Color Memory: A Yang-Mills Analog of Gravitational Wave Memory

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A transient color flux across null infinity in classical Yang-Mills theory is considered. It is shown that a pair of test "quarks" initially in a color singlet generically acquire net color as a result of the flux. A nonlinear formula is derived for the relative color rotation of the quarks. For a weak color flux, the formula linearizes to the Fourier transform of the soft gluon theorem. This color memory effect is the Yang-Mills analog of the gravitational memory effect.

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Introduction.—The gravitational memory effect in general relativity [1–3] concerns a subtle and beautiful aspect of the behavior of inertial detectors in the weak field region far from gravitating sources. It is a key observational link interconnecting asymptotic symmetries and soft theorems [4]. In this Letter, we derive the analog of this effect in classical non-Abelian gauge theory. The memory effect in Abelian gauge theory was discussed in Refs. [5–7].

The color memory effect can be seen by two test "quarks" in a color singlet stationed at a fixed large radius near future null infinity $\mathcal{I}^+$ and fixed angles Θ_α, where the label $\alpha = 1, 2$ distinguishes the two quarks. In order for the statement that they are in a color singlet to have any meaning, we must specify a flat connection $A = iUdU^{-1}$ on $\mathcal{I}^+$. Here U is an element in the gauge group G. For simplicity, we take the initial value at retarded time u_i to be

$$U(u_i) = 1, \tag{1}$$

although the generalization is straightforward. Now consider the effect of color flux or radiation through $\mathcal{I}^+$, which we take to begin after the initial time u_i and end before some final time u_f. Over this time interval, while the position of each quark is pinned to a fixed radius and angle, the color of each quark Q_α evolves according to

$$\partial_u Q_\alpha = iA_u(\Theta_\alpha)Q_\alpha, \tag{2}$$

where A_u denotes a component of the gauge field near $\mathcal{I}^+$. It is convenient to use the temporal gauge

$$A_u = 0, \tag{3}$$

so that the quarks do not change their colors.

By assumption, at late times $u > u_f$ the field strength vanishes and the connection is again flat. However, we will see that the classical constraint equation on $\mathcal{I}^+$ implies that generically

$$U(u_f) \neq 1. \tag{4}$$

This is color memory: The connection "remembers" some aspects of the color flux. This means that our two initially color-singlet quarks will no longer be in a color singlet after the passage of the color flux. Parallel transport from Θ_1 to Θ_2 to compare their colors will reveal a relative color rotation between the quarks:

$$U(u_f, \Theta_2)U^{-1}(u_f, \Theta_1). \tag{5}$$

This conclusion does not depend on the temporal gauge choice. The main result of this Letter is a nonlinear formula for $U(u_f)$ in terms of the color flux through $\mathcal{I}^+$.

Classical vacua in non-Abelian gauge theory are degenerate and labeled by the flat connections on the "celestial sphere" at $\mathcal{I}^+$ [8]. These vacua are related by the action of spontaneously broken "large" gauge symmetries that do not die off at infinity. The color flux through $\mathcal{I}^+$ is a domain wall which induces transitions between the degenerate vacua. The color memory effect measures these transitions. If the celestial sphere is initially tiled with test quarks pointing in the same direction in color space, after the passage of the color flux they will point in different directions upon parallel transport by (5). This enables one to read off the difference between the initial and final flat connections.

Gravity and Abelian gauge theory essentially become free theories at $\mathcal{I}^+$. In these cases, the memory effect is a linear function of the appropriate flux at $\mathcal{I}^+$ and is given by the Fourier transform of the soft theorem. This is not the case for non-Abelian gauge theory, for which nonlinear effects, albeit in a substantially weakened form, persist all the way to the boundary of Minkowski space. Finding the finite classical memory effect requires solving an interesting nonlinear partial differential equation (PDE) on the sphere. We present and solve this equation to the first two orders in weak field perturbation theory, with the first order given by the Fourier-transformed soft gluon theorem. (It would be interesting to either find closed form solutions or prove that they exist.) Interestingly, the full nonlinear PDE has appeared previously in the QCD literature. See, for example, the work of McLerran and collaborators [9], who were studying gluon distribution functions inside hadrons at small Bjorken scale x. This suggests that the color memory effect has already been encountered in some form, just not called by that name or related to the vacuum degeneracy of flat connections. This relation merits further investigation.

Efforts to measure the gravitational memory effect are underway at LIGO [10] and the pulsar timing array [11,12]. The measurement of $SU(3)$ color memory is of course difficult because of confinement. The measurement must take place on an energy scale above the confinement scale yet below that of the dynamical process. Indeed, as the basic equation of color memory has been previously encountered [9], color memory may already have been measured. A promising context for experimental applications of color memory is the color-glass condensate reviewed in Ref. [13]. Such applications are outside the scope of the present work and left to future studies.

Preliminaries.—In this section, we present notations, conventions, and an asymptotic expansion of the field equations.

