Epi-Two-Dimensional Fluid Flow: A New Topological Paradigm for Dimensionality

Z. Yoshida[1,*] and P. J. Morrison[2,†]
[1]*Department of Advanced Energy, The University of Tokyo, Kashiwa, Chiba 277-8561, Japan*
[2]*Department of Physics, The University of Texas at Austin, Austin, Texas 78712, USA*
(Received 27 March 2017; published 11 December 2017)

While a variety of fundamental differences are known to separate two-dimensional (2D) and three-dimensional (3D) fluid flows, it is not well understood how they are related. Conventionally, dimensional reduction is justified by an *a priori* geometrical framework; i.e., 2D flows occur under some geometrical constraint such as shallowness. However, deeper inquiry into 3D flow often finds the presence of local 2D-like structures without such a constraint, where 2D-like behavior may be identified by the integrability of vortex lines or vanishing local helicity. Here we propose a new paradigm of flow structure by introducing an intermediate class, termed epi-two-dimensional flow, and thereby build a topological bridge between 2D and 3D flows. The epi-2D property is local and is preserved in fluid elements obeying ideal (inviscid and barotropic) mechanics; a local epi-2D flow may be regarded as a "particle" carrying a generalized enstrophy as its charge. A finite viscosity may cause "fusion" of two epi-2D particles, generating helicity from their charges giving rise to 3D flow.

DOI: 10.1103/PhysRevLett.119.244501

Phenomenologically, two-dimensional (2D) fluid flow is very different from three-dimensional (3D) flow in that the former is less turbulent and more capable of generating and sustaining large-scale vortical structures [1]. This is because the dynamics of vortices in 2D systems is constrained, resulting in the suppression of some essential mechanisms of turbulence. Here we generalize by replacing the usual *geometrical* constraint by a *topological* constraint that extracts the essential property of 2D flow. We call such a constrained flow epi-2D.

Conventionally, the 2D geometrical constraint is believed appropriate for a fluid with limited depth and slow variation of physical quantities in the vertical direction compared to those in the horizontal directions. However, 2D-like behavior, our epi-2D flow, may occur in flows without being strictly 2D. For example, sometimes in 3D systems, flow may not be totally 3D by having subdomains in which the flow is 2D-like [2].

In this work, we precisely formulate the concept of epi-2D flow. The difference in the invariants of 3D and 2D systems serves as a guide: As is well known, the helicity is a constant of motion in an ideal 3D flow [3,4], while in 2D geometry the helicity degenerates to zero, being compensated by the enstrophy (or its generalization as described below). Usually, these two different invariants are regarded as attributes of different dimensionality [5], but we switch viewpoints and use the invariants as discriminants of dimensionality. Interestingly, the class of flows that conserve (appropriately generalized) enstrophy is much larger than the geometrical 2D class. This extended class is what we refer to as epi-2D. We will see that the "fusion" of two epi-2D flows may yield a 3D flow, transmuting the corresponding enstrophies into helicity. In fact, the epi-2D behavior is a local property, so we can formulate a "particle picture" of transmutation.

Consider first the basic equations and conservation laws of fluid mechanics. Let M be a 3D domain containing an ideal (inviscid) barotropic fluid. We assume $M = \mathbb{T}^3$, the 3-torus, and ignore boundary effects [6]. Denoting by ρ the mass density, $\boldsymbol{V}$ the fluid velocity, and P the pressure, the governing equations are

$$\partial_t \rho = -\nabla \cdot (\boldsymbol{V}\rho), \tag{1}$$

$$\partial_t \boldsymbol{V} = -(\boldsymbol{V}\cdot\nabla)\boldsymbol{V} - \nabla h, \tag{2}$$

where $\rho^{-1}\nabla P = \nabla h$ with an enthalpy $h = h(\rho)$. The energy of the system is

$$H = \int_M \left[\frac{1}{2}|\boldsymbol{V}|^2 + \varepsilon(\rho)\right]\rho d^3x, \tag{3}$$

where $\varepsilon(\rho)$ is the internal energy per unit mass and $\partial(\rho\epsilon)/\partial\rho = h$. It follows by direct calculation that the energy H, the helicity

$$C = \int_M \boldsymbol{V}\cdot\boldsymbol{\omega} d^3x, \tag{4}$$

with $\boldsymbol{\omega} = \nabla\times\boldsymbol{V}$ being the vorticity, and the total mass $N = \int_M \rho d^3x$ are conserved.

