Turbulence of Weak Gravitational Waves in the Early Universe

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(Received 17 October 2016; revised manuscript received 13 October 2017; published 28 November 2017)

We study the statistical properties of an ensemble of weak gravitational waves interacting nonlinearly in a flat space-time. We show that the resonant three-wave interactions are absent and develop a theory for four-wave interactions in the reduced case of a $2.5 + 1$ diagonal metric tensor. In this limit, where only plus-polarized gravitational waves are present, we derive the interaction Hamiltonian and consider the asymptotic regime of weak gravitational wave turbulence. Both direct and inverse cascades are found for the energy and the wave action, respectively, and the corresponding wave spectra are derived. The inverse cascade is characterized by a finite-time propagation of the metric excitations—a process similar to an explosive nonequilibrium Bose–Einstein condensation, which provides an efficient mechanism to ironing out small-scale inhomogeneities. The direct cascade leads to an accumulation of the radiation energy in the system. These processes might be important for understanding the early Universe where a background of weak nonlinear gravitational waves is expected.

DOI: 10.1103/PhysRevLett.119.221101

Introduction.—The recent direct observations of gravitational waves (GWs) by the LIGO-Virgo collaboration [1], a century after their prediction by Einstein [2], is certainly one of the most important events in astronomy, which opens a new window onto the Universe, the so-called GW astronomy. In the modern Universe, shortly after being excited by a source, e.g., a merger of two black holes, GWs become essentially linear and therefore noninteracting during their subsequent propagation. In the very early Universe, different mechanisms have been proposed for the generation of primordial GWs, like e.g., phase transition [3–9], self-ordering scalar fields [10], cosmic strings [11], and cosmic defects [12]. Production of GWs is also expected to have taken place during the cosmological inflation era [13–15], and many efforts are currently made to detect indirectly their existence [16]. The physical origin of the exponential expansion of the early Universe is, however, not clearly explained and still under investigation [17,18]. Formally, it was incorporated into the general relativity equations simply through adding a positive cosmological constant.

The primordial GWs were, presumably, significantly more nonlinear than the GWs in the modern Universe (like the GWs observed recently by LIGO-Virgo) as they had much larger energy packed in a much tighter space [19]. Although not firmly validated, a scenario was suggested in which a first-order phase transition proceeds through the collisions of true-vacuum bubbles creating a potent source of GWs [20–22]. According to this scenario, at the time of the grand-unified-theory (GUT) symmetry breaking ($t_* \sim 10^{-36}$ sec, $T_* \sim 10^{15}$ GeV), the ratio of the energy density in GW ($\rho_{\rm GW}$) to that in radiation ($\rho_{\rm rad}$) after the transition is about 5% [21]. From the expressions given in [21] and using as a time scale t_* (and also $g_* \sim 100$), we find the following estimate for the GW amplitude: $h \sim 0.3$. Supposedly, such waves were covering the Universe quasiuniformly rather than being concentrated locally in space and time near an isolated burst event, and it is likely that their distribution was broad in frequencies and propagation angles. At some stage of expansion of the Universe, the GWs had become rather weak, but still nonlinear enough for having nontrivial mutual interactions. Importance of the nonlinear nature of the GWs was pointed out in the past for explaining, e.g., the memory effect [23] or part of the dark energy [24]. The possibility to get a turbulent energy cascade of the primordial gravitons was also mentioned [25,26], but, to date, no theory has been developed. A turbulence theory seems to be particularly relevant for GWs because they are nonlinear, and their dissipation is negligible. Recent works [27,28] explore some ideas on similar lines: they investigate numerically the turbulent nature of black holes, define a gravitational Reynolds number, and show that the system can display a nonlinear parametric instability with transfers reminiscent of an inverse cascade (see also Refs. [29,30]).

