Entropy Production and Fluctuation Theorems for Active Matter

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Active biological systems reside far from equilibrium, dissipating heat even in their steady state, thus requiring an extension of conventional equilibrium thermodynamics and statistical mechanics. In this Letter, we have extended the emerging framework of stochastic thermodynamics to active matter. In particular, for the active Ornstein-Uhlenbeck model, we have provided consistent definitions of thermodynamic quantities such as work, energy, heat, entropy, and entropy production at the level of single, stochastic trajectories and derived related fluctuation relations. We have developed a generalization of the Clausius inequality, which is valid even in the presence of the non-Hamiltonian dynamics underlying active matter systems. We have illustrated our results with explicit numerical studies.

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Active matter systems are composed of constitutive elements that are capable of self-propulsion. Either through an internal mechanism or by extracting energy from their environment, these elements exhibit self-induced motion in the absence of any externally applied force. Examples include solutions containing single cellular organisms such as bacteria or protozoa, synthetic colloidal systems, and vibrated monolayers of granular matter [1–6]. In fact, active matter models have been used to describe flocking, schooling, and herding behavior in animal movement [7–11]. Active systems are attracting growing interest due to their relevance for understanding live biological systems and their potential applications to the design of synthetic colloidal systems with controllable properties [12–18]. Moreover, they exhibit novel collective properties such as phase separation in the absence of explicit attractive interactions [19–26], rectification of random fluctuations [27], and spontaneous self-organization and pattern formation [28,29], which make their study of interest in its own right.

Active matter systems constitute a new class of condensed matter systems that are inherently out of equilibrium and thereby not describable by the standard, Gibbsian framework. While the collective behavior of active particles has been modeled by hydrodynamic equations based on conservation and symmetry principles [15], and individual active particles by various Brownian dynamics [18], a systematic framework for the *nonequilibrium* thermodynamics and statistical mechanics of active matter is still in development. Many of the published studies so far have focused on utilizing equilibrium thermostatic notions, often based on approximating active systems by passive systems [30–39]. In this Letter we propose an alternative approach based on stochastic thermodynamics [40,41], which is an emerging framework for the description of thermodynamics and statistical mechanics of stochastic systems far from equilibrium. Stochastic thermodynamics has enabled us to define thermodynamic quantities such as energy, work, heat, entropy, and entropy production at the level of individual realizations of stochastic dynamics. Moreover, one obtains exact analytical results for the fluctuations of entropy production in the form of *equalities*, as opposed to the inequalities of the second law of thermodynamics. These equalities are more popularly known as fluctuation relations [42–47].

In this Letter, we choose the active Ornstein-Uhlenbeck process (AOUP), alternatively called the Gaussian colored noise model, to illustrate our approach [48–54]. Like other active matter models, this model is known to exhibit motility-induced phase separation (MIPS) [55]. The AOUP model is different from many other models of active matter [56–58] in that there is no explicit internal drive that forces the system out of equilibrium; the active behavior of the system arises from the nonequilibrium nature of the forces from the environment. In particular, the damping and fluctuating forces from the environment do not satisfy the fluctuation-dissipation relation (FDR). This fact makes it a challenge to develop stochastic thermodynamics for the AOUP, as the usual approaches rely heavily on the equilibrium nature of the environment. We overcome this challenge by proposing an exact mathematical mapping of the AOUP, which is an overdamped Langevin model, to a passive, underdamped Langevin model with effective reservoir forces that satisfy the FDR. We derive our generalizations of both the first and the second laws of thermodynamics in reference to this mapped system, the latter giving rise to a modified Clausius inequality. Moreover, we derive both integral and detailed fluctuation relations for entropy production. These in turn allow us to make exact and verifiable predictions for the behavior of the original active matter system.

This Letter is inspired in part by a recent study of the AOUP [49], in which the authors studied entropy production of the AOUP based on its path-integral representation. The entropy production in this context involves the time reversal of a stochastic process with nonconservative forces, which is still a debated issue [59,60]. Our work introduces a particular microscopic outlook to the resolution of this issue through an explicit model of an effective heat bath. This leads to an expression for entropy production [Eq. (9)] that is different from that found in Ref. [49]. In particular, we find that it is nonzero and positive even for an AOUP in a simple harmonic potential, whereas the authors in Ref. [49] report zero average entropy production in such cases. The crucial difference between our treatment and theirs lies in the very definition of entropy production along a trajectory: In our definition [Eq. (8)] we consider the time reversal of the dynamics in addition to the reversal of trajectories, in accordance with the framework of stochastic thermodynamics [41,61]; in Ref. [49] the time reversal of the dynamics had not been considered [see Eq. (9) in Ref. [49]]. Moreover, we have considered a time-dependent scenario in which the potential energy of the system may vary with time—leading to a renormalized potential energy in the underdamped dynamics, different from the original one—a case not considered in the earlier study. The fact that other authors have obtained different results for the AOUP model [49] suggests further study is needed to determine definitively whether the results we present here are consistent with every microscopic system described by the AOUP.

