms [28] and with laser beam pulses in nonlinear media under conditions of collapse [29]. Closed vortex lines were found at the periphery of 2D-conservative solitons in a medium with saturable nonlinearity of the refractive index [30]. Recently, we have predicted asymmetric rotating and precessing 3D-topological dissipative optical solitons in homogeneous one-component media with saturable amplification and absorption [31]. A thorough review of the current state in the field is given in Ref. [32]; see also a recent experiment [33]. Important is that an additional, compared to the case of conservative solitons, factor of energy inflow and outflow balance leads to much higher stability of dissipative solitons as attractors.

The goal of this Letter is to present a new wide family of topological solitons in a homogeneous one-component large size laser medium with saturable amplification and absorption. These tangle structures include a number of unclosed (infinite) and closed vortex lines. The scheme is similar to that studied in Ref. [31], but the structures differ radically: only one unclosed vortex line, without any closed ones, was present in Ref. [31], and initial field structures form now by embedding in 3D space of various 2D-laser vortex structures found earlier [34–38].

The scheme is described by the scalar (the case of radiation linear polarization) parabolic (dimensionless) equation for the slowly varying field envelope E [31]:

$$\frac{\partial E}{\partial z} = \left((i+d_\perp)\nabla_\perp^2 + (i+d_\parallel)\frac{\partial^2}{\partial \tau^2}\right)E + f_{nl}(|E|^2)E. \quad (1)$$

Evolution variable is the Cartesian coordinate along the direction of radiation predominant propagation z. Diffraction is reflected by the transverse Laplacian $\nabla_\perp^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ with transverse Cartesian coordinates x and y; $\tau = t - z/v_g$ is the time in the copropagating system of coordinates moving along the z axis with the group velocity v_g. Positive and small "diffusion coefficients" describe frequency dispersion of medium gain or loss ($d_\parallel$) and angular selectivity of amplification or absorption ($d_\perp$). The nonlinear function $f_{nl}(I)$ of radiation intensity $I = |E|^2$ reflects the balance of laser gain and losses for two-level schemes of fast active (laser gain) and passive (saturable absorption) centers doped in the medium with exact frequency tuning [39]:

$$f_{nl}(|E|^2) = -1 + \frac{g_0}{1+|E|^2/\beta} - \frac{a_0}{1+|E|^2}. \quad (2)$$

Here, g_0 and a_0 are small-signal gain and absorption coefficients, respectively; intensity I is taken in units of the absorption saturation intensity; β denotes the ratio of saturation intensities for gain and absorption, and the term -1 corresponds to nonresonant linear absorption, using normalization of the longitudinal coordinate z. Because of fast, exponential field decay at periphery, $|x|+|y|+|\tau| \to \infty$, of bright localized structures considered here, the choice of boundary conditions to Eq. (1) is not critical; the results are the same for zero-derivative and periodic boundary conditions.

Let us describe the choice of the field initial distribution in the case of equal diffusion coefficients $d_\perp = d_\parallel = d$, when Eq. (1) has spherical symmetry in the space $\mathbf{r} = (x, y, \tau)$. We use structures $E_2(x, y, z=0)$ as a 2D-laser soliton or a "solidlike" solitons' complex. The complex can have an N-fold axis of symmetry [34–38] and include a number of vortices—points with vanishing field $E = 0$, around which the phase varies by $2\pi m_i$ with integer m_i, topological charge. The total topological charge of 2D structure $m_{2D} = \sum_i m_i$. We set such distribution in the cross section $y = 0$ of 3D-initial distribution as $E_3(x, y=0, \tau, z=0) = E_2(x - x_c, \tau - y_c, z=0)$ with constant (x_c, y_c). Next, we rotate this structure around axis τ by polar angle φ by 2π; the trace of the structure center forms a circle. Simultaneously we twist the structure around its center in the poloidal plane; twist angle $\theta_c = \varphi s/N$ with integer s and N. Then we introduce, for the field E_3, a phase multiplier $\exp(iM_0\varphi)$ with an integer M_0. In result, we obtain a continuous toroidal initial field distribution with one or a number of infinite, unclosed vortex lines characterized by the total topological charge $M = M_0 + m_{2D}s/N$, and one or a number of closed vortex lines, including knots. Hence, for 3D structures we get two new, as compared with the generating 2D structures, topological indices: charge M and fractional twist index s/N. Such preparation of initial conditions differs from that in Refs. [40,41] by the introduction of an unclosed vortex line with charge M_0; additionally, our generating 2D structure is a single 2D-dissipative soliton or their complex. In the simulations, we fix values $a_0 = 2$ and $\beta = 10$ varying laser gain g_0, diffusion coefficients $d_\parallel$ and $d_\perp$, and using various 2D solitons for construction of 3D-initial distributions. The 3D-structure stability is checked numerically, as well as in Ref. [7]; see the Supplemental Material [42], Sec. 2, where we present also the 3D-solitons' characterization, Sec. 1, their dynamics, Figs. S3–S10, S12, and energy flows, Figs. S13–S15.