The nonlinear properties of the GWs, especially the primordial GWs mentioned above, call for using the wave turbulence approach considering statistical behavior of random weakly nonlinear waves [31,32]. The energy transfer between such waves occurs mostly within resonant sets of waves, and the resulting energy distribution, far from a thermodynamic equilibrium, is often characterized by exact power law solutions similar to the Kolmogorov spectrum of hydrodynamic turbulence—the so-called Kolmogorov–Zakharov (KZ) spectra [31,32]. The wave turbulence approach has been successfully applied to many diverse physical systems like, e.g., capillary and gravity waves [33–37], superfluid helium and processes of Bose-Einstein condensation [38], nonlinear optics [39], rotating fluids [40], geophysics [41], elastic waves [42], or astrophysical plasmas [43] (see [31] for a more detailed list of references).

In this Letter, we develop a theory of weak GW turbulence at the level of four-wave interactions in a reduced setup of a $2.5+1$ diagonal metric tensor. The physical properties of such a system are first rigorously derived. Then, in the last section, we present a nonrigorous discussion of a potential connection to the physics of the very early Universe.

Absence of resonant three-wave interactions.—We shall consider Einstein's general relativity equations (free of the cosmological constant) for an empty space $R_{\mu\nu}=0$, where $R_{\mu\nu}$ is the Ricci curvature tensor. We will be interested in weak space-time ripples on the background of a flat space. Respectively, the metric tensor will be assumed to have the form $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$, where $h_{\mu\nu}\ll 1$, and $\eta_{\mu\nu}$ is the Poincaré-Minkowski flat space-time metric. In the linear approximation with the gauge conditions, Einstein's vacuum equations give rise to two GW modes: the plus- and cross-polarized ones [44]. Next order in small amplitudes leads to terms with quadratic nonlinearities, which are often associated with triadic resonant interactions. To describe such triadic interactions of GWs, we need to consider the quadratic part of the Ricci tensor $R_{\mu\nu}=R^{(1)}_{\mu\nu}+R^{(2)}_{\mu\nu}$, with $R^{(1)}_{\mu\nu}=-\Box h_{\mu\nu}$, and [45]

$$\begin{aligned}R^{(2)}_{\mu\nu}=&+\frac{1}{4}\left(2\frac{\partial h^\alpha_\sigma}{\partial x^\alpha}-\frac{\partial h^\alpha_\alpha}{\partial x^\sigma}\right)\left(\frac{\partial h^\sigma_\mu}{\partial x^\nu}+\frac{\partial h^\sigma_\nu}{\partial x^\mu}-\frac{\partial h_{\mu\nu}}{\partial x_\sigma}\right)\\&-\frac{1}{2}h^{\lambda\alpha}\left(\frac{\partial^2 h_{\lambda\alpha}}{\partial x^\nu\partial x^\mu}-\frac{\partial^2 h_{\mu\alpha}}{\partial x^\nu\partial x^\lambda}-\frac{\partial^2 h_{\lambda\nu}}{\partial x^\alpha\partial x^\mu}+\frac{\partial^2 h_{\mu\nu}}{\partial x^\alpha\partial x^\lambda}\right)\\&-\frac{1}{4}\left(\frac{\partial h_{\sigma\nu}}{\partial x^\lambda}+\frac{\partial h_{\sigma\lambda}}{\partial x^\nu}-\frac{\partial h_{\lambda\nu}}{\partial x^\sigma}\right)\left(\frac{\partial h^\sigma_\mu}{\partial x_\lambda}+\frac{\partial h^{\sigma\lambda}}{\partial x^\mu}-\frac{\partial h^\lambda_\mu}{\partial x_\sigma}\right).\end{aligned}\tag{1}$$