Consider a suspension of N active colloidal particles with $\mathbf{x}_i$ denoting the position of the ith particle. In the absence of the medium, the dynamics of the particles are governed by the possibly time-dependent potential $\Phi(\mathbf{X}, t)$, where $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N)$ refers to the configuration space of the whole system. There are two forces from the medium: a damping force, $-\dot{\mathbf{x}}_i/\mu$ for particle i, and a Gaussian random force, $\mathbf{v}_i/\mu$, the latter having the following properties [62]:

$$\langle v_{ia} \rangle = 0, \qquad \langle v_{ia}(0) v_{jb}(t) \rangle = \delta_{ij}\delta_{ab}\frac{D}{\tau}e^{-|t|/\tau}, \tag{1}$$

for all i, j, a, and b. Here, the angular brackets $\langle \dots \rangle$ denote the noise average (i.e., the average with respect to many realizations of the random forces $\mathbf{v}_i$); v_{ia} denotes the ath component of $\mathbf{v}_i$; δ_{xy} denotes the Kronecker delta function; τ is the persistence time of the noise; and D is the diffusion coefficient. Equation (1) implies, in particular, that the random forces felt by different particles in different directions are independent of each other. The Langevin equation of the ith particle is given by

$$\dot{\mathbf{x}}_i = -\mu \nabla_i \Phi + \mathbf{v}_i, \tag{2a}$$

$$\tau \dot{\mathbf{v}}_i = -\mathbf{v}_i + \sqrt{2D}\boldsymbol{\eta}_i, \tag{2b}$$

where $\boldsymbol{\eta}_i$ denotes a Gaussian random noise with the properties $\langle \eta_{ia} \rangle = 0$ and $\langle \eta_{ia}(0)\eta_{jb}(t) \rangle = \delta_{ij}\delta_{ab}\delta(t)$ with $\delta(t)$ denoting the Dirac delta distribution.

Because the noise force $\mathbf{v}_i/\mu$ has exponential memory whereas the damping force $-\dot{\mathbf{x}}_i/\mu$ is memoryless, the model violates the FDR, and we have to conclude that the medium is out of equilibrium. As pointed out in Ref. [49], in the limit of vanishing persistence time ($\tau \to 0$) the model reduces to an equilibrium model satisfying the FDR with respect to temperature $T \equiv D/(k_B \mu)$ where k_B is the Boltzmann constant. Motivated by this observation, we replace D in the following discussion by $\mu k_B T$. We also use the notation $\beta = 1/k_\mathrm{B} T$.

As remarked in the introduction, in the absence of the FDR we cannot utilize the framework of stochastic thermodynamics as is. In particular, we cannot interpret the heat given to the medium divided by T to be the change in entropy of the medium for finite persistence time τ. Fortunately, it is possible to overcome this challenge due to a surprising property of the fluctuations of this active matter system: the overdamped AOUP can be mapped *exactly* to an underdamped Langevin process where the new, effective medium (*reservoir*) is in equilibrium. The effective underdamped process is given by (Sec. I in the Supplemental Material [63])

$$\dot{\mathbf{x}}_i = \frac{\mathbf{p}_i}{m}, \tag{3a}$$

$$\dot{\mathbf{p}}_i = -\nabla_i \Psi - \frac{\mathbf{p}_i}{\mu m} + \sqrt{\frac{2}{\mu\beta}}\boldsymbol{\eta}_i - \mu(\mathbf{P}\cdot\nabla)\nabla_i \Phi, \tag{3b}$$