The 2D geometrical reduction can be obtained as follows. Let z be a "perpendicular" coordinate in the Cartesian (x, y, z) system and let $\boldsymbol{e}_z = \nabla z$. The reduction with $\boldsymbol{e}_z \cdot \boldsymbol{V} = 0$ and $\partial_z = 0$ yields the 2D system on the x-y plane (a flat torus $\mathbb{T}^2$). Using $\boldsymbol{V} = (v_x, v_y, 0)^T$ and

$\boldsymbol{v} = (v_x, v_y)^T$, the vorticity becomes $\boldsymbol{\omega} = \nabla \times \boldsymbol{V} = \omega \boldsymbol{e}_z$, where $\omega = \partial_x v_y - \partial_y v_x$. Because $\boldsymbol{V} \cdot \boldsymbol{\omega} = 0$ for this 2D reduction, helicity conservation is now trivial: $C \equiv 0$. Interestingly, however, a different invariant emerges: The generalized enstrophy

$$Q = \int_M f(\vartheta)\rho d^2x, \tag{5}$$

with the potential vorticity $\vartheta = \omega/\rho$, is now a constant of motion (f being an arbitrary smooth function). It is also easy to show the constancy of the "local" enstrophy that is defined by replacing the domain M of the integral (5) by an arbitrary comoving (i.e., transported by the flow $\boldsymbol{v}$) subdomain $\Sigma(t)$. For the simple choice $f(\vartheta) = \vartheta$, the local enstrophy reads $Q = \int_{\Sigma(t)} \omega d^2x = \oint_{\partial\Sigma(t)} \boldsymbol{v} \cdot d\boldsymbol{x}$, and the constancy of this Q is known as Kelvin's circulation theorem. For an incompressible flow ($\nabla \cdot \boldsymbol{v} = 0$), we may assume $\rho = \text{const}$, and then Q has a special form $\int \omega^2 d^2x$, which is the usual enstrophy.

Our formulation of epi-2D flow begins by inquiring into the root cause of these invariants, deemed as a reflection of some symmetry. In particular, we replace the geometrical symmetry $\boldsymbol{e}_z\cdot = 0$ and $\partial_z = 0$ that characterizes the 2D system by a gauge symmetry that yields an equivalent enstrophy invariant.

The fluid equations (1) and (2) are a Hamiltonian field theory on a phase space $\mathcal{V}$ of fluid variables $\boldsymbol{u} = (\rho, \boldsymbol{V})^T$ [7,8]. The constancy of the Hamiltonian (energy) H is due to $\partial_t H = 0$. In contrast, the conservation of the total mass N and the helicity C is independent of the choice of Hamiltonian, implying that they are not related to any explicit symmetry of the system. Such constants of motion are called Casimir invariants. A possible mechanism that yields a Casimir invariant is a gauge symmetry in some representation of $\mathcal{V}$. If there is an underlying phase space X of fundamental variables $\boldsymbol{\xi}$, with the physical variables $\boldsymbol{u}$ represented by some specific combinations of $\boldsymbol{\xi}$ where the map $\boldsymbol{\xi} \mapsto \boldsymbol{u}$ is redundant, then a gauge freedom occurs and a Casimir invariant is a Noether charge of the gauge symmetry. One example of this is the relabeling symmetry of the Lagrangian to Eulerian fluid representations [9], but for the purposes here the relevant symmetry is that of the Clebsch parametrization [10–14], for which it was shown that N and C are Noether charges [15].

