Asymptotic One-Point Functions in Gauge-String Duality with Defects

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We take the first step in extending the integrability approach to one-point functions in AdS/dCFT to higher loop orders. More precisely, we argue that the formula encoding all tree-level one-point functions of SU(2) operators in the defect version of $\mathcal{N}=4$ supersymmetric Yang-Mills theory, dual to the D5-D3 probe-brane system with flux, has a natural asymptotic generalization to higher loop orders. The asymptotic formula correctly encodes the information about the one-loop correction to the one-point functions of nonprotected operators once dressed by a simple flux-dependent factor, as we demonstrate by an explicit computation involving a novel object denoted as an amputated matrix product state. Furthermore, when applied to the Berenstein-Maldacena-Nastase vacuum state, the asymptotic formula gives a result for the one-point function which in a certain double-scaling limit agrees with that obtained in the dual string theory up to wrapping order.

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Introduction.—Apart from observables which are protected by supersymmetry, the AdS/CFT correspondence has not provided us with many examples of quantities which can be explicitly calculated to all orders in the coupling constant in both the string theory and the field theory and successfully matched. The main examples are the cusp anomalous dimension [1] and the expectation value of the circular Maldacena-Wilson loop [2–4]. An instructive attempt to arrange for a situation which could allow an all-order comparison between the gauge and string theory was made with the invention of the Berenstein-Maldacena-Nastase (BMN) limit, where a certain double-scaling parameter combining the 't Hooft coupling constant λ with a large angular momentum quantum number was introduced and certain observables being close to protected were considered [5]. However, it turned out that for the observables considered the BMN expansion became inconsistent starting at four-loop order in the field theory [6–8].

In a variant of the AdS/CFT correspondence which involves a D5-D3 probe-brane setup on the string-theory side and a codimension-one defect in the $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory, another double-scaling limit has recently been proposed [9]. It consists of sending the 't Hooft coupling as well as a certain background gauge field flux k to infinity while keeping a certain ratio involving the two parameters fixed. While the study of the BMN expansion acted as a seed for the development of the integrability approach to the $\mathcal{N}=4$ SYM theory [10], at the present stage we already have available a vast number of integrability tools that we can make use of when investigating the defect setup and the associated novel double-scaling limit. In addition, in the defect case we have an entirely new collection of observables including one-point functions, two-point functions between operators of unequal conformal dimension, and correlators between bulk and boundary fields [11]. In particular, we can consider the BMN vacuum states of the $\mathcal{N}=4$ SYM theory whose two- and three-point functions do not get quantum corrections in the pure $\mathcal{N}=4$ SYM theory but whose one-point functions are nonvanishing and receive quantum corrections in the defect theory.

One-point functions of protected operators were calculated at the tree level in the above-mentioned defect CFT in Ref. [12] and in a closely related theory building on a nonsupersymmetric D7-D3 probe-brane system in Ref. [13]. Furthermore, exploiting the integrability structure of the $\mathcal{N}=4$ SYM theory and introducing an appropriate boundary state in the form of a matrix product state, one-point functions of nonprotected operators were calculated at the tree level for the SU(2) sector in Refs. [14,15]. This approach was generalized to the SU(3) sector [16] as well as to the SO(6) sector of the above-mentioned nonsupersymmetric defect CFT [17,18]. Most recently, the one-loop correction to the one-point function of the BMN vacuum was calculated [19,20] and shown to match the string-theory prediction of Ref. [12]. In addition, a strategy for computing the one-loop correction to the one-point functions of nonprotected operators was presented [20]. This involved the introduction of a new object denoted as the amputated matrix product state.

In the present Letter, we will argue that the integrability approach to one-point functions suggests a certain generalization of the tree-level formula for the SU(2) sector to higher loop orders. We shall furthermore concretely implement the above-mentioned strategy for the calculation of one-loop corrections to one-point functions and show that the results can be accounted for by the suggested asymptotic formula when dressed by a simple flux-dependent factor. This flux factor leads to a breakdown of the above-mentioned double-scaling limit for nonprotected operators already at one-loop order. For protected operators, the flux factor is absent, and we will show that the proposed formula implies that the one-point function of the BMN vacuum state has an expansion that in the double-scaling limit, up to terms of wrapping order, matches an expansion derived in the string-theory language using a supergravity approximation.