where $\mathbf{p}_i$ is the auxiliary momentum of particle i; $m = \tau/\mu$ is the effective mass; $\Psi = \Phi + \mu m (\partial/\partial t)\Phi$ is an effective potential; and $\mathbf{P} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N)$ and $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N)$, respectively, refer to the phase-space momenta and coordinates of the whole system. Also, we have used ∇_i to denote the gradient with respect to $\mathbf{x}_i$ and $\nabla = (\nabla_1, \nabla_2, \dots, \nabla_N)$ to denote the spatial gradient in the phase space of the whole system. The damping and the noise terms, $-\mathbf{p}_i/\mu m$ and $\sqrt{2/\mu\beta}\boldsymbol{\eta}_i$, respectively, satisfy the FDR with respect to temperature T. In the following we interpret them to be forces from the effective, equilibrium reservoir. The nonequilibrium nature of this mapped dynamics arise from the momentum-dependent force $\mathbf{F}_{i,m} \equiv -\mu(\mathbf{P}\cdot\nabla)\nabla_i\Phi$. In some sense, we have decomposed the forces of the nonequilibrium medium into those of an underlying equilibrium reservoir and explicit forces. We now develop the results of stochastic thermodynamics around this model.

To begin, we consider the first law of thermodynamics—namely, conservation of energy. The total energy of the system is given by $E = (1/2)P^2/m + \Psi$, kinetic plus potential energy. A trajectory Γ over the interval $[0, t]$ is defined to be the sequence of points $\Gamma = \{\mathbf{X}_{0:t}, \mathbf{P}_{0:t}\} = \{\mathbf{X}(t'), \mathbf{P}(t') | 0 \le t' \le t\}$. Work done on the system along any Γ is given by [64]:

$$W[\Gamma] = \int_0^t dt' \left(\frac{\partial \Psi}{\partial t'} - \mu(\mathbf{P}\cdot\nabla)\nabla\Phi \circ \frac{\mathbf{P}}{m} \right), \tag{4}$$

where the first term denotes the thermodynamic work corresponding to the conservative force, $-\nabla_i\Psi$, and the second term denotes the mechanical work done by the nonconservative force, $\mathbf{F}_{i,m}$. Here, the circle ($\circ$) denotes Stratonovich multiplication [65], the necessity of which follows from the chain rule of derivatives [64,66]. The heat given to the reservoir over Γ is the amount of work done against the reservoir forces:

$$Q^{\mathrm{res}}[\Gamma] = -\int_0^t dt' \sum_{i=1}^{N} \left(-\frac{\mathbf{p}_i}{\tau} + \sqrt{\frac{2}{\mu\beta}}\boldsymbol{\eta}_i\right) \circ \frac{\mathbf{p}_i(t')}{m}. \tag{5}$$

The consistency of these definitions can be seen through the following relation, the first law of thermodynamics for the current system (Sec. II in the Supplemental Material [63]):

$$E(t) - E(0) = W[\Gamma] - Q^{\mathrm{res}}[\Gamma], \tag{6}$$

which holds true at the level of individual stochastic trajectories, not just on the average.

We now consider the second law of thermodynamics. Let $\rho(\mathbf{X},\mathbf{P};t)$ be the phase space probability density of the system at any time t. If the system is at $[\mathbf{X}(t),\mathbf{P}(t)]$ at time t, following the developments in stochastic thermodynamics we can define the *stochastic entropy* $s(t)$ at time t to be $s(t) = -\ln\rho(t)$ with $\rho(t) \equiv \rho[\mathbf{X}(t),\mathbf{P}(t);t]$ [46]. The average entropy of the system $H(t)$ at any time t is given by the average of $s(t)$ with respect to $\rho(\mathbf{X},\mathbf{P};t)$, which is also the Shannon information $H(t) = -\int d\mathbf{X}d\mathbf{P}\rho(\mathbf{X},\mathbf{P};t)\ln\rho(\mathbf{X},\mathbf{P};t)$. The change in entropy of the reservoir over the interval $[0,t]$, on the other hand, is given by the Clausius formula $\beta Q^{\mathrm{res}}[\Gamma]$, because the reservoir is in equilibrium [Eq. (5)]. Unlike the usual second law of thermodynamics for passive systems, however, the total entropy production over $[0,t]$ is not just the sum of the change in the stochastic entropy of the system, Δs, and the Clausius entropy change of the medium, Q^{res}/T. To see this we need to first define the time reversal of the mapped process in Eq. (3) (Sec. III in the Supplemental Material [63]):

$$\dot{\mathbf{x}}_i = \frac{\mathbf{p}_i}{m}, \tag{7a}$$

$$\dot{\mathbf{p}}_i = -\nabla_i\Psi - \frac{\mathbf{p}_i}{\mu m} + \sqrt{\frac{2}{\mu\beta}}\boldsymbol{\eta}_i + \mu(\mathbf{P}\cdot\nabla)\nabla_i\Phi, \tag{7b}$$

