Solvable Hydrodynamics of Quantum Integrable Systems

Vir B. Bulchandani,[1] Romain Vasseur,[1,2,4] Christoph Karrasch,[3] and Joel E. Moore[1,2]
[1]Department of Physics, University of California, Berkeley, Berkeley, California 94720, USA
[2]Materials Sciences Division, Lawrence Berkeley National Laboratory, Berkeley, California 94720, USA
[3]Dahlem Center for Complex Quantum Systems and Fachbereich Physik, Freie Universität Berlin, 14195 Berlin, Germany
[4]Department of Physics, University of Massachusetts, Amherst, Massachusetts 01003, USA
(Received 17 April 2017; published 30 November 2017)

The conventional theory of hydrodynamics describes the evolution in time of chaotic many-particle systems from local to global equilibrium. In a quantum integrable system, local equilibrium is characterized by a local generalized Gibbs ensemble or equivalently a local distribution of pseudomomenta. We study time evolution from local equilibria in such models by solving a certain kinetic equation, the "Bethe-Boltzmann" equation satisfied by the local pseudomomentum density. Explicit comparison with density matrix renormalization group time evolution of a thermal expansion in the *XXZ* model shows that hydrodynamical predictions from smooth initial conditions can be remarkably accurate, even for small system sizes. Solutions are also obtained in the Lieb-Liniger model for free expansion into vacuum and collisions between clouds of particles, which model experiments on ultracold one-dimensional Bose gases.

DOI: 10.1103/PhysRevLett.119.220604

Introduction.—Understanding the dynamics of interacting, many-body quantum systems far from equilibrium remains one of the most challenging problems in modern physics. In recent decades, this problem has taken on a new urgency thanks to rapid progress in the experimental construction of ultracold atomic systems. The tools available for strongly nonequilibrium dynamics with nonuniform initial conditions, even in integrable models whose equilibrium properties can be calculated exactly, have been restricted to low temperature and conformal invariance [1,2], to specific quantities [3], or to long-time asymptotic behavior [4–7]. Quantum integrable models include experimentally relevant examples like the Heisenberg antiferromagnet and the Lieb-Liniger gas in one dimension. They possess extensively many conserved quantities, which prevent them from thermalizing like generic ergodic systems and often result in ballistic transport properties.

The fact that integrable models can have unusual "generalized hydrodynamics" due to an infinite number of local conservation laws was first understood in the context of classical particle systems [8,9]. This stands in contrast to conventional hydrodynamics, which describes transport of only three conserved quantities, namely, mass, momentum, and energy. The generalized hydrodynamics of quantum integrable models was developed recently in studies of the nonequilibrium steady state [1–3,10–19] that is established at the junction between two infinite reservoirs [4,5]. An important insight is that making a local-density-type approximation for *all* local conserved charges implies a conservation law at the level of the local pseudomomentum distribution. Thus in the context of integrable models, the hydrodynamic equations imply a fundamental "Bethe-Boltzmann equation," which is an inversion of the logic familiar from conventional statistical mechanics.

The completeness of this equation for the two-reservoir steady state of the *XXZ* model, i.e., that the Bethe-Boltzmann equation correctly captures the physics of unusual quasilocal conservation laws [20–25], was tested by comparing hydrodynamic predictions to known results for spin transport in the linear-response limit [7,26]. It was observed at the end of Ref. [7] that the ansatz for two reservoirs introduced in Ref. [5] was actually valid to first order in time for arbitrary smooth, locally equilibrated initial conditions. Hydrodynamics in the two-reservoir case is a function of only one variable (say x/t) because of the absence of any length scale in the initial condition, and consequently a first-order solution is sufficient; every other nontrivial initial condition yields dynamics that is a function of two variables, space and time. The present work builds on this earlier observation to develop *converged* solutions for finite-time hydrodynamical evolution from general smooth, locally equilibrated initial conditions.

This allows us to make novel and detailed physical predictions for finite-time dynamics in a wide range of physical systems. For example, we obtain the first practical hydrodynamic technique for the one-dimensional Bose gas [27–32] that applies to arbitrary local generalized Gibbs ensemble (GGE) initial conditions and takes into account the higher conservation laws of the underlying quantum system. This allows us to obtain detailed profiles for the evolution of the Lieb-Liniger gas from collision type initial conditions, which could, in principle, be tested in the laboratory. At the same time, our approach allows for a hydrodynamic description of finite-time spin dynamics in the *XXZ* chain, in excellent agreement with results obtained

from density matrix renormalization group (DMRG) techniques [33–37].