Our proposal.—The defect version of the $\mathcal{N}=4$ SYM theory which is dual to the D5-D3 probe-brane system with flux k is characterized by having a codimension-one defect, say, at $x_3=0$, separating two regions of space, $x_3>0$ and $x_3<0$, where the gauge group is, respectively, (broken) $U(N)$ and $U(N-k)$. The difference in the rank of the gauge group implies assigning the following vacuum expectation values, for $x_3>0$, to three out of the six scalar fields of the $\mathcal{N}=4$ SYM theory:

$$\langle\phi_i\rangle_{\text{tree}}=-\frac{1}{x_3}t_i\oplus 0_{(N-k)\times(N-k)},\qquad i=1,2,3,\quad (1)$$

where the t_i are the generators of a k-dimensional irreducible representation of SU(2). For a precise description of the holographic setup, we refer to Ref. [20] as well as the original papers [21,22].

As usual, we identify two complex scalars of the $\mathcal{N}=4$ SYM theory with spins of an integrable SU(2) spin chain as $\uparrow\equiv X=\phi_1+i\phi_4$ and $\downarrow\equiv Y=\phi_2+i\phi_5$. A Bethe (eigen)state of this spin chain is characterized by two Dynkin labels L and M corresponding, respectively, to the length and the number of excitations and, in addition, by M rapidities $\{u_i\}$ that satisfy certain Bethe equations. For a given eigenstate $|\mathbf{u}\rangle$, we define the corresponding single-trace operator from the SU(2) sector as

$$\mathcal{O}\equiv\left(\frac{4\pi^2}{\lambda}\right)^{L/2}\frac{\mathcal{Z}}{\sqrt{L}}\frac{\operatorname{tr}\prod_{l=1}^{L}(\langle\uparrow_l|\otimes X+\langle\downarrow_l|\otimes Y)|\mathbf{u}\rangle}{\sqrt{\langle\mathbf{u}|\mathbf{u}\rangle}}.\quad (2)$$

Far away from the defect, the tree-level two-point function of $\mathcal{O}$ is normalized to unity, and we will use the freedom in the choice of the finite part of the renormalization constant $\mathcal{Z}$ to enforce this also at the loop level. The one-point function then takes the form

$$\langle\mathcal{O}(x)\rangle=\left(\frac{4\pi^2}{\lambda}\right)^{L/2}\frac{C_k}{\sqrt{L}}\frac{1}{x_3^{\Delta}},\quad (3)$$

where Δ denotes the scaling dimension of the operator. The calculation of C_k will be the subject of this Letter.

Tree level.—At the tree level, the one-point function can be written as the overlap of a Bethe eigenstate of the Heisenberg spin chain with a matrix product state [14,15]. The corresponding Bethe equations read

$$1=\left(\frac{u_k-\frac{i}{2}}{u_k+\frac{i}{2}}\right)^L\prod_{\substack{j=1\\ j\neq k}}^{M}\frac{u_k-u_j+i}{u_k-u_j-i}\equiv\exp[i\Phi_k].\quad (4)$$

Using the algebraic Bethe ansatz approach [23], the Bethe state can be built from the ferromagnetic vacuum $|0\rangle_L$ with all spins up via the creation operators $B(u)$:

$$|\mathbf{u}\rangle=B(u_1)\ldots B(u_M)|0\rangle_L.\quad (5)$$

Defining the matrix product state as

$$\langle\text{MPS}|=\operatorname{tr}\prod_{l=1}^{L}(\langle\uparrow_l|\otimes t_1+\langle\downarrow_l|\otimes t_2),\quad (6)$$

the tree-level one-point function of $\mathcal{O}$ is given as

$$C_k=\frac{\langle\text{MPS}|\mathbf{u}\rangle}{\sqrt{\langle\mathbf{u}|\mathbf{u}\rangle}}.\quad (7)$$