Weak turbulence theory predicts that resonant n-wave interactions play the dominant role for the nonlinear evolution. For the three-wave interactions, we have the conditions $\mathbf{k}=\mathbf{k}_1+\mathbf{k}_2$ and $\omega_{\mathbf{k}}=\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2}$, with the dispersion relation $\omega_{\mathbf{k}}=c|\mathbf{k}|=ck$, where c is the speed of light ($c=1$ thereafter), and $\mathbf{k}$ is the wave vector. These resonant conditions are formally identical to the respective conditions for the acoustic wave turbulence problem for which it is well known that all the resonant triads consist of collinear $\mathbf{k}$'s [32]. Therefore, in the physical space, the three-wave resonant interactions split the 3D dynamics into individual 1D systems independent for all particular directions. Let us choose one of such directions, and let our z axis be parallel to the chosen direction. We shall use the transverse-traceless gauge; i.e., $h^\mu_\mu=0$, $\partial_\mu h_{\mu\nu}=0$, and $h^{0\nu}=0$ [44]. Then, the normal mode structure is $h^+_{11}=-h^+_{22}=a$, corresponding to the plus-polarized GW, and $h^\times_{12}=h^\times_{21}=b$, corresponding to the cross-polarized waves (all the other tensor components are zero). Evolution equations for a and b follow from taking the respective projections in equation $\Box h_{\mu\nu}=2R^{(2)}_{\mu\nu}$, which gives $\Box a=R^{(2)}_{11}-R^{(2)}_{22}$ and $\Box b=R^{(2)}_{12}+R^{(2)}_{21}$. Substitute here the respective components of $R^{(2)}_{\mu\nu}$ from expression (1) in which only derivatives with respect to t and z are left; this gives, after some calculations (we define $\dot a=\partial_t a$ etc.), $R^{(2)}_{11}=R^{(2)}_{22}=\frac{1}{2}[\dot a^2+\dot b^2-(\partial_z a)^2-(\partial_z b)^2]$ and $R^{(2)}_{12}=R^{(2)}_{21}=0$ so that $\Box a=0$ and $\Box b=0$. Therefore, three-wave interactions of weak GWs are absent, and the dominant resonant interactions in weak GW turbulence is four-wave or higher.

Theory for four-wave interactions.—To calculate the four-wave interactions, one has to expand Einstein's equations up to the third-order nonlinearity and perform a canonical transformation to eliminate the quadratic nonlinearities. In the general case, this seems to be a laborious task, dealing with which we postpone to future. In this Letter, we will simplify our treatment of interacting GWs by considering a $2.5+1$ diagonal reduction recently studied in the framework of strong GW [46]. This is probably the simplest metric that contains nonlinear properties sufficient for deriving a nontrivial wave turbulence theory of random weakly nonlinear GWs engaged in four-wave interactions. In the past, diagonal metrics were used for describing a wide range of phenomena like, e.g., the Schwarzschild black hole [47] or the Friedmann-Robertson-Walker model of cosmology [45]. Note that this reduced form (with fields depending on two space variables only but have nonzero components for the third spatial direction) is different from the $2+1$ case which does not support GW [48].

Let us consider the vacuum space-time evolution described by the diagonal metric tensor [46]

$$g_{\mu\nu}=\begin{pmatrix}-(H_0)^2 & 0 & 0 & 0\\ 0 & (H_1)^2 & 0 & 0\\ 0 & 0 & (H_2)^2 & 0\\ 0 & 0 & 0 & (H_3)^2\end{pmatrix},\tag{2}$$

where Lamé coefficients H_0, H_1, H_2, and H_3 are functions of $x_0=t$, $x_1=x$, and $x_2=y$, and independent of $x_3=z$. Corresponding $2.5+1$ vacuum Einstein's equations were recently proven to be compatible in a sense that the dynamics preserves the assumed form of the metric tensor [46]. This provides us with a significantly simplified setup for the description of GW. The simplification comes at a cost: only plus-polarized and not cross-polarized GWs are included in the description. The cross-polarized waves are absent initially and are not excited during the evolution. Also, in this framework, we are restricted to a 2D dependence of the physical space variables. However, the $2.5+1$ dynamical vacuum system appears to be a good starting point for studying the properties of interacting GWs and developing a wave turbulence theory. Following [46], we further define

$$H_0=e^{-\lambda}\gamma,\quad H_1=e^{-\lambda}\beta,\quad H_2=e^{-\lambda}\alpha,\quad H_3=e^{\lambda}.\tag{3}$$