The first generating 2D structure is an axially symmetric fundamental soliton ($N = \infty$, $m = 0$) at the distance from the axis τ equal to one-half of the intersoliton distance in the pair of weakly coupled antiphase laser solitons [34,35]. After its rotation in 3D space and for $M = 1$, we get a torus whose axis is an infinite vortex line coinciding with the axis τ. This axially symmetric structure is metastable and transforms, with our parameters, for large z to an asymmetric and precessing 3D soliton with a single unclosed (infinite) curved vortex line with topological charge 1 [31]. This "precesson" is stable in the parameters' domain I indicated in Fig. 2.

For the next structure we use a single axially symmetric vortex soliton ($N = \infty$, $m = 1$, 3D structures with other m decay [42], Fig. S3) with distance from the axis τ one-half of the distance between two weakly coupled antiphase vortex solitons with opposite charges [34,35]. The twist is absent, $s = 0$, and index $M = 1$. The resulting "simple topologically charged torus," or "apple" has a rigid, solidlike axially symmetric intensity distribution without deformations during propagation. There are now one unclosed (infinite straight line) and one closed (ring) vortex lines with topological charge 1; see Figs. 1(a) and 1(b). The soliton is stable in the domain II in Fig. 2. A mirror image of the soliton (replacement $x \to -x$) is a similar soliton with inversion of the directions of phase rotation and energy flows.

Near the stability domain's boundaries, with increase of gain g_0, the apple vortex lines deform, as shown in Figs. 2(a), 2(b), 2(c), and Ref. [42], Fig. S4. The vortex loop and the intensity distribution in Fig. 2(a) have a sixfold symmetry axis. The intensity distribution is solidlike and turns with constant angular velocity. At the upper boundary of the stability domain, asymmetric structures arise like (a), (b) in Fig. 2; some of them have solidlike intensity distribution. For larger d, solitons can decay: The symmetric apple decays with scenario (c): the unclosed vortex line curves, the structure oscillates and elongates progressively in the τ direction [42], Fig. S5. An asymmetric apple decays with the scenario shown in Ref. [42], Fig. S6, to a final chaoticlike structure. Increase of gain induces disappearance