obtained by keeping the reservoir terms unchanged and replacing t and $\mathbf{P}$ by $-t$ and $-\mathbf{P}$, respectively, in the rest of the terms. We also need to consider the time reversal of the phase space trajectory Γ, given by $\Gamma^r = \{\mathbf{X}^r_{0:t}, \mathbf{P}^r_{0:t}\}$ with $\mathbf{X}^r(t') = \mathbf{X}(t-t')$ and $\mathbf{P}^r(t') = -\mathbf{P}(t-t')$. Next, we need to consider $P(\Gamma)$, the probability of Γ in process (3) and $P^{\mathrm{r}}(\Gamma^r)$, the probability of the time-reversed trajectory Γ^r in the time-reversed process [Eq. (7)]. If there is any explicit time dependence in Ψ and Φ, the time dependence has to be reversed as well in the time-reversed process. Because entropy production is a measure of time-reversal symmetry breaking, entropy production $\Sigma[\Gamma]$ along Γ is given by

$$\Sigma[\Gamma] \equiv k_B \ln\frac{P[\Gamma]}{P^r[\Gamma^r]}. \tag{8}$$

Using Eqs. (3) and (7), we can derive an explicit path integral expression (Sec. IV in the Supplemental Material [63]):

$$\frac{\Sigma[\Gamma]}{k_B} = \Delta s + \beta Q^{\mathrm{res}}[\Gamma] + \frac{\mu^2\beta}{2}\int (d\mathbf{P}\cdot\nabla)^2\Phi. \tag{9}$$

It is easy to prove that the average of $\Sigma(\Gamma)$ is non-negative as required by the second law of thermodynamics. This follows from the fact that the average of Σ can be written as $\langle\Sigma\rangle/k_B = \sum_\Gamma P[\Gamma]\ln[P[\Gamma]/P^{\mathrm{r}}(\Gamma^{\mathrm{r}})]$, which takes the form of a relative entropy with the known property that it is never negative [67]. In fact, we can derive the following expression for the average entropy production (Sec. V in the Supplemental Material [63]):

$$\langle\Sigma\rangle(t) = k_B\Delta H + \frac{\langle Q^{\mathrm{res}}\rangle}{T} + k_B\mu\int_0^t dt'\langle\nabla^2\Phi\rangle \geq 0. \tag{10}$$

This is the central result of our Letter. It expresses the second law of thermodynamics as a modified Clausius inequality. Each quantity in the inequality can be calculated even for the original dynamics as the mapped system is mathematically equivalent to the AOUP. The inequality therefore constitutes a prediction for the original system.

The difference between the usual Clausius inequality and our generalization in Eq. (10) is embodied by the last term $k_B\mu\int_0^t d]t'\langle\nabla^2\Phi\rangle$. Even though this term arises from the momentum-dependent force $\mathbf{F}_{i,m}$, it is not the average work done by $\mathbf{F}_{i,m}$ as can be seen from Eq. (4). If the reservoir terms are taken out from Eq. (3), due to $\mathbf{F}_{i,m}$, the dynamics is still not Hamiltonian and the phase space volume is not conserved under the dynamics. The last term in Eq. (10) is the integral of the average phase space contraction rate due to $\mathbf{F}_{i,m}$ (Sec. VI in the Supplemental Material [63]). For deterministic thermostats, this is interpreted as the entropy production [68–72]. Moreover, recent developments in stochastic thermodynamics have demonstrated that the usual Clausius inequality has to be modified in the presence of feedback control [73–84]. The momentum-dependent force $\mathbf{F}_{i,m}$ can be seen as a spatially inhomogeneous feedback cooling operation. The last term in Eq. (10) refers to the extra contribution to entropy production from the feedback controller. Towards the end of this Letter we will demonstrate that it is crucial to include this extra term. In its absence the inequality may not be satisfied. Mathematically, the appearance of force-divergence terms in fluctuation relations and consequent inequalities should be expected whenever the "conjugate" process involves a reversal of forces [85].