In terms of fields α, β, γ, and λ, Einstein's equations $R_{01} = R_{02} = R_{12} = 0$ and $R_{\mu\mu} = 0$ become, respectively,

$$\partial_x\dot{\alpha} = -2\alpha\dot{\lambda}(\partial_x\lambda) + \frac{\dot{\beta}(\partial_x\alpha)}{\beta} + \frac{\dot{\alpha}(\partial_x\gamma)}{\gamma},$$

$$\partial_y\dot{\beta} = -2\beta\dot{\lambda}(\partial_y\lambda) + \frac{\dot{\alpha}(\partial_y\beta)}{\alpha} + \frac{\dot{\beta}(\partial_y\gamma)}{\gamma},$$

$$\partial_x\partial_y\gamma = -2\gamma(\partial_x\lambda)(\partial_y\lambda) + \frac{(\partial_x\alpha)(\partial_y\gamma)}{\alpha} + \frac{(\partial_x\gamma)(\partial_y\beta)}{\beta},$$

and

$$\partial_t\left(\frac{\alpha\beta}{\gamma}\dot{\lambda}\right) - \partial_x\left(\frac{\alpha\gamma}{\beta}\partial_x\lambda\right) - \partial_y\left(\frac{\beta\gamma}{\alpha}\partial_y\lambda\right) = 0.$$

In the linear approximation, we have $\alpha = \beta = \gamma = 1$ and $\ddot{\lambda} - \partial_{xx}\lambda - \partial_{yy}\lambda = 0$. This equation has a wave solution $\lambda = c_1 \exp(-i\omega_{\mathbf{k}} t + i\mathbf{k}\cdot\mathbf{x}) + c_2 \exp(i\omega_{\mathbf{k}} t + i\mathbf{k}\cdot\mathbf{x})$, where $\mathbf{k} = (p, q)$ is a 2D wave vector, whereas c_1 and c_2 are arbitrary constants.

Let us introduce the perturbed variables $\tilde{\alpha}=\alpha-1$, $\tilde{\beta}=\beta-1$, and $\tilde{\gamma} = \gamma - 1$. We can see from the Einstein's equations that the leading order of each of these perturbations is quadratic in the wave amplitude λ, which is of order ϵ. Thus, in the leading order, we obtain

$$\partial_x\dot{\tilde{\alpha}} = -2\dot{\lambda}(\partial_x\lambda),$$

$$\partial_y\dot{\tilde{\beta}} = -2\dot{\lambda}(\partial_y\lambda), \qquad \partial_x\partial_y\tilde{\gamma} = -2(\partial_x\lambda)(\partial_y\lambda) \tag{4}$$

and

$$\partial_t[(1+\tilde{\alpha}+\tilde{\beta}-\tilde{\gamma})\dot{\lambda}] = \partial_x[(1+\tilde{\alpha}-\tilde{\beta}+\tilde{\gamma})\partial_x\lambda] + \partial_y[(1-\tilde{\alpha}+\tilde{\beta}+\tilde{\gamma})\partial_y\lambda]. \tag{5}$$

One can obtain our dynamical equations from a variational principle for the so-called Einstein-Hilbert action defined by the Lagrangian density [46]

$$\mathcal{L} = \frac{1}{2}\left[\frac{\alpha\beta}{\gamma}\dot{\lambda}^2 - \frac{\alpha\gamma}{\beta}(\partial_x\lambda)^2 - \frac{\beta\gamma}{\alpha}(\partial_y\lambda)^2 - \frac{\dot{\alpha}\dot{\beta}}{\gamma} + \frac{(\partial_x\alpha)(\partial_x\gamma)}{\beta} + \frac{(\partial_y\beta)(\partial_y\gamma)}{\alpha}\right] \approx \mathcal{L}_{\rm free} + \mathcal{L}_{\rm int}, \tag{6}$$

where $\mathcal{L}_{\rm free} = \frac{1}{2}[\dot{\lambda}^2 - (\nabla\lambda)^2]$, and