Bethe-Boltzmann equation.—The Bethe-Boltzmann equation is a hydrodynamic description of quantum integrable systems [4,5], which aims to capture nonequilibrium dynamics in such systems using the thermodynamic Bethe ansatz (TBA) [38,39]. The Bethe-Boltzmann equation takes its simplest form for the Lieb-Liniger interacting Bose gas, which was used in Ref. [4]. We briefly summarize the physical assumptions leading to this equation below, following the presentation given in Ref. [7] for the *XXZ* chain. Thus consider a one-dimensional Bose gas with delta-function interactions, placed on a line of length L. This has Hamiltonian

$$H = \int_0^L dx \Psi^\dagger \left(-\frac{\hbar^2}{2m}\nabla^2 - \mu \right) \Psi + c\Psi^\dagger\Psi^\dagger\Psi\Psi, \quad (1)$$

and the field operators satisfy canonical commutation relations $[\Psi^\dagger(x), \Psi(y)] = \delta(x-y)$. It is useful to set $\hbar = 2m = 1$. This system is called *integrable* for all values of the interaction strength c because every N-body scattering process with $N > 2$ factorizes as the product of two-body scattering processes. Integrability in this sense is reflected by the existence of infinitely many conserved charges. In a given macrostate of the Lieb-Liniger gas, with occupied density of states ρ_k at pseudomomentum k, these can be written as $Q_n = \int_{-\infty}^{\infty} dk \rho_k q_n(k)$,with $q_n(k) = k^n/n$ and $n = 0, 1, \ldots$ Let us now consider evolution of the Lieb-Liniger gas from *locally equilibrated* initial conditions; we assume that this is captured by a spatiotemporally varying pseudomomentum distribution $\rho_k(x,t)$ [40]. This yields a spatiotemporal distribution of charge density, given by

$$Q_n(x,t) = \int_{-\infty}^{\infty} dk \rho_k(x,t) q_n(k). \quad (2)$$

We note that in order for this expression to be defined (and indeed for the hydrodynamic approach as understood in Refs. [4,5] to be consistent for the Lieb-Liniger gas), $\rho_k(x,t)$ must decrease more rapidly in k than any power of k, for all x and t [43]. Motivated by conservation of Q_n at the quantum mechanical level, let us postulate the local conservation law

$$\partial_t Q_n(x,t) + \partial_x J_n(x,t) = 0. \quad (3)$$

Surprisingly, the physically correct formula for $J_n(x,t)$ turns out to be given in terms of the quasiparticle velocity $v_k[\rho(x,t)]$ of collective excitations of the state with pseudomomentum distribution $\rho_k(x,t)$, which is complicated but known from the TBA [44]; it has been found that [4,5]

$$J_n(x,t) = \int_{-\infty}^{\infty} dk \rho_k(x,t) q_n(k) v_k[\rho(x,t)], \quad (4)$$

in the hydrodynamic limit, which is connected to the validity of earlier conjectures for the Drude weight [45]. Substituting this expression into Eq. (3) and appealing to completeness of conserved charges in integrable models, one deduces a conservation law for the local pseudomomentum distribution, given by

$$\partial_t \rho_k(x,t) + \partial_x \{\rho_k(x,t) v_k[\rho]\} = 0. \quad (5)$$

We call this the *Bethe-Boltzmann equation*, as it has the structure of a dissipationless Boltzmann equation for the occupied pseudomomentum density. Intuitively, the Bethe-Boltzmann equation has the meaning that "occupied quantum numbers are locally conserved." We emphasize that for integrable systems, the generalized hydrodynamic Eqs. (3) imply the fundamental Bethe-Boltzmann Eq. (5), in sharp contrast with the logic familiar from conventional statistical mechanics.