In Ref. [14], it was shown that only operators with L and M even and with paired rapidities $\{u_i\}=\{-u_i\}$ have nontrivial one-point functions [24]. For $k=2$, the tree-level one-point function can be elegantly described in terms of the Bethe function Φ introduced above. Let us order the roots as $\{u_1,\ldots,u_{M/2},-u_1,\ldots,-u_{M/2}\}$ and introduce the following $(M/2)\times(M/2)$ dimensional matrices $G_\pm$:

$$G_\pm=\partial_m\Phi_n\pm\partial_{m+(M/2)}\Phi_n,\quad (8)$$

with $\partial_m\equiv(\partial/\partial u_m)$. Then, the one-point function for $k=2$ can be written as

$$C_2=2^{1-L}\sqrt{\frac{Q(\frac{i}{2})}{Q(0)}}\sqrt{\frac{\det G_+}{\det G_-}},\quad (9)$$

where $Q(u)=\prod_{i=1}^{M}(u-u_i)$ is the Baxter polynomial.

According to Ref. [15], the one-point function for $k>2$ then takes the form

$$C_k=i^LT_{k-1}(0)\sqrt{\frac{Q(\frac{i}{2})Q(0)}{Q^2(\frac{ik}{2})}}\sqrt{\frac{\det G_+}{\det G_-}},\quad (10)$$

where

$$T_n(u)=\sum_{a=-(n/2)}^{n/2}(u+ia)^L\frac{Q(u+\frac{n+1}{2}i)Q(u-\frac{n+1}{2}i)}{Q(u+(a-\frac{1}{2})i)Q(u+(a+\frac{1}{2})i)}\quad (11)$$

can be identified as the transfer matrix of the Heisenberg spin chain in the $(n+1)$-dimensional representation [25].

Quantization.—Bearing in mind the integrability approach to the spectral problem of the $\mathcal{N}=4$ SYM theory, it is natural to introduce the coupling constant dependence via the Zhukovsky variable x [26]:

$$x+\frac{1}{x}=\frac{u}{g},\qquad x=\frac{u}{g}-\frac{g}{u}+O(g^2),\quad (12)$$

where the effective planar coupling constant g^2 is related to the 't Hooft coupling $\lambda=Ng_{YM}^2$ as $g^2=(\lambda/16\pi^2)$ and where the cut of the function $x(u)$ is taken to be the straight line $[-2g,2g]$. The all-loop asymptotic Bethe equations which determine the conformal operators of the $\mathcal{N}=4$

SYM theory and their anomalous dimensions are then given by [7]

$$1=\left(\frac{x(u_k-\frac{i}{2})}{x(u_k+\frac{i}{2})}\right)^L\prod_{j\neq k}\frac{u_k-u_j+i}{u_k-u_j-i}\exp[2i\theta(u_k,u_j)]$$
$$\equiv\exp[i\tilde{\Phi}_k], \tag{13}$$

where $\exp[2i\theta(u_k,u_j)]$ is the so-called dressing phase. A natural generalization of (9) is obtained by replacing the classical Bethe function Φ by the quantum Bethe function $\tilde{\Phi}$. Furthermore, a natural generalization of the transfer matrix is the following one:

$$\tilde{T}_n(u)=g^L\sum_{a=-(n/2)}^{n/2}x(u+ia)^L \times\frac{Q(u+\frac{n+1}{2}i)Q(u-\frac{n+1}{2}i)}{Q(u+(a-\frac{1}{2})i)Q(u+(a+\frac{1}{2})i)}. \tag{14}$$

This gives a natural expression for (10) at the quantum level. Of course, the roots u_i appearing in the Baxter polynomials satisfy the all-loop Bethe equations (13). It is not excluded that further modifications of the transfer matrix are necessary, but for the consistency checks that we perform the present modification suffices. Furthermore, the phase factor in (13) does not come into play in these checks. However, we do need to allow for a flux factor $\mathbb{F}_k$, such that we find

$$C_k=i^L\tilde{T}_{k-1}(0)\sqrt{\frac{Q(\frac{i}{2})Q(0)}{Q^2(\frac{ik}{2})}}\sqrt{\frac{\det\tilde{G}_+}{\det\tilde{G}_-}}\mathbb{F}_k. \tag{15}$$

The flux factor $\mathbb{F}_k$ is 1 for protected operators, and its general form at one-loop order turns out to be

$$\mathbb{F}_k=1+g^2\left[\Psi\left(\frac{k+1}{2}\right)+\gamma_E-\log 2\right]\Delta^{(1)}+O(g^4), \tag{16}$$

where $\Delta^{(1)}=2\sum_{i=1}^M(1/u_i^2+\frac{1}{4})$ is the one-loop correction to the scaling dimension. Note that the Euler digamma function Ψ can be reexpressed in terms of the harmonic number H, which is generalized to noninteger arguments via $H_x=\Psi(x+1)+\gamma_E$.