$$\mathcal{L}_{\rm int} = \frac{1}{2}[(\tilde{\alpha}+\tilde{\beta}-\tilde{\gamma})\dot{\lambda}^2 + (-\tilde{\alpha}+\tilde{\beta}-\tilde{\gamma})(\partial_x\lambda)^2 + (\tilde{\alpha}-\tilde{\beta}-\tilde{\gamma})(\partial_y\lambda)^2 - \dot{\tilde{\alpha}}\dot{\tilde{\beta}} + (\partial_x\tilde{\alpha})(\partial_x\tilde{\gamma}) + (\partial_y\tilde{\beta})(\partial_y\tilde{\gamma})],$$

representing the linear (free-wave) dynamics and the (leading order of) the wave interaction, respectively. Let us deal with fields which are periodic with period L in both x and y (limit $L \to +\infty$ to be taken later) and introduce Fourier coefficients $\lambda_{\mathbf{k}}(t) = L^{-2}\int_{\rm square}\lambda(\mathbf{x},t)\exp(-i\mathbf{k}\cdot\mathbf{x})dxdy$, etc. Then,

$$\int \mathcal{L}_{\rm free}d\mathbf{x} = \frac{1}{2}\sum_{\mathbf{k}}(|\dot{\lambda}_{\mathbf{k}}|^2 + k^2|\lambda_{\mathbf{k}}|^2). \tag{7}$$

We introduce the normal variables as

$$\lambda_{\mathbf{k}} = \frac{a_{\mathbf{k}} + a^*_{-\mathbf{k}}}{\sqrt{2k}}, \qquad \dot{\lambda}_{\mathbf{k}} = \frac{\sqrt{k}(a_{\mathbf{k}} - a^*_{-\mathbf{k}})}{i\sqrt{2}} \tag{8}$$

so that

$$\int \mathcal{L}_{\rm free}d\mathbf{x}dt = \frac{i}{2}\int dt\sum_{\mathbf{k}}(a^*_{\mathbf{k}}\dot{a}_{\mathbf{k}} - a_{\mathbf{k}}\dot{a}^*_{\mathbf{k}}) - \int H_{\rm free}dt,$$

$$\int \mathcal{L}_{\rm int}d\mathbf{x}dt = -\int H_{\rm int}dt,$$

where

$$H_{\rm free} = \sum_{\mathbf{k}} k|a_{\mathbf{k}}|^2 \tag{9}$$

and

$$H_{\rm int} = \frac{1}{2}\sum_{1,2,3}\delta_{123}\{(-\tilde{\alpha}_1 - \tilde{\beta}_1 + \tilde{\gamma}_1)\dot{\lambda}_2\dot{\lambda}_3 - [(\tilde{\alpha}_1 - \tilde{\beta}_1 + \tilde{\gamma}_1)p_2p_3 + (-\tilde{\alpha}_1 + \tilde{\beta}_1 + \tilde{\gamma}_1)q_2q_3]\lambda_2\lambda_3\} + \frac{1}{2}\sum_{\mathbf{k}}[\dot{\tilde{\alpha}}_{\mathbf{k}}\dot{\tilde{\beta}}^*_{\mathbf{k}} - (p^2\tilde{\alpha}_{\mathbf{k}} + q^2\tilde{\beta}_{\mathbf{k}})\tilde{\gamma}^*_{\mathbf{k}}], \tag{10}$$

are the free and interaction Hamiltonians, respectively. Here, we use shorthand notations $\sum_{1,2,3} = \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}$, $\delta_{123} = \delta_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3}$ (Kronecker delta), $\lambda_1 = \lambda_{\mathbf{k}_1}$, etc.

Now we are ready to pass to the Hamiltonian description. Taking variation of the action with respect to $a^*_{\mathbf{k}}$, we have the required Hamiltonian equation

$$i\dot{a}_{\mathbf{k}} = \frac{\partial H}{\partial a^*_{\mathbf{k}}}, \quad \text{where } H = H_{\rm free} + H_{\rm int}.$$