Finite time scheme.—In practice, it is useful to change variables to the local Fermi factor $\vartheta_k(x,t)$, defined as the ratio of occupied quantum numbers at pseudomomentum k. This yields the *advection form* of the Bethe-Boltzmann equation [4,5]

$$\partial_t \vartheta_k(x,t) + v_k[\hat{\vartheta}] \partial_x \vartheta_k(x,t) = 0. \quad (6)$$

Whereas this equation has so far only been used to analyze self-similar nonequilibrium steady states whose properties depend only on x/t, the purpose of this Letter is to illustrate how it can be solved efficiently at finite time for arbitrary initial conditions. We propose a numerical solution to Eq. (6), based on a backwards implicit numerical scheme [7,46] which for time step $dt > 0$, determines $\vartheta_k(x,t)$ from $\vartheta_k(x,t-dt)$ via the implicit equation

$$\vartheta_k(x,t) = \vartheta_k(x - v_k[\hat{\vartheta}(x,t)]dt, t - dt). \quad (7)$$

This solves Eq. (6) up to order $O(dt^2)$. We emphasize that the velocity in the right-hand side of Eq. (7) depends nonlinearly on all of the Fermi factors at (x, t), making Eq. (7) an implicit equation that can be solved by numerical iteration. Details of our implementation can be found in the Supplemental Material [44]. Achieving convergence of numerical schemes for nonlinear conservation laws in general, even in the low-dimensional setting, is known to be difficult [47]. It is therefore remarkable that the above scheme, applied to an extremely high-dimensional system [48], converges at all. Moreover, the scheme [Eq. (7)] is found to converge quickly as $dt \to 0$, so that one can obtain accurate results even for large time steps dt. From the solution of Eq. (6), one can readily compute physical quantities of interest (such as charge and current densities) using Eqs. (2) and (4).

We note that in general, nonlinear systems of equations of conservation type [Eq. (5)] or advection type [Eq. (6)]

are difficult to understand analytically, because of the possibility of shock formation from smooth initial conditions. From the viewpoint of mathematical rigor, the conservation form [Eq. (5)] is better defined, but existing analytical [49] methods for understanding conservation laws have little practical utility in the present high-dimensional limit. Ordinarily, one can make little analytical progress with nonlinear advection equations. However, somewhat surprisingly, the advection form [Eq. (6)] lies in a special class of such systems which *are* possible to understand analytically. These are the "semi-Hamiltonian" or "rich" systems of hydrodynamic type [50–53], and possess several interesting geometrical properties related to integrability (see Supplemental Material [44]).

Hydrodynamics for the XXZ spin chain.—The Bethe-Boltzmann formalism can be extended to study nonequilibrium dynamics and transport in any integrable system or integrable quantum field theory. A particularly interesting example is provided by the spin-1/2 *XXZ* chain with Hamiltonian

$$H = J\sum_j S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z, \tag{8}$$

where predictions from hydrodynamics can be compared to DMRG results [33,34]. Here, we set the coupling to $J = 1$, and parameterize the anisotropy of the theory by $\Delta = \cos\gamma$. The Bethe-Boltzmann formalism for the gapless phase ($-1 < \Delta < 1$) of this model is discussed in detail elsewhere in the literature [5,7,26] (see Supplemental Material [44]). For the purposes of comparison with the DMRG, we restrict to $\Delta = \frac{1}{2}$ (other values of Δ were considered in previous works [5,7,26] for nonequilibrium steady states). We also focus on nonequilibrium energy transport, in particular the evolution of local energy density, given by $n_E(x,t) = \sum_{j=1}^{N_t} \int d\lambda e_j(\lambda)\rho_j(x,t,\lambda)$ in the hydrodynamic limit (see Supplemental Material [44]).

To illustrate the range of validity of the method, we consider a strongly nonequilibrium example, namely, the Gaussian initial temperature profile

$$\beta(x) = \beta_0 - (\beta_0 - \beta_M)e^{-x^2/L^2}, \tag{9}$$

with $\beta_0 > \beta_M$. Physically speaking, this corresponds to a perturbation β_M^{-1} of a background temperature β_0^{-1}, localized over a typical length $\sim L$. We first illustrate the convergence of our numerical scheme [Eq. (6)] by taking such a Gaussian initial state and letting the time step $dt \to 0$. This is depicted in Fig. 1. As dt is lowered, the numerical solution at long times (say, $t = 20$) converges very quickly, and remarkably, even one-step or two-step schemes (e.g., $dt = 20$ or $dt = 10$) yield good approximations to the converged solution.