Let X be the phase space of Clebsch parameters

$$\boldsymbol{\xi} = (\varrho, \varphi, p, q, r, s)^T \in X, \tag{6}$$

where each ξ_j ($j = 1, \ldots, 6$) is a function on the base space $M = \mathbb{T}^3$ [16]. On the space of observables (i.e., smooth functionals on X), we define a canonical Poisson bracket

$$\{F, G\} = \langle \partial_{\boldsymbol{\xi}} F, J \partial_{\boldsymbol{\xi}} G \rangle, \tag{7}$$

where $\langle \boldsymbol{\eta}, \boldsymbol{\xi} \rangle = \int_M \boldsymbol{\eta} \cdot \boldsymbol{\xi} d^3x$, $\partial_{\boldsymbol{\xi}} F$ is the gradient of F in X, and J is the symplectic operator

$$J = J_c \oplus J_c \oplus J_c, \qquad J_c = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{8}$$

Given a Hamiltonian H, the adjoint representation of Hamiltonian dynamics is $dF/dt = \{F, H\}$, which is equivalent to Hamilton's equation of motion

$$\partial_t \boldsymbol{\xi} = J \partial_{\boldsymbol{\xi}} H. \tag{9}$$

We relate the physical quantity $\boldsymbol{u} \in \mathcal{V}$ and $\boldsymbol{\xi} \in X$ by $\rho \Leftrightarrow \varrho$ and (denoting $\check{p} = p/\varrho$ and $\check{r} = r/\varrho$)

$$\boldsymbol{V} \Leftrightarrow \wp = \nabla\varphi + \check{p}\nabla q + \check{r}\nabla s. \tag{10}$$

Writing a vector as (10) is called the Clebsch parametrization [10–12]. The five Clebsch parameters $(\varphi, \check{p}, q, \check{r}, s)$ are sufficient to represent every 3-vector (1-form in 3D space) [13]. Inserting (10) into the fluid energy (3), we obtain the Hamiltonian

$$H(\boldsymbol{\xi}) = \int_M \left[\frac{1}{2} \left| \nabla\varphi + \frac{p}{\varrho}\nabla q + \frac{r}{\varrho}\nabla s \right|^2 + \varepsilon(\varrho) \right] \varrho d^3x. \tag{11}$$

With this H, the equation of motion (9) reads [6]

$$\begin{aligned} &\partial_t \varrho + \nabla \cdot (\boldsymbol{V}\varrho) = 0, \qquad \partial_t \varphi + \boldsymbol{V} \cdot \nabla\varphi = h - \frac{1}{2}V^2, \\ &\partial_t p + \nabla \cdot (\boldsymbol{V}p) = 0, \qquad \partial_t q + \boldsymbol{V} \cdot \nabla q = 0, \\ &\partial_t r + \nabla \cdot (\boldsymbol{V}r) = 0, \qquad \partial_t s + \boldsymbol{V} \cdot \nabla s = 0. \end{aligned} \tag{12}$$

The first equation of (12) is nothing but the mass conservation law (1). Evaluating $\partial_t \boldsymbol{V}$ by inserting (10) and using (12), we obtain (2). Hence, Hamilton's equation (9) with the Hamiltonian (11) describes fluid motion obeying (1) and (2) [11,17,18]. The invariance of the helicity C also follows from (12).

We examine how the 2D geometrical reduction works in the Hamiltonian formalism. In a 2D system ($M = \mathbb{T}^2$), we can parameterize a general 2D velocity as

$$\boldsymbol{V} \Leftrightarrow \wp = \nabla\varphi + \check{p}\nabla q \quad (\check{p} = p/\varrho). \tag{13}$$

Here, only three Clebsch parameters φ, $\check{p}$, and q are sufficient [13]. The vorticity is $\omega = (\nabla\check{p} \times \nabla q) \cdot \boldsymbol{e}_z$. The helicity cannot be defined in the 2D space. The potential vorticity is a scalar $\vartheta = \omega/\varrho$, and the generalized enstrophy reads $Q = \int_{\Sigma(t)} f(\omega/\varrho)\varrho d^2x$, where a subdomain $\Sigma(t)$ is moved by the group action of $e^{t\mathrm{v}}$. We can easily verify $dQ/dt = 0$ by (12).

Epi-2D flow is obtained in the 3D setting with the phase space X on the base space $M = \mathbb{T}^3$ by setting $r = 0$ [19]. The corresponding physical fields are $\rho \Leftrightarrow \varrho$ and

$$\boldsymbol{V} \Leftrightarrow \wp = \nabla\varphi + \check{p}\nabla q \quad (\check{p} = p/\varrho). \tag{14}$$

This yields a 2D-like representation, but there is a difference between (13) for 2D flow and (14) for epi-2D flow, for the latter resides in the 3D domain $\mathbb{T}^3$.