In the linear approximation, when $H_{\rm int}$ is neglected, we have the free GW solution, $a_{\mathbf{k}} \sim \exp(-ikt)$. To find $H_{\rm int}$, in addition to expressing $\lambda_{\mathbf{k}}$ and $\dot{\lambda}_{\mathbf{k}}$ in terms of $a_{\mathbf{k}}$ and $a^*_{\mathbf{k}}$, we have to express there $\tilde{\alpha}_{\mathbf{k}}, \tilde{\beta}_{\mathbf{k}}, \dot{\tilde{\alpha}}_{\mathbf{k}}, \dot{\tilde{\beta}}_{\mathbf{k}}$, and $\tilde{\gamma}_{\mathbf{k}}$ in terms of the same normal variables. This can be easily done in the Fourier space (see the Supplemental Material [49]). After the introduction of these expressions and relations (8) into Eq. (10), we obtain $H_{\rm int}$ in terms of variables $a_{\mathbf{k}}$ and $a^*_{\mathbf{k}}$. All terms in $H_{\rm int}$ are quartic in $a_{\mathbf{k}}$ and $a^*_{\mathbf{k}}$, which indicates that the leading-order interaction process is four wave. The terms with products of four $a_{\mathbf{k}}$'s or four $a^*_{\mathbf{k}}$'s can be dropped as they correspond to an empty $4 \to 0$ process. The remaining terms can be grouped into two parts: $H_{\rm int} = H_{3\to1} + H_{2\to2}$. Part $H_{3\to1}$ contains products of three $a_{\mathbf{k}}$ and one $a^*_{\mathbf{k}}$ and vice versa—these represent a $3 \to 1$ process. Part $H_{2\to2}$ contains products of two $a_{\mathbf{k}}$ and two $a^*_{\mathbf{k}}$—these represent a $2 \to 2$ process. Let us first consider the $3 \to 1$ process. The $3 \to 1$ resonance conditions are satisfied only by wave quartets which are collinear (for the same reason as in the $2 \to 1$ process; see also [32]). Thus, in this case, we can consider contributions to the Hamiltonian from the resonant manifold only, where the quartets are collinear, which drastically simplifies the calculation (e.g., $p_5/p_1 - q_5/q_1 = 0$ etc.). Then, by a straightforward but lengthy calculation (see the Supplemental Material [49]), we find that all the $3 \to 1$ terms cancel (on the $3 \to 1$ resonant manifold), i.e., $H_{3\to1} = 0$, whereas for the $2 \to 2$ process, we obtain the following expression:

$$H_{2\to2} = \sum_{1,2,3,4} T^{12}_{34}\delta^{12}_{34} a_1 a_2 a_3^* a_4^*, \tag{11}$$

with $T^{12}_{34} = \frac{1}{4}(W^{12}_{34} + W^{21}_{34} + W^{12}_{43} + W^{21}_{43})$, $W^{12}_{34} = Q^{12}_{34} + Q^{34}_{12}$, and

$$Q^{12}_{34} = \frac{1}{4\sqrt{k_1k_2k_3k_4}}\left\{2\left(\frac{p_4}{p_1-p_3} - \frac{q_4}{q_1-q_3}\right)\frac{k_2(p_1p_3-q_1q_3)}{k_1-k_3} - 2\left(\frac{p_4}{p_1-p_3} + \frac{q_4}{q_1-q_3}\right)\frac{k_1k_2k_3}{k_1-k_3} \right.$$
$$\left. + \left(\frac{p_2}{p_1+p_2} - \frac{q_2}{q_1+q_2}\right)\frac{k_1(p_3p_4-q_3q_4)}{k_1+k_2} - \left(\frac{p_2}{p_1+p_2} + \frac{q_2}{q_1+q_2}\right)\frac{k_1k_3k_4}{k_1+k_2} + \frac{2k_1k_3p_2q_4}{(p_1+p_2)(q_1+q_2)} + \frac{2k_1p_3(q_2k_4+k_2q_4)}{(p_1-p_3)(q_1-q_3)}\right\}. \tag{12}$$