We now compare the predictions of the Bethe-Boltzmann equation against DMRG calculations [33,34],

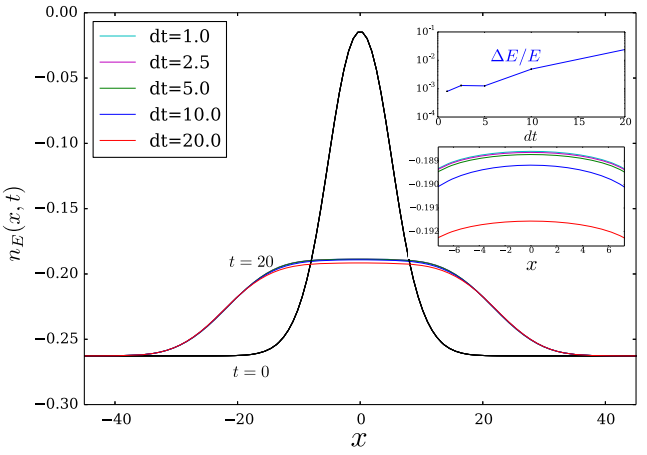

FIG. 1. Convergence of the method as $dt \to 0$ for an initial Gaussian temperature profile given by Eq. (9) with $\beta_0 = 2.0$, $\beta_M = 0.1$, and $L = 8$ in the *XXZ* spin chain at $\Delta = \frac{1}{2}$. The numerical solution at time $t = 20$ is rapidly converging as dt is decreased, with $dt \sim 10$ being already quite accurate. Insets: Top: Relative error in total energy, showing energy conservation as $dt \to 0$. Bottom: Close-up of the main figure showing convergence.

with the initial condition [Eq. (9)] prepared using standard finite-temperature methods [35–37]. This is shown in Fig. 2. We find an excellent agreement between DMRG and hydrodynamic results with $dt = 2.5$ for quite different initial temperature profiles—Gaussian [Eq. (9)] and $\tanh\beta(x) = (\beta_0 + \beta_M)/2 + [(\beta_0 - \beta_M)/2]\tanh(x/L)$ functions. Provided that the initial condition is smooth enough for the DMRG and the thermodynamic Bethe ansatz calculations to agree at $t = 0$, subsequent agreement at later times is essentially perfect. In fact, at low temperatures where it is hard to obtain smooth initial conditions in the DMRG, the main source of error comes from slight disagreements in the initial conditions between the two approaches.

Hydrodynamics for the interacting Bose gas.—An existing hydrodynamic description of the quasi-one-dimensional Bose gas, based on a local density approximation for the first three conserved charges of the Lieb-Liniger model, has proved effective for capturing nonequilibrium dynamics for such systems [27–32,54,55]. In its domain of physical validity [55,56], the present approach improves the existing theory by allowing for local GGE initial conditions and respecting all higher conservation laws implied by integrability. The extension of this hydrodynamic approach to other important aspects of 1D Bose gas physics, including dynamics in external potentials [31,32,57–60] and correlation functions [56,60–62], is currently an active area of research [63,64].

Two nonequilibrium quenches which are of particular experimental interest are sudden expansions of Bose gases into vacuum and collisions of clouds of ultracold bosons [65–75]. We find that our numerical solution to Eq. (6)