A major contribution from stochastic thermodynamics is the surprising result that the inequalities of the second law can be replaced by exact equalities, called the fluctuation relations. These are in a sense more refined versions of the inequalities because the latter can be derived by the application of Jensen's inequality. We can derive the following integral fluctuation relation for entropy production in our active matter system:

$$\langle e^{-\Sigma/k_B}\rangle = \sum_{\Gamma} P[\Gamma] e^{-\Sigma[\Gamma]/k_B} = \sum_{\Gamma^r} P^r[\Gamma^r] = 1, \tag{11}$$

where the second relation follows from the definition of entropy production [Eq. (8)] and the third relation follows from the normalization of $P^r[\Gamma^r]$. Equation (11) is our second main result. The modified Clausius inequality $\langle\Sigma\rangle \geq 0$ follows from the application of Jensen's inequality to Eq. (11). In fact, there is a more detailed equality underlying Eq. (11) (Sec. VII in the Supplemental Material [63]):

$$P(\Sigma = \sigma) = e^{\sigma/k_B} P^r(\Sigma^r = -\sigma), \tag{12}$$

where $P(\Sigma = \sigma)$ denotes the probability density of $\Sigma = \sigma$ in the process described by Eq. (3) and $P^{\mathrm{r}}(\Sigma^{\mathrm{r}} = \sigma)$ denotes the same quantity for the reverse process [Eq. (7)]. We have to assume that the initial condition for the reverse process is obtained from the final condition of the first process by reversing the sign of the momenta. Equation (12) is the detailed fluctuation relation for entropy production.

There are three qualitatively different fluctuation relations (both integral and detailed) in stochastic thermodynamics for passive, overdamped dynamics: that of total entropy production, excess entropy production, and housekeeping heat. We have already extended the relation for total entropy production to the AOUP. For the sake of completeness we will address the latter two cases in the following, starting with the excess entropy production. Let the steady state distribution of the dynamics in Eq. (3) in the absence of any time dependence of Φ be $\rho_s(\mathbf{X}, \mathbf{P}; \boldsymbol{\alpha})$, where $\boldsymbol{\alpha}$ represents fixed, external parameters of the system. The excess entropy production along any trajectory Γ between two nonequilibrium steady states is given by $-\Delta \ln \rho_s + \int_0^t dt' \dot{\mathbf{Z}} \circ \nabla_{\mathbf{Z}} \ln \rho_s$, with $\mathbf{Z}(t') = [\mathbf{X}(t'), \mathbf{P}(t')]$. The first term denotes the change in stochastic entropy of the system and the second term the generalized work done against the generalized (nonequilibrium) force $-k_B T \nabla_{\mathbf{Z}} \ln \rho_{\mathrm{s}}$ in units of $k_B T$, which is also called the excess heat $Q^{\mathrm{res,ex}}$ (in units of $k_B T$). The excess entropy production can also be written as $-\int_0^t dt' \dot{\boldsymbol{\alpha}} \cdot \nabla_{\boldsymbol{\alpha}} \ln \rho_s$. The corresponding integral fluctuation relation

$$\langle e^{\int_0^t dt' \dot{\boldsymbol{\alpha}} \cdot \nabla_{\boldsymbol{\alpha}} \ln \rho_s} \rangle = 1 \tag{13}$$

is a consequence of the Markovian nature of the dynamics as shown in Ref. [44]. By applying Jensen's inequality we get

$$\Delta H_{\mathrm{s}} + \beta \langle Q^{\mathrm{res,ex}} \rangle \geq 0, \tag{14}$$

which can be a stricter bound for ΔH compared to that of Eq. (10). Unlike total entropy production, there is generally no detailed fluctuation relation for excess entropy production because the steady state distribution ρ_s is generally not time-reversal symmetric, $\rho_s(\mathbf{X}, -\mathbf{P}) \neq \rho_s(\mathbf{X}, \mathbf{P})$ [86]. The concept of housekeeping heat for the current system is ambiguous due to the momentum-dependent force $\mathbf{F}_{i,m}$. It has been recently shown that such systems have many different notions of housekeeping heat each with its own fluctuation relation and consequent inequality [87]. Given this ambiguity, we reserve a detailed discussion of the relevant results for a future study.