Given the standard form of the interaction Hamiltonian (11), derivation of the kinetic equation (KE) of weak wave turbulence is straightforward and can be found, e.g., in chapter 6 of [31]. The result is

$$\dot{n}_{\mathbf{k}} = 4\pi \int |T^{\mathbf{k}\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}|^2 n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} n_{\mathbf{k}} \left[\frac{1}{n_{\mathbf{k}}} + \frac{1}{n_{\mathbf{k}_3}} - \frac{1}{n_{\mathbf{k}_1}} - \frac{1}{n_{\mathbf{k}_2}}\right]$$
$$\times \delta(\mathbf{k} + \mathbf{k}_3 - \mathbf{k}_1 - \mathbf{k}_2)\delta(\omega_{\mathbf{k}} + \omega_{\mathbf{k}_3} - \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2})$$
$$\times d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3,$$

where the wave action spectrum is defined as

$$n_{\mathbf{k}} = \lim_{L\to\infty} \frac{L^2}{4\pi^2}\langle|a_{\mathbf{k}}|^2\rangle, \tag{13}$$

and where $\langle\,\rangle$ denotes the ensemble average. It is worth reminding that the KE is valid under assumptions of small nonlinearity (in our case $h \ll 1$), random phases, and taking the infinite box limit while keeping the mean wave energy density constant. Assuming the mirror symmetry of the spectrum $n_{\mathbf{k}} = n_{-\mathbf{k}}$, we have in terms of the original variables $n_{\mathbf{k}} = k\lim_{L\to\infty}\frac{L^2}{4\pi^2}\langle|\lambda_{\mathbf{k}}|^2\rangle \sim h^2\ell$, where h is the typical size of the metric ripples, and ℓ is the typical length scale. The KE has the following isotropic constant-flux stationary KZ solutions [see, e.g., Eqs. (9.36) and (9.37) in [31] reproduced via a dimensional derivation in the Supplemental Material [49]]

$$n_{\mathbf{k}} \sim k^{-2} \quad \text{and} \quad n_{\mathbf{k}} \sim k^{-5/3}, \tag{14}$$

corresponding, respectively, to the direct cascade of the vacuum ripple energy from small to large k's, and to the inverse cascade of the wave action (number of gravitons) from large to small k's. Extension to the 3D isotropic geometry (also given in the Supplemental Material [49]) gives $n_{\mathbf{k}} \sim k^{-3}$ and $n_{\mathbf{k}} \sim k^{-8/3}$. There is also a solution corresponding to thermodynamic equilibrium (in any geometry), the Rayleigh-Jeans spectrum, $n_{\mathbf{k}} = T/(k+\mu)$, $(T, \mu = \text{const})$.

Interestingly, the 1D spectra (14) can be recovered with a simple phenomenology. For four-wave interactions, the typical time scale of the cascade is $\tau_{\text{cas}} \sim \tau_{\text{GW}}/\epsilon^4$, where the small parameter, $\epsilon \sim \tau_{\text{GW}}/\tau_{\text{NL}} \ll 1$, measures the time scale separation between the wave period $\tau_{\text{GW}} \sim 1/\omega$ and the nonlinear time $\tau_{\text{NL}} \sim \ell/(hc)$, which follows from the perturbed Ricci tensor. The KE conserves the total energy $\mathcal{E} = \int \omega_{\mathbf{k}} n_{\mathbf{k}} d\mathbf{k}$ and the total wave action $\mathcal{N} = \int n_{\mathbf{k}} d\mathbf{k}$ (per unit area). Let us consider the energy $\mathcal{E}_\ell$ within the scales greater than ℓ, namely $\mathcal{E}_\ell = \int_{k'<k} E^{(1\text{D})}_{k'} dk'$, where we have introduced the 1D energy spectrum $E^{(1\text{D})}_k$. We have $\mathcal{E}_\ell \sim (c^4/32\pi G)(h^2/\ell^2)$ [44] so that

$$\varepsilon \sim \frac{\mathcal{E}_\ell}{(\frac{\tau_{\text{NL}}}{\tau_{\text{GW}}})^3 \tau_{\text{NL}}} \sim \frac{\mathcal{E}_\ell}{(\frac{\ell}{h})^4 \omega^3} \sim \frac{\mathcal{E}_\ell^3}{k^3} \sim E_k^{(1D)3}, \tag{15}$$

where ε is the constant energy flux. This gives the spectrum $E^{(1\text{D})}_k \sim \varepsilon^{1/3} k^0$. With the wave action flux ζ, we obtain in the same manner, $N^{(1\text{D})}_k \sim \zeta^{1/3} k^{-2/3}$.