Checks.—We now test (15)—first at one-loop order for nonprotected operators in the SU(2) sector and then at higher orders for the BMN vacuum. Finally, we will discuss the flux factor and the fate of the double-scaling limit.

SU(2) at one loop.—In Ref. [20], we have shown that the one-loop one-point function is given by the sum of three contributions: (a) the manifestly finite overlap of the Bethe eigenstate with a special spin-chain state, denoted as an amputated matrix product state, (b) an ultraviolet (UV)-divergent contribution proportional to the one-loop dilatation operator, which requires operator renormalization, and (c) the one-loop correction to the Bethe state. Demanding that the two-point function far away from the defect remains unit normalized also at one-loop order fixes the renormalization constant to be $\mathcal{Z}=1+g^2(\Delta^{(1)}/2)[(1/\epsilon)+1+\gamma_E+\log\pi]+O(g^4)$; see, for instance, Ref. [27]. The one-loop one-point function then reads [20]

$$C_k=\frac{(\langle\mathrm{MPS}|+g^2\langle\mathrm{AMPS}|)|\mathbf{u}\rangle}{\sqrt{\langle\mathbf{u}|\mathbf{u}\rangle}}\times\left\{1+g^2\left[\Psi\left(\frac{k+1}{2}\right)+\gamma_E-\log 2+\frac{1}{2}\right]\Delta^{(1)}\right\}+O(g^4), \tag{17}$$

where $|\mathrm{AMPS}\rangle$ denotes the amputated matrix product state, to be explicated below, and $|\mathbf{u}\rangle$ denotes the loop-corrected Bethe state. In order to evaluate (17) explicitly, we need two ingredients. We need to evaluate the overlap of $\langle\mathrm{AMPS}|$ with the Bethe state, and we need to compute the first correction to the Bethe state, i.e., the two-loop Bethe eigenstate.

Overlap with $\langle\mathrm{AMPS}|$.—The amputated matrix product state $\langle\mathrm{AMPS}|$ is defined as [20]

$$\langle\mathrm{AMPS}|=\sum_{l=1}^L\mathcal{A}_{l,l+1}\langle\mathrm{MPS}|, \tag{18}$$

where $\mathcal{A}_{i,i+1}$ removes the matrices at positions i and $i+1$ (with $L+1\sim 1$) if they are identical and otherwise kills the trace; cf. (6).

Let us consider the overlap between a Bethe state and the amputated matrix product state. The overlap is nonzero only for an even number of magnons M, and in the coordinate formulation it reads

$$\langle\mathrm{AMPS}|\mathbf{u}\rangle=\sum_{n\in\{n\}_M}\Psi_B(n,\{u_i\})\sum_{l=1}^L\mathcal{A}_{l,l+1}\times\mathrm{tr}\left[\prod_{i=1}^M(t_1^{n_{(i+1)i}-1}t_2)\right], \tag{19}$$

where $\{n\}_M$ denotes the usual set of ordered magnon positions $(n_1<\cdots<n_M)$ and $\Psi_B(n,\{u_i\})$ is the Bethe wave function. Furthermore, the shorthand notation $n_{ij}\equiv n_i-n_j$ and $n_{M+1}\equiv n_1+L$ is used throughout.