We now illustrate our results with a simple case study. Consider a single active particle in a one-dimensional simple harmonic potential, $\Phi = x^2/2$, where we have set the spring constant to one. For simplicity, we assume the other parameters, μ, τ, T, and the constant k_B, to be unity as well. In Fig. 1 we have plotted the probability distribution of entropy production Σ over 0.5 units of time in the steady state of the system. We see that it is possible to have negative entropy production over individual trajectories, but the average over sufficiently many trajectories is always non-negative. In particular, the average entropy production in the current case turns out to be $\langle\Sigma\rangle = 0.250 \pm 0.005$. This is in contrast to the entropy production formula proposed in Ref. [49], which predicts the entropy production in the current scenario to be zero. To show the importance of the phase space contraction term in entropy production, the last term in Eq. (10), we have also measured the incomplete entropy production $\Sigma' = \Delta s + Q^{\mathrm{res}}/T$. In this case we find $\langle\Sigma'\rangle = -0.250 \pm 0.005$. The negativity of the average implies that the usual Clausius inequality is not satisfied even for an AOUP in one dimension with a simple harmonic potential.

A popular approach to analyzing the AOUP is to use a perturbative expansion in τ [34,49]. In contrast, our results are applicable for all values of τ. In particular, the expression for

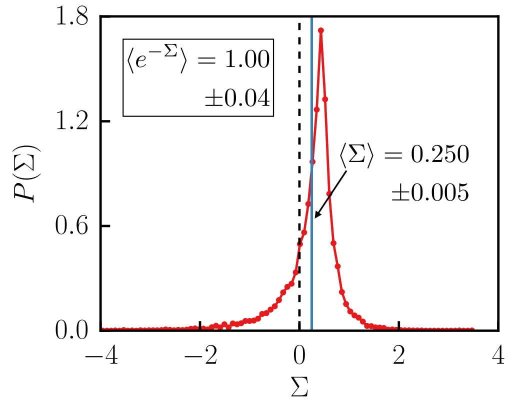

FIG. 1. Numerical distribution of entropy production Σ of a single active particle in a one-dimensional simple harmonic potential, $\Phi = x^2/2$. We have used temperature units to set $k_B = 1$. We see that entropy production can be negative for individual realizations. However, the average entropy production is positive, as shown by the blue, solid vertical line and the associated numerical value. Moreover, the integral fluctuation relation for entropy production [Eq. (11)] is satisfied.

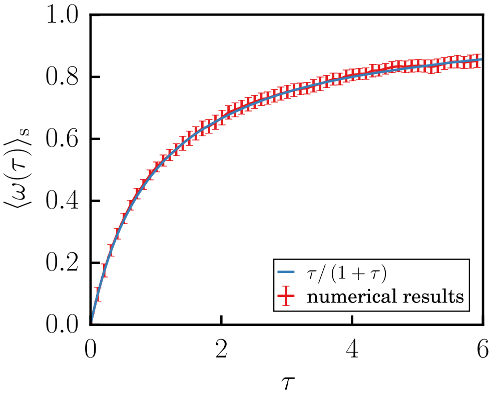

FIG. 2. Dependence of average steady-state entropy production rate $\langle \omega(\tau) \rangle_s$ on the persistence time τ for an active particle in a simple harmonic well. As in the case of Fig. 1, in the plot we have assumed mobility μ and temperature T to be unity and a temperature scale that ensures $k_B = 1$.

entropy production [Eq. (10)] is valid no matter how large the value of τ. To demonstrate this, we have considered in detail the same example as above. We have derived an exact analytical expression for the average steady state entropy production rate, $\langle \omega(\tau) \rangle_s \equiv \lim_{t\to\infty} \langle \Sigma_t \rangle_s / t$, as a function of τ given by $\langle \omega(\tau) \rangle_s = \tau/(1+\tau)$. [We have assumed the parameters k (spring constant), μ, and T, and the constant k_B to be unity.] We have tested this formula by numerically integrating the equation of motion, as demonstrated in Fig. 2. As expected, we see that $\langle \omega(\tau) \rangle_s$ is zero for $\tau = 0$, because the system is in equilibrium in this case. Somewhat counterintuitively, the $\langle \omega(\tau) \rangle_s$ approaches a finite value 1 (more generally $\mu k_B T$ if no special values for μ and T and special temperature unit are considered) as τ tends to infinity. This means that larger τ does not imply a larger entropy production rate (farther from equilibrium) when τ becomes large compared to the relaxation time scale of the system.

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Note added.—While the Letter was in review we became aware of Refs. [88,89], the first one dealing with the dynamics of harmonically confined beads in an active bath and the second one dealing with the Clausius relation for active matter. We have discussed the stochastic thermodynamics of the model in Ref. [88] in Sec. VIII of our Supplemental Material [63]. This discussion is consistent with that in the second reference [89].

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