Epi-2D flow is generated by the reduced Hamiltonian

$$H(\boldsymbol{\xi}) = \int_M \left[\frac{1}{2}\left|\nabla\varphi + \frac{p}{\varrho}\nabla q\right|^2 + \varepsilon(\varrho)\right]\varrho d^3x, \tag{15}$$

giving the 3D equations (1) and (2). While s does not appear in (15), it obeys the same equation (12) but with $\boldsymbol{V}$ independent of s. Such a field, comoving with the epi-2D flow, is called a phantom field [20]. Or s is a gauge field with the observables blind to its initial value.

As previously remarked, (14) cannot represent an arbitrary 3D flow: Epi-2D flow may have a finite vorticity $\boldsymbol{\omega} = \nabla\times\wp = \nabla\check{p}\times\nabla q$, but its helicity density $\wp\cdot(\nabla\times\wp) = \nabla\varphi\cdot(\nabla\check{p}\times\nabla q) = \nabla\cdot(\varphi\nabla\check{p}\times\nabla q)$ is an exact differential, implying zero helicity: $C = \int_M \wp\cdot(\nabla\times\wp)d^3x = 0$.

As for 2D, this degeneracy of the helicity is compensated by a different invariant obtained by extending the generalized enstrophy to 3D. With the phantom s,

$$Q \coloneqq \int_{\Omega(t)} f(\vartheta)\varrho d^3x, \qquad \vartheta = \frac{\boldsymbol{\omega}\cdot\nabla s}{\varrho}, \tag{16}$$

with arbitrary f and an arbitrary comoving 3D volume element $\Omega(t) \subset M$, is seen to be conserved upon making use of (12). If we choose $s = z$, then (16) reduces to the 2D form $Q = \int_{\Sigma(t)} f(\omega/\varrho)\varrho d^2x$. In what follows, we choose the simplest case $f(\vartheta) = \vartheta$.

Local epi-2D regions within a 3D flow can be exploited to define particlelike behavior. With the general 3D parametrization $\boldsymbol{V} \Leftrightarrow \wp = \nabla\varphi + \check{p}\nabla q + \check{r}\nabla s$, a region in which $\check{r} = 0$ may be called an epi-2D domain. Since $\check{r}$ comoves with the fluid, every infinitesimal volume element, say, $\Omega_j(t)$ with elements indexed by j, included in an epi-2D domain may be viewed as a quasiparticle, which we call an epi-2D particle. The generalized enstrophy evaluated for the vorticity $\boldsymbol{\omega}_+ = \nabla\check{p}\times\nabla q$ in $\Omega_j(t)$, denoted by $Q_+(\Omega_j)$, is a constant of motion. Here the index $+$ is used to distinguish from counterpart domains where $\check{p} = 0$, for which $Q_-(\Omega_j) = \int_{\Omega_j(t)} \vartheta_-\varrho d^3x$, where $\vartheta_- = (\boldsymbol{\omega}_-\cdot\nabla q)/\varrho$ with the vorticity $\boldsymbol{\omega}_- = \nabla\check{r}\times\nabla s$.

We call $Q_\pm(\Omega_j)$ the charge of the epi-2D particle Ω_j. It is remarkable that both Q_+ and Q_- are invariant even in 3D flows. Both charges essentially measure the "circulations" of the decomposed components of the flow [cf. the comment following (5)]. In fact, Kelvin's circulation theorem applies in general 3D flows. The merit of the use of the charges $Q_\pm$, in comparison with the conventional circulation, is in that they can delineate the local flow structure. As far as a particle (volume element) carries only Q_+ or Q_-, it is epi-2D. However, when the vorticity is in a "mixed state", i.e., $\boldsymbol{\omega} = \boldsymbol{\omega}_+ + \boldsymbol{\omega}_- = \nabla\check{p}\times\nabla q + \nabla\check{r}\times\nabla s$, the particle becomes 3D, and then helicity is created from the charges Q_+ and Q_-. Indeed, the integrands of Q_+ and Q_- (the charge densities) are, respectively,