For any even M and $k=2$, one can compute directly the action of $\sum\mathcal{A}_{l,l+1}$ on the traces in (19):

$$\sum_{l=1}^L\mathcal{A}_{l,l+1}\mathrm{tr}\prod_{i=1}^M(t_1^{n_{(i+1)i}-1}t_2)$$
$$\overset{k=2}{=}(-1)^{(M/2)+\sum_i n_i}2^{3-L}\left[L+2\sum_{i=1}^M(\delta_{n_{(i+1)i}=1}-1)\right]. \tag{20}$$

Using this, the rest of the computation can be carried out symbolically by brute force in *Mathematica*, at least for smaller values of M. This was done for $M=2,4$ and leads to the conjecture

$$\langle\mathrm{AMPS}|\mathbf{u}\rangle\overset{k=2}{=}(4L-\Delta^{(1)})\langle\mathrm{MPS}|\mathbf{u}\rangle, \tag{21}$$

which was subsequently tested numerically up to and including $M = 6$ and $L = 16$. A closed formula for $M = 2$ and any k can likewise be obtained.

Two-loop Bethe states.—The first loop correction to the Bethe state, i.e., the two-loop Bethe state, can be generated via the so-called Θ morphism [28]. To this end, we consider the Heisenberg spin chain with impurities θ_i. The one-loop Bethe state can again be constructed using the algebraic Bethe ansatz approach:

$$|\theta;\mathbf{u}\rangle = \hat{B}(u_1)...\hat{B}(u_M)|0\rangle_L, \tag{22}$$

where the $\hat{B}$ operator is

$$\hat{B}(u) = \langle\uparrow|\bigotimes_{j=1}^{L}\left(\mathbb{1}_{j,0} + \frac{i}{u-\theta_j-\frac{i}{2}}\mathbb{P}_{j,0}\right)|\downarrow\rangle. \tag{23}$$

The two-loop eigenstate is then

$$|\mathbf{u}\rangle \equiv \left(1 - g^2\frac{\Delta^{(1)}}{2}\mathbb{H}_{L,1}\right)\{|\theta;\mathbf{u}\rangle\}_\Theta, \tag{24}$$

where $\mathbb{H}_{j,j+1} = \mathbb{1}_{j,j+1} - \mathbb{P}_{j,j+1}$ is the Heisenberg spin-chain Hamiltonian density. The Θ morphism $\{\}_\Theta$ is defined via

$$\{f\}_\Theta \equiv f + \frac{g^2}{2}\sum_{i=1}^{L}\left[\frac{\partial}{\partial\theta_i} - \frac{\partial}{\partial\theta_{i+1}}\right]^2 f + O(g^4)\Big|_{\theta_j\to 0}. \tag{25}$$

The rapidities $\{u_i\}$ have to satisfy the two-loop Bethe equations (13). For instance, the easiest case is $M = 2$, $k = 2$, where we find for the overlap with the matrix product state

$$\frac{\langle\mathrm{MPS}|\mathbf{u}\rangle}{\sqrt{\langle\mathbf{u}|\mathbf{u}\rangle}} = \sqrt{\frac{L}{L-1}\frac{u^2+\frac{1}{4}}{u^2}\frac{1+g^2\frac{4}{u^2+(1/4)}}{1+\frac{g^2}{L-1}\frac{6u^2-\frac{1}{2}}{[u^2+(1/4)]^2}}}. \tag{26}$$

A closed expression for $M = 2$ and any k can similarly be derived.

General formula.—Now that we have all the ingredients, we are ready to check if (15) reproduces (17). Indeed, one can analytically show that for $M = 2$ both formulas agree. Moreover, we numerically compared (15) and (17) for $L = 8$ and $M = 4$ excitations for various values of k and again found perfect agreement.

BMN vacuum at all loop orders.—A particularly simple situation arises if we consider the spin-chain vacuum, which corresponds to the protected operator $\mathrm{tr}(X^L)$.

For the vacuum, there are no Bethe roots, and our proposal (10) reduces to

$$C_k = i^L T_{k-1}(0) = \sum_{a=(1-k)/2}^{(k-1)/2}[igx(ia)]^L; \tag{27}$$

i.e., the only contribution stems from the transfer matrix for the vacuum. We notice, in particular, that the contribution from the flux factor trivializes.