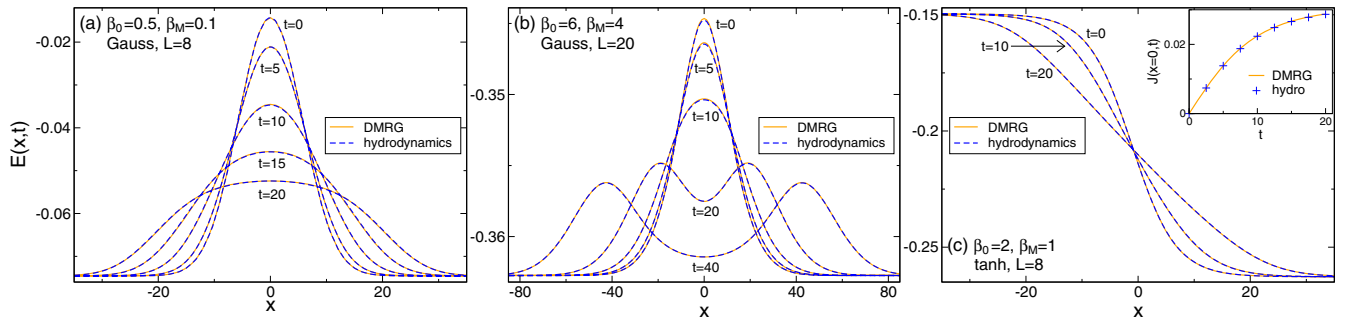

FIG. 2. Time evolution of energy density from various initial temperature profiles at $t = 0$ for the XXZ spin chain at $\Delta = \frac{1}{2}$. Left: High temperature Gaussian initial state. Middle: Low temperature Gaussian initial state. Right: Two-reservoir setup with temperatures $\beta_0 = 2$, $\beta_M = 1$ connected through a $\tanh(x/L)$ interpolation with $L = 8$. Inset: Energy current at $x = 0$ showing the approach to a nonequilibrium steady state at long times.

converges for initial conditions modeling both of these scenarios. From the resulting evolution in $\theta(x, t, k)$, one can track the evolution of any local conserved charge of the model. In Fig. 3, we plot the time evolution of particle and energy densities [defined by $n = Q_1$ and $n_E = Q_2$ in Eq. (2), respectively] in a Lieb-Liniger model with interaction strength $c = 1$ from free expansion and collision type initial conditions. The initial states are prepared at temperature $T = 1$ using a chemical potential profile interpolating between $\mu = 5$ inside a box and a large negative value $\mu = -50$ outside, with the edges of the box smoothed out using tanh functions. For the collision protocol, two clouds of bosons initially prepared as in the free expansion quench are given opposite momenta $k = \pm k_0$ with $k_0 = 2.5$.

General features.—We hope that the previous examples have established that the Bethe-Boltzmann equation is a

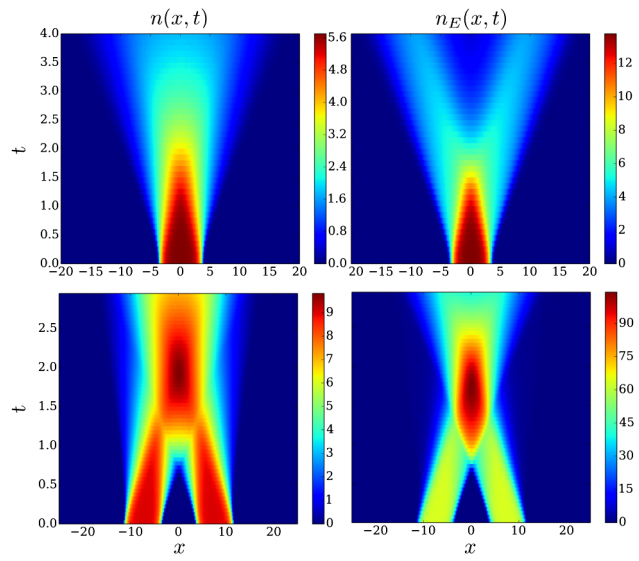

FIG. 3. Hydrodynamic evolutions in the Lieb-Liniger model with interaction strength $c = 1$: the top panel depicts free expansion initial conditions ($dt = 0.1$) and the lower panel models a collision between clouds of bosons with opposite initial momenta ($dt = 0.05$).

valuable tool for specific computations. More generally, it is natural to ask which phenomena in integrable models are missed by this hydrodynamical approach and how it differs qualitatively from conventional hydrodynamics. Clearly there is a significant assumption that the initial condition is well described by a local GGE [76,77]. In fact, the "thermalization problem" for quantum integrable models, namely, the question of determining the GGE to which a given quantum state converges in the long-time limit, is a difficult problem which remains unsolved in general. Nevertheless, when local equilibrium initial conditions can be imposed, hydrodynamics does seem to capture the leading behavior at long time and length scales, as illustrated in the examples above. There can also be other important subleading behaviors, beyond the approach to local equilibrium. An example is the behavior of the Lieb-Liniger model in the low-temperature limit, where it can be described by a conformal field theory or bosonization. Conformal invariance [1] and other methods [78] both predict a Schwarzian derivative term in the time evolution from locally thermal initial conditions, which might be an example of subleading behavior beyond hydrodynamics.