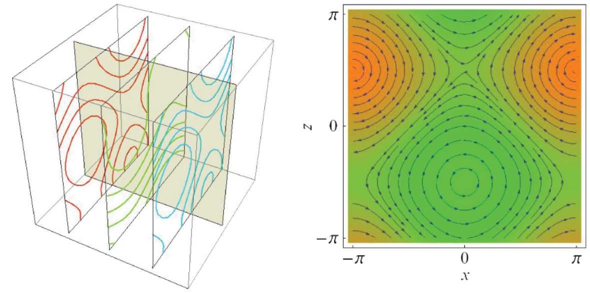

FIG. 1. An epi-2D flow given by $\boldsymbol{V}_+$. (Left) The integrable vortex lines. (Right) Contours of $\mathscr{Q}_+$ and the flow vector on the surface $y = 0$ indicated by the gray cross section in the left figure (the color code ranges from orange $= 2$ to green $= -2$).

$$\mathscr{Q}_+ = (\nabla\check{p}\times\nabla q)\cdot\nabla s, \qquad \mathscr{Q}_- = (\nabla\check{r}\times\nabla s)\cdot\nabla q,$$

while the integrand of C (the helicity density) is

$$\mathscr{C} = \wp\cdot(\nabla\times\wp) = \check{r}\mathscr{Q}_+ + \check{p}\mathscr{Q}_- + \mathscr{C}_{\rm ex}, \tag{17}$$

where $\mathscr{C}_{\rm ex} = \nabla\varphi\cdot(\nabla\check{p}\times\nabla q + \nabla\check{r}\times\nabla s)$ is the exact part of the helicity density. The residual helicity density $\mathscr{C}_r = \mathscr{C} - \mathscr{C}_{\rm ex} = \check{r}\mathscr{Q}_+ + \check{p}\mathscr{Q}_-$ describes the coupling of epi-2D particles; evidently, either $\check{r} = 0$ or $\check{p} = 0$ causes $\mathscr{C}_r$ to vanish. Conversely, the combination of two charges $\mathscr{Q}_+$ and $\mathscr{Q}_-$ yields $\mathscr{C}_r$. Therefore, $\mathscr{C}_r = 0$ can be used as an alternative definition of epi-2D flow.

We give an illustrative example to visualize epi-2D flow and its transition to 3D. Let (x, y, z) be Cartesian coordinates, and $\boldsymbol{V} = \alpha\boldsymbol{V}_+ + \beta\boldsymbol{V}_-$ with

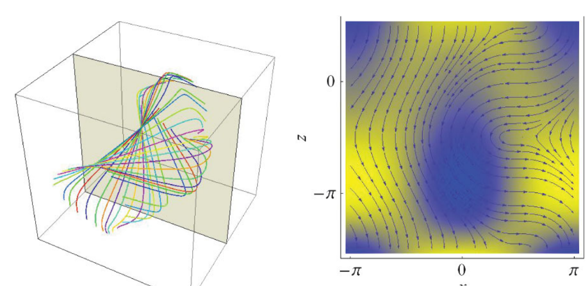

FIG. 2. The combination $\boldsymbol{V} = \boldsymbol{V}_+ + \boldsymbol{V}_-$ yields a 3D flow. (Left) The vortex lines become chaotic (nonintegrable). (Right) Contours of $\mathscr{C}_r$ together with the surface-aligned component of the flow vector on the surface $y = 0$ indicated by the gray cross section in the left figure (the color code ranges from blue $= 5$ to yellow $= -1$).

$$\boldsymbol{V}_+ = \begin{pmatrix} b\sin y - c\cos z \\ 0 \\ a\sin x \end{pmatrix}, \qquad \boldsymbol{V}_- = \begin{pmatrix} 0 \\ c\sin z - a\cos x \\ -b\cos y \end{pmatrix},$$

where α, β, a, b, and c are arbitrary real constants and the vorticities satisfy $\boldsymbol{\omega}_\pm = \nabla\times\boldsymbol{V}_\pm = \boldsymbol{V}_\mp$. When $\alpha=\beta=1$, $\boldsymbol{V}$ is the famous ABC flow [21], satisfying $\nabla\times\boldsymbol{V}=\boldsymbol{V}$. We may cast $\boldsymbol{V}$ into the Clebsch form (10) with

$$\begin{aligned} \varphi &= (\alpha a\sin x - \beta b\cos y)z, \\ \check{p} &= \alpha[b\sin y - c\cos z - (a\cos x)z], \qquad q = x, \\ \check{r} &= \beta[c\sin z - a\cos x - (b\sin y)z], \qquad s = y. \end{aligned} \tag{18}$$