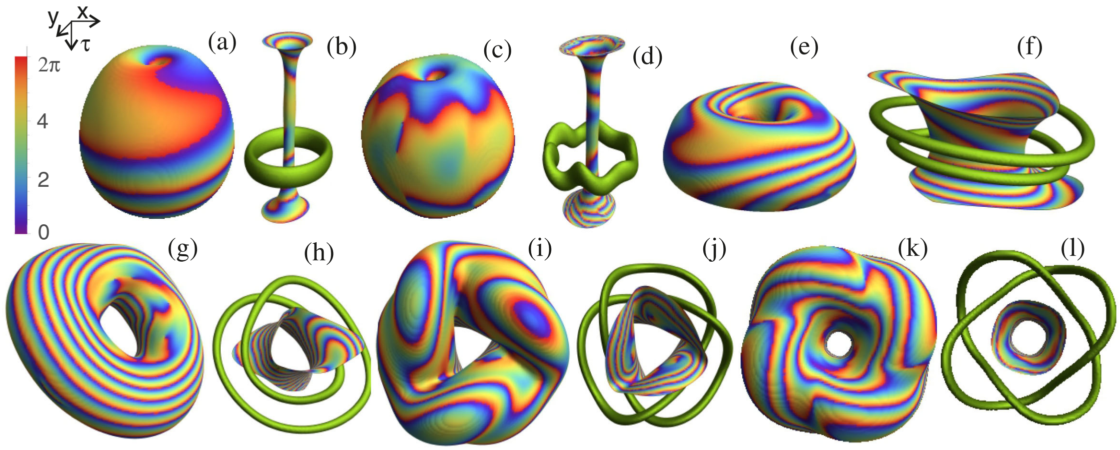

FIG. 1. "Hula-hoop" solitons: Isointensity surfaces at propagation distance $z = \text{const}$ (a),(c),(e),(g),(i), intensity $I/I_{\max} = 0.05$, the surfaces illustrate the soliton localization. (b),(d),(f),(h),(j), $I/I_{\max} = 0.15$, and the corresponding skeletons indicate vortex lines; all skeletons include unclosed (infinite) vortex lines. Vertical (color) scale shows the field phase. (a),(b) A simple topologically charged torus, or an apple with solidlike axially symmetric intensity distribution not varying during its propagation with constant velocity. Generating 2D structure is a single vortex soliton with charge $m = 1$. Topological charge for the unclosed vortex line $M = 1$, topological twist index $s/N = 0$. The skeleton (b) includes a single axially symmetric closed vortex line (ring); $d = 0.06$, $g_0 = 2.11$. (c), (d) The same as in (a),(b) above, for structures indicated by (a) and (b) in Fig. 2. The closed vortex line is curved; $d = 0.035$, $g_0 = 2.127$. (e),(f) A doubly twisted torus with two closed vortex lines passing with increase of z periodically one through the other; $d = 0.04$, $g_0 = 2.115$. (g),(h) A soliton with an unknotted closed vortex line; $s/N = -1$, $M = 3$, $d = 0.06$, $g_0 = 2.115$. (i),(j) A soliton with trefoil closed vortex line; $s/N = -3$, $M = 3$, $d = 0.05$, $g_0 = 2.114$. (k),(l) A soliton with Solomon's knot (two linked) closed vortex lines; $s/N = -4$, $M = 2$, $d = 0.068$, $g_0 = 2.118$.

of solitons via propagation of the lasing mode front over the whole space. Slow variations of parameters induce transformations of the solitons' type and hysteretic phenomena.

The next four structures are generated by a 2D pair of strongly coupled solitons with equal topological charges $m_1 = m_2 = 1$ ($N = 2$) [34,35] and $M_0 = 3$ (complexes with other M_0 decay for used s). Then tracks from the 2D-vortices' rotation are two strongly coupled closed 3D-vortex lines. Without twist ($s = 0$), an axially symmetric torus is metastable, and, after propagation over $z \sim 10^4$, a double-ring soliton forms with an infinite unclosed and two closed strongly coupled vortex lines periodically passing one through the other [Figs. 1(e), 1(f), and Ref. [42], Fig. S7]. The structure moves along the axis τ with small variations of its shape and velocity. It is stable in the domain shown in Fig. 2(d). Its stability domain is more narrow than domain V in Fig. 2(d). With twist $s/N = -1/2$, we get an unknotted soliton [Figs. 1(g), 1(h), and Ref. [42], Fig. S8] with a single closed vortex line turning twice around the infinite, unclosed vortex lines. The intensity distribution moves and rotates with weakly oscillating velocity. Unknotted solitons are stable in domain IV in Fig. 2. Next, with $s/N = -3/2$, a trefoil soliton forms that includes a single-closed trefoil vortex line turning two times round axis τ; the corresponding pair of the coupled line's branches is twisted 3 times, Figs. 1(i), 1(j), and Ref. [42], Fig. S9. The total charge of the unclosed vortex lines $M = 3$, they present three unlinked lines with topological charge 1 [42], Fig. S11. The soliton is chiral, its mirror image obtained by replacement $x \to -x$ cannot be transformed to the initial one by rotation. Trefoil solitons are stable in domain III of Fig. 2.