For even k and even L, the one-point function formula can be readily expanded as a power series in g with the result (up to the order of g^{2L}, the result is identical for odd k)

$$C_k(g) = 2\sum_{n=0}^{L/2}\binom{L-n}{n}\frac{L}{L-n}\frac{B_{L-2n+1}(\frac{1+k}{2})}{L-2n+1}g^{2n} + g^{2L}\sum_{n=0}^{\infty}\frac{L[\Psi^{(L+2n-1)}(\frac{1+k}{2}) - \Psi^{(L+2n-1)}(\frac{1-k}{2})]}{(-1)^n n!(L+n)!}g^{2n}, \tag{28}$$

where B_n is the Bernoulli polynomial with index n and $\Psi^{(n)}$ is the polygamma function. We notice that the term occurring in the second line of (28) starts contributing only at wrapping order. For even k and odd L, the one-point function vanishes as $x(u)$ is an odd function (away from the cut). At the one-loop level, we find that (28) exactly agrees with Ref. [20]. For odd k, the contribution from $a = 0$ in (27) should be understood in the following way:

$$[gx(ia)]^L|_{a=0} \equiv [gx(+0i)]^L + [gx(-0i)]^L. \tag{29}$$

This prescription can be motivated by the fact that it leads to the correct result for $\langle\mathrm{tr}X^2\rangle$ at one-loop order for odd k and in addition ensures that the one-point function vanishes for odd L, also when k is odd.

String theory.—We can compare this result to a string-theory prediction in the double-scaling limit proposed in Ref. [9]. This limit consists in taking

$$\lambda\to\infty, \qquad k\to\infty, \qquad \frac{\lambda}{k^2}\ \text{fixed and small}, \tag{30}$$

on top of the planar limit. In Ref. [12], the one-point function of a specific $\mathrm{SO}(3)\times\mathrm{SO}(3)$-invariant chiral primary was calculated by a variant of the Witten prescription, in particular, implying a supergravity approximation, which is justified here due to the assumption of $\lambda\to\infty$. As explained in Ref. [20], the result of this computation can be turned into a prediction for the one-point function we are considering divided by its tree-level value.

The prediction from the string theory reads

$$\frac{C_k(g)}{C_k(0)}\Big|_{st} = \frac{\Gamma(L+\frac{1}{2})}{\kappa^{L+1}\sqrt{\pi}\Gamma(L)}[\kappa^2+1]^{3/2} \times\int_{-\arctan\kappa}^{\pi/2}d\theta\cos^{2L-1}\theta(\kappa+\tan\theta)^{L-2}. \tag{31}$$

The leading two terms of the integral above in the large $\kappa = (\pi k/\sqrt{\lambda})$ expansion were already given in Ref. [12], and we can even evaluate the integral exactly to get

$$\frac{C_k(g)}{C_k(0)}\Big|_{st} = \frac{(\kappa+\sqrt{\kappa^2+1})^L(L\sqrt{\kappa^2+1}-\kappa)}{2^L(L-1)\kappa^{L+1}}. \tag{32}$$

Comparison.—Let us ignore the second line of (28), which as mentioned above starts contributing only at wrapping order. In the large-k limit, we have $B_n(1+k/2) \to (k/2)^n$, and the first line of (28) organizes itself as a power series in $(g/k)^2$:

$$\left.\frac{C_k(g)}{C_k(0)}\right|_{gt} \to 1 + \sum_{n=1}^{L/2} \binom{L-n}{n-1} \frac{L}{n} \frac{L+1}{L-n} \left(\frac{2g}{k}\right)^{2n} + O(g^{2L}), \quad \text{as } k \to \infty. \tag{33}$$

Remarkably, this agrees with the string-theory prediction up to wrapping order after identifying $\kappa = (k/4g) = (\pi k/\sqrt{\lambda})$. The terms in the second line of (28) have a scaling behavior in k which violates the double-scaling limit. It is tempting to attribute these terms to wrapping interactions.

Flux factor.—The flux factor in our proposal (15) has no counterpart at the tree level and depends on the anomalous scaling dimension $\Delta - L$ such that it vanishes for protected operators.