When $\alpha=1$ and $\beta=0$, we obtain an epi-2D flow $\boldsymbol{V}_+$ with $\mathscr{Q}_+ = c\sin z - a\cos x$. The copresence of $\boldsymbol{V}_+$ and $\boldsymbol{V}_-$ creates a 3D flow; when $\alpha=\beta=1$, $\boldsymbol{V}$ has the residual helicity density $\mathscr{C}_r = r\mathscr{Q}_+ + p\mathscr{Q}_-$. In Fig. 1, we show the structure of $\boldsymbol{V}_+$ and the distribution of $\mathscr{Q}_+$ on a surface $y=0$. Figure 2 depicts the 3D flow $\boldsymbol{V}=\boldsymbol{V}_+ + \boldsymbol{V}_-$ and $\mathscr{C}_r$ on a cross section of $y=0$.

Table I summarizes our newly proposed classification of flows. The smallest set hosts a vorticity-free, potential flow (or lamellar field [22]). The next of the hierarchy includes "weighted" potential flows (or a complex lamellar field) that have vorticity but still zero helicity density. A further generalization yields the epi-2D flows that have only exact helicity density (hence, $\mathscr{C}_r=0$). The epi-2D class subsumes conventional 2D systems where we may take $s=z$ (the perpendicular coordinate); this is possible since $\boldsymbol{\omega}$ is aligned to the fixed vector $\boldsymbol{e}_z$ [23]. As the generalization of the *a priori* base space of a 2D system, an epi-2D flow has intrinsic vortex surfaces (cf. [24]). While the direction of $\boldsymbol{\omega}$ changes dynamically, the vortex lines remain integrable, keeping the similarity to 2D flows (cf. [2]). Contrary to 2D flow, however, epi-2D flow allows for vortex stretching, which may make the epi-2D particle thinner (thus, the possibility of singularity generation is not precluded). A general 3D flow may be viewed as a mixed state of epi-2D particles, with each particle carrying a charge of Q_+ or Q_- [25]. When particles with Q_+ and Q_- occupy the same volume element, they produce a helicity to make the volume 3D (cf. [26] for an experimental visualization of knotted vortices). Otherwise, the volume is epi-2D. The epi-2D property is topologically invariant; i.e., an epi-2D volume remains so under ideal fluid motion. If some nonideal process, such as vortex reconnection, occurs [27–29], however, two epi-2D particles can fuse to generate helicity.

TABLE I. Topological classification of flows. $\boldsymbol{V}$, 3D flow; $\boldsymbol{V}_r = \boldsymbol{V} - \nabla\varphi$, solenoidal component; $\mathscr{Q}_\pm$, generalized enstrophy density; $Q_\pm = \int_{\Omega(t)} \mathscr{Q}_\pm d^3x$, generalized enstrophy; $\mathscr{C}$, helicity density; $\mathscr{C}_r$, residual helicity density; $C = \int_M \mathscr{C}_r d^3x$, helicity. Notice that the local integral $\int_{\Omega(t)} \mathscr{C}_r d^3x$ is not a constant of motion.

Classification	Representation	Invariants
Vorticity-free ($\nabla\times\boldsymbol{V}=0$)	$\boldsymbol{V}=\nabla\varphi$	$\mathscr{C}=0$ $\mathscr{Q}=0$
Helicity-free ($\boldsymbol{V}\cdot\nabla\times\boldsymbol{V}=0$)	$\boldsymbol{V}=\check{p}\nabla q$	$\mathscr{C}=0$ $Q_\pm$
Epi-2D ($\boldsymbol{V}_r\cdot\nabla\times\boldsymbol{V}_r=0$)	$\boldsymbol{V}=\nabla\varphi+\check{p}\nabla q$	$\mathscr{C}_r=0$ $Q_\pm$
General	$\boldsymbol{V}=\nabla\varphi+\check{p}\nabla q+\check{r}\nabla s$	C $Q_\pm$

The class of locally epi-2D flows is capable of describing strongly heterogeneous 3D vortex dynamics where the helicity density is localized in a narrow subdomain (such local structures often manifest as coherent vortices and are called worms) [30]. We note that vortex stretching can happen in any of these subdomains.