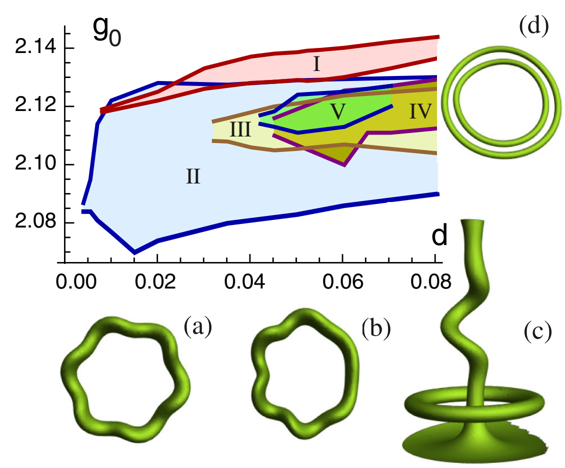

FIG. 2. Overlapping stability domains in the plane of parameters (g_0, d) for precessons (I, no closed vortex lines), axially symmetric apples (II, single closed vortex line), trefoils (III, knotted vortex line), unknots (IV, closed vortex line is unknotted), and Solomon's knots with two doubly linked unknots (V); double-ring solitons, inset (d), exist in the domain more narrow than V. Near the upper boundary of domain II (apples), the vortex lines lose their symmetry as shown in the insets (a) (the closed line has a sixfold symmetry axis), (b) (deformation of the closed line includes 5th and 6th angular harmonics), and (c) (waved unclosed vortex line).

Near its upper boundary, the branches transform to spirals whose interaction destroys the soliton. Finally, with $M_0 = 6$ and $s = -8$, we get the soliton with two doubly linked closed vortex lines—the Solomon's knot and a pair of unlinked unclosed vortex lines with charge 1; their total charge $M = 2$, see Figs. 1(k), 1(l), and Ref. [42], Fig. S10.

In a more realistic case of different diffusion coefficients $d_\perp \neq d_\parallel$, the medium is anisotropic in 3D-space $\mathbf{r}$ and soliton orientation is not arbitrary. Simulations show that the unusual structures found survive weak anisotropy: The topological structure of intensity distribution and energy flows remains the same, and anisotropy results only in the soliton's reorientation demonstrated for the apple soliton in the Supplemental Material [42], Fig. S12. For different initial directions of angular velocity, it is oriented finally along the axis τ for $d_\perp < d_\parallel$ and orthogonally to the axis for $d_\perp > d_\parallel$, similarly to the case of the precessons [31].

Concluding, we have predicted a new wide class of topological 3D-dissipative solitons—hula-hoop solitons—in one-component homogeneous media that can be realized in large size (larger than about 10 μm) lasers with fast nonlinearity. Their extreme stability (attractors) is confirmed by numerical simulation. Taking into account the finite medium response time and involving regimes of self-induced transparency [43], it would be possible to generate new and even subcycle solitons [37] with various topology; this extends the capabilities of optical information processing, including coding of topologically stable 3D symbols.

As shown in the Supplemental Material [42], Figs. 13–15, distribution of energy flow in the solitons underlines their complicated internal structure with strong coupling of various vortex lines. The skeletons of the solitons, i.e., the combination of one or two closed (loops) and one, two, or three unclosed vortex 3D lines, are tangles [1]—the topology not found earlier for dissipative solitons and, probably, for any solitons in homogeneous nonlinear media. It is natural to expect the existence of similar topological 3D-dissipative solitons in different physical, chemical, and biological dissipative systems, including reaction-diffusion systems [7,44].

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[42] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.119.263901 for characterization of 3D solitons (Sec. 1), demonstration of their stability (Sec. 2 and Figs. S1, S2), and illustrations of solitons' formation, propagation, evolution, and decay (movies S3–S7.avi).
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