At one-loop order, the corresponding contribution in (17) has been calculated in Ref. [20]. It is the finite part of the UV-divergent integral whose UV divergence is subtracted by the renormalization constant and yields the one-loop scaling dimension $\Delta^{(1)}$. Since UV divergences exponentiate, it is possible that the flux factor exponentiates as well, and the following form of the higher loop flux factor seems natural:

$$\mathbb{F}_k = 2^{L-\Delta} \exp\left\{ (\Delta - L) \left[\Psi\left(\frac{k+1}{2}\right) + \gamma_E \right] \right\}. \tag{34}$$

A direct field-theoretic check of (34) at two-loop order would clearly be desirable, though very demanding.

An independent consequence of the flux factor is that it leads to a breakdown of the double-scaling limit for nonprotected operators starting already at one-loop order. As an example, let us consider the Konishi operator, which has $L=4$, $M=2$, and $u_1 = -u_2 = (1/2\sqrt{3}) + O(g^2)$. Its one-loop one-point function can be explicitly worked out to be

$$C_k = \frac{k(k^2-1)}{12\sqrt{3}} \left\{ 1 + 12 g^2 \left[\Psi\left(\frac{k+1}{2}\right) + \gamma_E - \log 2 + \frac{5}{6} \right] \right\}, \tag{35}$$

where we used that $\Delta^{(1)} = 12$. Since $\Psi(k+1/2) \sim \log k$ for large k, the perturbative expansion in the double-scaling limit does not arrange itself in powers of (λ/k^2).

Conclusions and outlook.—We have argued that the recently derived, integrability-based formula for tree-level one-point functions in the SU(2) sector of a specific defect version of the $\mathcal{N}=4$ SYM theory points towards a natural higher-loop generalization. The generalization is based on an idea which worked successfully for the spectral problem of the $\mathcal{N}=4$ SYM theory and which consists of introducing the coupling constant via a Zhukovski transformation of the Bethe roots characterizing the conformal operators. More precisely, the Zhukovski variables should replace the Bethe roots in both the Bethe equations and the transfer matrix of the system, and the Bethe equations should be equipped with the usual phase factor of the $\mathcal{N}=4$ SYM theory. Furthermore, in the present case, an additional flux factor contributing to the higher-loop one-point function formula is needed.

We have performed a number of nontrivial consistency checks of the generalized one-point function formula, and these have come out positive. First, we have compared the higher-loop one-point function formula to an honest field-theory calculation of the one-loop one-point function of nonprotected operators in the SU(2) sector. This calculation is technically demanding, involving the evaluation of the overlap of an uncorrected Bethe eigenstate and a so-called amputated matrix product state as well as the overlap between a loop-corrected Bethe eigenstate and an uncorrected matrix product state [20]. Results can be obtained analytically for BMN operators with two excitations, whereas for more complicated operators one has to resort to numerical computations. For all cases tested, the field-theory computation agreed with the proposed higher-loop formula. As a second test, we have carried out an analysis of the higher-loop formula when applied to the BMN vacuum state $\mathrm{tr}(X^L)$. For this state, the one-point function consists of two contributions: one which comes into play only at wrapping order and one for which it is possible to impose the double-scaling limit, proposed in Ref. [9], and obtain a power series expansion in the double-scaling parameter. This power series expansion can be compared to a similar expansion obtained by a string-theory analysis using a supergravity approximation, and agreement is found up to wrapping order for any length L of the BMN vacuum state. These two consistency checks constitute a strong indication that we are on the right track when trying to move towards higher loop orders.

The flux factor we propose depends on the anomalous dimension of the operator considered and leads to a breakdown of the double-scaling limit in the case of nonprotected operators starting already at one-loop order. While the exponentiation of the flux factor is certainly natural from the one-loop point of view, an explicit field-theoretic check at two-loop order is clearly required.

The presented higher-loop one-point function formula is expected to be only an asymptotic formula, in the sense that we expect there to be further corrections from wrapping interactions as was the case for the $\mathcal{N}=4$ SYM theory [29,30]. It would be very interesting to investigate the possible wrapping corrections in the present defect CFT or to study the theory using the thermodynamical Bethe ansatz approach to clarify whether the second line of (28) can indeed be understood as wrapping terms.

It would likewise be interesting to investigate whether the integrability approach can be used to infer some properties of the higher-loop contributions to other observables in the present defect CFT such as Wilson loops

[9,31,32] or less-studied objects such as two-point functions of operators of unequal conformal dimension [33].

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