In conclusion, the newly formulated epi-2D vector fields are useful for delineating between mixed states of order and disorder, which indeed appear as intermittency, coherent vortices, or various local structures in fluid systems. Previously, the framework of 2D geometry was the only one for describing simple (integrable) vortex structures and discussing their moderate (or ordered) dynamics. However, such structures or dynamics can manifest themselves without this *a priori* geometrical constraint; they are more flexible and ubiquitous in general 3D space, as we do observe in actual phenomena. The epi-2D class abstracts the topological characteristics of the usual 2D flows; it persists under deformations by ideal fluid motion (including stretching); being a local property, it is suitable for characterizing the mixture of epi-2D and true 3D dynamics; it bridges 2D and 3D by elucidating how 3D flow is created from the epi-2D prototype or, conversely, how epi-2D degenerates to 2D. Here we discussed fluid mechanics, but the paradigm of Table I for 3D vectors applies to a variety of fields, including magnetic fields [31] and optical vortices [32], as well as chiral charge-density waves [33].

Z. Y. was supported by JSPS KAKENHI Grant Nos. 15K13532, 17H01177, and P. J. M. by U.S. Department of Energy DE-FG02-04ER-54742. P. J. M. thanks George Miloshevich for useful suggestions.

*yoshida@ppl.k.u-tokyo.ac.jp
†morrison@physics.utexas.edu

[1] A. Hasegawa, Adv. Phys. **34**, 1 (1985).
[2] The alignment of vorticity vectors is one of the characteristics of 2D-like flow, causing the suppression of blowup singularity [P. Constantin and C. Fefferman, Indiana University Mathematics Journal **42**, 775 (1993)].
[3] J. J. Moreau, C. R. Acad. Sci. Paris **252**, 2810 (1961).