We consider a non-Abelian gauge theory with gauge group G and elements g_R in representation R. Hermitian generators T_R^a in the representation R obey

$$[T_R^a, T_R^b] = i f^{abc} T_R^c, \tag{6}$$

where the a runs over the dimension of the group and the sum over repeated indices is implied. We denote the four-dimensional gauge potential $\mathcal{A}_\mu = \mathcal{A}_\mu^a T_R^a$ with spacetime index $\mu = 0, 1, 2, 3$.

Since we will be interested in the asymptotic expansions of fields near future null infinity ($\mathcal{I}^+$), it is convenient to introduce retarded coordinates in which the Minkowski metric reads

$$ds^2 = -du^2 - 2du dr + 2r^2 \gamma_{z\bar{z}} dz d\bar{z}, \tag{7}$$

where $(z, \bar{z})$ are stereographic coordinates on the celestial sphere with $\gamma_{z\bar{z}} = [2/(1+z\bar{z})^2]$ the unit round metric. The equations of motion are

$$\nabla^\nu \mathcal{F}_{\nu\mu} - i[\mathcal{A}^\nu, \mathcal{F}_{\nu\mu}] = g_{\mathrm{YM}}^2 j_\mu^M, \tag{8}$$

where j_μ^M is the matter color current, g_{YM} is the gauge coupling, and the field strength is

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - i[\mathcal{A}_\mu, \mathcal{A}_\nu]. \tag{9}$$

The theory is invariant under the gauge transformations

$$\mathcal{A}_\mu \to g_R \mathcal{A}_\mu g_R^{-1} + i g_R \partial_\mu g_R^{-1}, \qquad j_\mu^M \to g_R j_\mu^M g_R^{-1}. \tag{10}$$

Working in temporal gauge (3), we expand the remaining components of the gauge field near $\mathcal{I}^+$ in inverse powers of r [14]:

$$\begin{aligned}\mathcal{A}_r(u, r, z, \bar{z}) &= \frac{1}{r^2} A_r(u, z, \bar{z}) + \mathcal{O}(r^{-3}),\\ \mathcal{A}_z(u, r, z, \bar{z}) &= A_z(u, z, \bar{z}) + \mathcal{O}(r^{-1}).\end{aligned} \tag{11}$$

These falloff conditions ensure finite charge and energy flux through $\mathcal{I}^+$, and they are preserved by large gauge transformations that approach $(z, \bar{z})$-dependent Lie group-valued functions on the celestial sphere [8,15]. The leading behavior of the field strength is then

$$\begin{aligned}&\mathcal{F}_{ur} = \frac{1}{r^2} F_{ur} + \mathcal{O}(r^{-3}), \qquad \mathcal{F}_{uz} = F_{uz} + \mathcal{O}(r^{-1}),\\ &\mathcal{F}_{z\bar{z}} = F_{z\bar{z}} + \mathcal{O}(r^{-1}),\end{aligned} \tag{12}$$

where

$$F_{ur} = \partial_u A_r, \quad F_{uz} = \partial_u A_z, \quad F_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z - i[A_z, A_{\bar{z}}]. \tag{13}$$

In retarded coordinates, the u component of (8) reads

$$\nabla^r \mathcal{F}_{ru} + \nabla^A \mathcal{F}_{Au} - i([\mathcal{A}^r, \mathcal{F}_{ru}] + [\mathcal{A}^A, \mathcal{F}_{Au}]) = g_{\mathrm{YM}}^2 j_u^M, \tag{14}$$

where here and hereafter $A, B, \ldots$ run over the S^2 coordinates $(z, \bar{z})$. At leading order in the large-r expansion, we find

$$-\partial_u F_{ru} + D^A F_{Au} = J_u, \tag{15}$$

where D_A is the covariant derivative on the unit S^2 and its indices are raised and lowered with γ_{AB}. The asymptotic color flux is

$$J_u = i\gamma^{AB}[A_B, F_{Au}] + g_{\mathrm{YM}}^2 \lim_{r\to\infty} [r^2 j_u^M]. \tag{16}$$

This includes a quadratic term from the gauge potential itself, as gluons contribute to the color flux. Note that the left-hand side of (15) is linear in the gauge potential.

Color memory effect.—We wish to compute the change in the vacuum, or flat connection, induced by color flux J_u through $\mathcal{I}^+$. For simplicity, we consider configurations with no color flux or magnetic fields prior to some initial retarded time u_i and after some final retarded time u_f:

$$F_{z\bar{z}}|_{u<u_i} = F_{uz}|_{u<u_i} = F_{z\bar{z}}|_{u>u_f} = F_{uz}|_{u>u_f} = 0, \tag{17}$$

and where the Coulombic component of the field strength is constant over the sphere at initial and final times:

$$F_{ru}(u_i, z, \bar{z}) = F_{ru}(u_i), \quad F_{ru}(u_f, z, \bar{z}) = F_{ru}(u_f). \tag{18}$$

The leading component of $F_{z\bar{z}}$, unlike those of F_{ru} or F_{uz}, does not linearize at $\mathcal{I}^+$. This is the source of the nonlinearity of color memory. In this case, (8) determines $A_z|_{u>u_f}$ in terms of the color flux J_u, $A_z|_{u<u_i}$, and the initial and final electric fields $F_{ru}|_{u<u_i}$ and $F_{ru}|_{u>u_f}$ (which may be set to zero in some applications). By (17), the transverse components of the boundary gauge fields are pure gauge and hence related by a large gauge transformation. The large gauge transformation determining the change in A_z across $\mathcal{I}^+$ can be found by solving (15) subject to the boundary conditions (17).