Discussion.—Which of the two KZ spectra is more relevant depends on the position of the forcing scale of the space-time ripples with respect to the available range of scales for the GWs. In turn, this may depend on the stage of the Universe evolution and on the respective physical processes generating the GWs at that particular epoch. As discussed in the introduction, a first-order phase transition could possibly provide an efficient mechanism to generate GWs with fairly high energy [20–22] at a critical time in the very early Universe. The validity of this scenario for our Universe is, however, still not clear and needs further investigations that could be done in the future. Having in mind these limitations, we can still discuss the potential consequences of the presence of a weak GW turbulence in the very early Universe. Before the inflation era, when all observable today parts of the Universe were within the horizon, the inverse cascade may have been an effective mechanism for “ironing out” the small-scale inhomogeneities, i.e., correlating the motions of the causally connected parts of the Universe. Indeed, this spectrum has a finite capacity: the integral defining the total wave action $\int n_{\mathbf{k}} d\mathbf{k}$ converges at $\mathbf{k} \to 0$. Such systems admit self-similar solutions of the second kind in which the spectral front traverses (explosively) the infinite range of scales in a finite time [43,50]. In our case, this means that the largest length scale of the system will be excited by the GW spectrum in a finite time t_*—a kind of nonequilibrium Bose-Einstein condensation, which was previously studied in the Gross-Pitaevskii model [31,51]. Based on the KE and putting back the original physical constants, we can estimate $t_* \sim \ell/(ch^4)$. Of course, the mode $k = 0$ (the infinite scale) is never reached. Indeed, the weak turbulence KE fails when the time ratio $\chi \equiv \tau_{\mathrm{GW}}/\tau_{\mathrm{NL}} \sim h$ becomes of order unity. By using the wave action spectrum, we find $\chi \sim \ell^{1/3}$ showing the existence of a maximal scale $\ell_{\max}$ beyond which turbulence becomes strong. The dilution of the GW energy due to the expansion of the Universe provides another constraint for $\ell_{\max}$. First of all, we have a restriction at the linear level: the GW cannot be longer than the Hubble horizon distance $d = c/H$. On the nonlinear level, the expansion acts as an effective large-scale dissipation, which arrests the inverse cascade at $k_{\min} \sim 1/(h^4 d)$ (see Supplemental Material [49]). The inverse cascade range forms if this wave number is smaller than the forcing scale.

The direct cascade of GWs is of the infinite capacity type; i.e., it corresponds to a growth of the GW physical-space energy density until this process is saturated at the value $\mathcal{E} \sim dh_0^6 c^4/(\ell_0^3 G)$ due to the expansion of the Universe (see Supplemental Material [49]). Here, h_0 and ℓ_0 refer to the values of the metric disturbance and their typical length at the forcing scale. Such a GW energy may play an important role for the overall expansion rate at the late stages of the inflation and the transition to the radiation dominated Universe. One could describe this effect in future by combining the coarse-graining method developed in [24] with the wave turbulence approach developed in our work.

Production of a GW background is a fundamental prediction of any cosmological inflationary model [14]. The features of such a fossil signal encode unique information about the physics of the early Universe that might be detected in the future [52,53]. Here, we predict the form of the GW spectra emerging from random nonlinear interactions. Our theory is, however, not strictly limited to the very early Universe. For example, turbulent black holes are potentially another application: our scenario based on four-wave interactions can be the explanation of the inverse cascade observed recently in [27–30]. Finally, we point out that in our case, the dual cascade behavior is caused by the fact that the leading-order interaction is four wave. We remind that absence of three-wave interactions was proven for the most general $3 + 1$ metric and, therefore, we expect existence of the inverse cascade solution in such a general case too.

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