[4] The first use of the helicity in the classical field theory was for characterizing stellar magnetic fields [L. Woltjer, Proc. Natl. Acad. Sci. U.S.A. **44**, 489 (1958)].
[5] The helicity and the enstrophy can be unified as cross helicities [Y. Fukumoto, Topologica **1**, 003 (2008)]. But here we focus on their difference and metamorphose.
[6] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.119.244501 for the problems pertinent to boundary conditions.
[7] P. J. Morrison and J. M. Greene, Phys. Rev. Lett. **45**, 790 (1980).
[8] P. J. Morrison, Rev. Mod. Phys. **70**, 467 (1998).
[9] N. Padhye and P. J. Morrison, Phys. Lett. A **219**, 287 (1996); Plasma Phys. Rep. **22**, 869 (1996).
[10] A. Clebsch, Z. Reine Angew. Math. **54**, 293 (1857).
[11] P. J. Morrison, AIP Conf. Proc. **88**, 13 (1982).
[12] J. Marsden and A Weinstein, Physica **7D**, 305 (1983).
[13] Z. Yoshida, J. Math. Phys. **50**, 113101 (2009).
[14] The Clebsch parameters mediate the correspondence between classical fields and quantum fields; cf. R. Jackiw, V. P. Nair, S.-Y. Pi, and A. P. Polychronakos, J. Phys. A **37**, R327 (2004); Z. Yoshida and S. M. Mahajan, J. Phys. A **49**, 055501 (2016).
[15] The variation of the action due to the Hamiltonian vector generated by C (or N) yields an exact current density (3-form in the 4D space-time), whose spatial integral is the Noether current. The 0th (temporal) component of the Noether current is the Noether charge which turns out to be C (or N) itself; K. Tanehashi and Z. Yoshida, J. Phys. A **48**, 495501 (2015).
[16] A deeper mathematical structure is elucidated by formulating the system in terms of differential forms; see Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.119.244501.
[17] C. C. Lin, in *Proceedings of the International School of Physics "Enrico Fermi" XXI* (Academic, New York, 1963).
[18] V. E. Zakharov and E. A. Kuznetsov, Sov. Phys. Dokl. **15**, 913 (1971).
[19] By epi-2D, we mean a class "over" 2D. It is different from the so-called "2.5D" class of fields that are defined on a 2D base space but have three components or the class of fields that have some continuous symmetry by which the number of independent variables can be reduced to two; see, for example, B. Ettinger and E. S. Titi, SIAM J. Math. Anal. **41**, 269 (2009).
[20] Z. Yoshida and P. J. Morrison, Fluid Dyn. Res. **46**, 031412 (2014).
[21] T. Dombre, U. Frisch, J. M. Greene, M. Hénon, A. Mehr, and A. M. Soward, J. Fluid Mech. **167**, 353 (1986).
[22] Historically, Kelvin called a vorticity-free flow ($\nabla \times \boldsymbol{V} = 0$, or $\boldsymbol{V} = \nabla\varphi$) a lamellar field and a helicity-free flow [$\boldsymbol{V} \cdot (\nabla \times \boldsymbol{V}) = 0$, or $\boldsymbol{V} = \check{p}\nabla q$] a complex lamellar field. The epi-2D flow is the combination of a lamellar field and a complex lamellar field. The Clebsch parameters φ, $\check{p}$, and q may be called Monge's potentials; see C. Truesdell, *The Kinematics of Vorticity* (Indiana University, Bloomington, 1954), Chap. I.
[23] An axisymmetric flow such that $\partial_\theta = 0$ and $\boldsymbol{e}_\theta \cdot \boldsymbol{V} = 0$ is also 2D, in which we may choose $s = \theta$, and then the enstrophy of $\omega_\theta = r\nabla\theta \cdot \boldsymbol{\omega}$ is constant. A helically symmetric flow such that $\iota\partial_\theta - \partial_z = 0$ and $\boldsymbol{h} \cdot \boldsymbol{V} = 0$ ($\boldsymbol{h} = \iota r\boldsymbol{e}_\theta - \boldsymbol{e}_z$ with a constant ι) is epi-2D, which can be written as $d\varphi + \check{p}d\psi$ with φ and $\check{p}$ being functions of r and $\psi = m(\theta + \iota z)$. It is not 2D, because $\boldsymbol{h}$ is not integrable to define a perpendicular 2D manifold; however, it is very close to 2D in that the vortex stretching vanishes; cf. [19].
[24] By Frobenius' theorem, the following two conditions are equivalent for a 1-form on a 3D manifold: (i) u is helicity free ($u \wedge du = 0$). (ii) u is locally integrable, i.e., $u = \alpha d\beta$ ($\exists\alpha, \beta$). An epi-2D flow consists of a helicity-free flow and a potential flow, which may be viewed as a Pfaffian form [C. Carathéodory, *Calculus of Variations and Partial Differential Equations of the First Order*, 3rd ed. (AMS Chelsea, Providence, 2002)]. See also Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.119.244501 (differential-geometrical formulation).
[25] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.119.244501 for graphical images of epi-2D particles and their fusion.
[26] D. Kleckner and W. T. M. Irvine, Nat. Phys. **9**, 253 (2013).
[27] S. Douady, Y. Couder, and M. E. Brachet, Phys. Rev. Lett. **67**, 983 (1991).
[28] Y. Hattori and H. K. Moffatt, Fluid Dyn. Res. **36**, 333 (2005).
[29] Y. Kimura and H. K. Moffatt, J. Fluid Mech. **751**, 329 (2014).
[30] M. Farge, G. Pellegrino, and K. Schneider, Phys. Rev. Lett. **87**, 054501 (2001).
[31] H. K. Moffatt, J. Plasma Phys. **81**, 905810608 (2015).
[32] K. Toyoda, F. Takahashi, S. Takizawa, Y. Tokizane, K. Miyamoto, R. Morita, and T. Omatsu, Phys. Rev. Lett. **110**, 143603 (2013).
[33] J. Ishioka, Y. H. Liu, K. Shimatake, T. Kurosawa, K. Ichimura, Y. Toda, M. Oda, and S. Tanda, Phys. Rev. Lett. **105**, 176401 (2010).