We now determine the change of A_z across $\mathcal{I}^+$. Integrating (15) over u, we find

$$-D^A \Delta A_A = \int_{u_i}^{u_f} du J_u + \Delta F_{ru}, \tag{19}$$

where

$$\Delta A_z = A_z(u_f, z, \bar{z}) - A_z(u_i, z, \bar{z}), \quad \Delta F_{ru} = F_{ru}(u_f) - F_{ru}(u_i). \tag{20}$$

Let us set $A_A(u_i) = 0$ by performing a large gauge transformation and define

$$J_{z\bar{z}} = -\gamma_{z\bar{z}} \left(\int_{u_i}^{u_f} du J_u + \Delta F_{ru} \right). \tag{21}$$

Then, (19) reduces to

$$\partial_z A_{\bar{z}}(u_f) + \partial_{\bar{z}} A_z(u_f) = J_{z\bar{z}}. \tag{22}$$

The general solution to the boundary conditions (17) is

$$A_z(u_f) = iU\partial_z U^{-1}, \tag{23}$$

where $U = U(z, \bar{z}) \in G$. Substituting this solution in (22), we obtain an equation for the large gauge transformation U relating initial and final flat connections on S^2:

$$i\partial_z(U\partial_{\bar{z}} U^{-1}) + i\partial_{\bar{z}}(U\partial_z U^{-1}) = J_{z\bar{z}}. \tag{24}$$

The color memory effect is defined by the solution to this equation. $U(z, \bar{z})$ determines the flat connection on S^2 after the color flux passes through $\mathcal{I}^+$.

To solve this equation perturbatively in $J_{z\bar{z}}$, we first invert (23):

$$U(z, \bar{z}) = \mathcal{P}\left\{ \exp\left(i \int_{(0,0)}^{(z,\bar{z})} dw A_w + d\bar{w} A_{\bar{w}} \right) \right\}, \tag{25}$$

where $\mathcal{P}$ denotes path ordering. $U(z, \bar{z})$ is independent of the path, because A is flat. Then, since the asymptotic gauge potential $A(u_f)$ is a 1-form on S^2, it can be parametrized as

$$A_z(u_f) = \partial_z(\alpha + i\beta), \qquad A_{\bar{z}}(u_f) = \partial_{\bar{z}}(\alpha - i\beta), \tag{26}$$

where α and β are Lie algebra-valued real functions on S^2. Substituting into (22), we find

$$2\partial_z \partial_{\bar{z}} \alpha = J_{z\bar{z}}, \tag{27}$$

which is solved by

$$\alpha(z) = \frac{1}{4\pi} \int d^2 w G(z, w) J_{w\bar{w}},$$
$$G(z, w) = \log \frac{|z - w|^2}{(1 + z\bar{z})(1 + w\bar{w})}. \tag{28}$$

The boundary conditions (17) give an additional differential equation for β:

$$2\partial_z \partial_{\bar{z}} \beta + [\partial_z \alpha, \partial_{\bar{z}} \alpha] + [\partial_z \beta, \partial_{\bar{z}} \beta] - i[\partial_z \alpha, \partial_{\bar{z}} \beta] + i[\partial_z \beta, \partial_{\bar{z}} \alpha] = 0. \tag{29}$$

This equation can be solved perturbatively in $J_{z\bar{z}}$ for β, which is given to leading order by

$$\beta(z) = -\frac{1}{(4\pi)^3} \int d^2 w d^2 w' d^2 w'' G(z, w) \partial_w G(w, w') \times \partial_{\bar{w}} G(w, w'') [J_{w'\bar{w}'}, J_{w''\bar{w}''}] + \mathcal{O}(J^3). \tag{30}$$

To this order in perturbation theory, the expression for the gauge field is

$$A_z = \int \frac{d^2 w}{4\pi} \partial_z G(z, w) \left[J_{w\bar{w}} - i \int \frac{d^2 w'}{4\pi} \frac{d^2 w''}{4\pi} \partial_w G(w, w') \partial_{\bar{w}} G(w, w'') [J_{w'\bar{w}'}, J_{w''\bar{w}''}] \right] + \mathcal{O}(J^3),$$
$$A_{\bar{z}} = \int \frac{d^2 w}{4\pi} \partial_{\bar{z}} G(z, w) \left[J_{w\bar{w}} + i \int \frac{d^2 w'}{4\pi} \frac{d^2 w''}{4\pi} \partial_w G(w, w') \partial_{\bar{w}} G(w, w'') [J_{w'\bar{w}'}, J_{w''\bar{w}''}] \right] + \mathcal{O}(J^3), \tag{31}$$

from which the large gauge transformation associated to the vacuum transition can be directly obtained via (25). It may be seen that the first term is the Fourier transform of the soft gluon theorem, while the second is the leading nonlinear correction for a finite color flux.

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