An important difference between generalized hydrodynamics in integrable models and conventional hydrodynamics concerns reversibility. The collision term in the standard Boltzmann equation induces dissipation and an increase of entropy. The Bethe-Boltzmann equation is dissipationless, and in fact its time evolution is *reversible*. The action of microscopic time reversal on a particular state is to invert all pseudomomenta k while fixing space, so that the time-reversed pseudomomenta are given by $\rho'_k(x, t) = \rho_{-k}(x, t)$. One can show despite the complex form of the velocity v_k in the Bethe-Boltzmann equations, it undergoes a simple sign change under this microscopic time reversal transformation [79].

So integrable models again present some surprises compared to ordinary kinetic theory: while the description by a local GGE is certainly a great reduction in complexity compared to an arbitrary quantum state and hence irreversible, the flow in the space of local GGEs described by

the Bethe-Boltzmann equation is reversible. Presumably this means that truly diffusive behavior, as is believed to appear for example in the gapped phase of the *XXZ* model, lies beyond this equation; for linear-response spin transport in *XXZ* model, which involves both ballistic and diffusive components [80,81], hydrodynamics correctly captures the ballistic part [7,26].

There are many possible mathematical questions regarding the existence and structure of solutions to the Bethe-Boltzmann equation [44], but we hope that the above results demonstrate its practical utility for applications to physics. It can be used as a starting point for comparison for other methods for quantum dynamics, or for incorporating integrability-breaking terms or driving. The hydrodynamical theory of integrable models is one of many examples in recent years of how the old vine planted by Yang and Yang [38] continues to bear fruit.

The authors acknowledge support from NSF DMR-1507141 and a Simons Investigatorship (V. B. B. and J. E. M.), the Emmy-Noether program of the Deutsche Forschungsgemeinschaft under KA 3360/2-1 (C. K.) and the Quantum Materials Program supported by the Director, Office of Science, Office of Basic Energy Sciences, Materials Sciences and Engineering Division, of the US DOE under Contract No. DE-AC02-05CH11231 (R. V.).

[1] S. Sotiriadis and J. Cardy, J. Stat. Mech. (2008) P11003.
[2] D. Bernard and B. Doyon, J. Phys. A **45**, 362001 (2012).
[3] R. Vasseur, C. Karrasch, and J. E. Moore, Phys. Rev. Lett. **115**, 267201 (2015).
[4] O. A. Castro-Alvaredo, B. Doyon, and T. Yoshimura, Phys. Rev. X **6**, 041065 (2016).
[5] B. Bertini, M. Collura, J. De Nardis, and M. Fagotti, Phys. Rev. Lett. **117**, 207201 (2016).
[6] A. De Luca, M. Collura, and J. De Nardis, Phys. Rev. B **96**, 020403 (2017).
[7] V. B. Bulchandani, R. Vasseur, C. Karrasch, and J. E. Moore, arXiv:1702.06146.
[8] C. Boldrighini, R. L. Dobrushin, and Y. M. Sukhov, J. Stat. Phys. **31**, 577 (1983).
[9] B. Doyon and H. Spohn, J. Stat. Mech. (2017) 073210.
[10] C. Karrasch, R. Ilan, and J. E. Moore, Phys. Rev. B **88**, 195129 (2013).
[11] B. Doyon, Nucl. Phys. **B892**, 190 (2015).
[12] A. De Luca, J. Viti, D. Bernard, and B. Doyon, Phys. Rev. B **88**, 134301 (2013).
[13] M. J. Bhaseen, B. Doyon, A. Lucas, and K. Schalm, Nat. Phys. **11**, 509 (2015).
[14] D. Bernard and B. Doyon, Annales Henri Poincaré **16**, 113 (2015).
[15] A. De Luca, J. Viti, L. Mazza, and D. Rossini, Phys. Rev. B **90**, 161101 (2014).
[16] A. Biella, A. De Luca, J. Viti, D. Rossini, L. Mazza, and R. Fazio, Phys. Rev. B **93**, 205121 (2016).
[17] D. Bernard and B. Doyon, J. Stat. Mech. (2016) 064005.
[18] R. Vasseur and J. E. Moore, J. Stat. Mech. (2016) 064010.
[19] E. Langmann, J. L. Lebowitz, V. Mastropietro, and P. Moosavi, arXiv:1701.06620.
[20] T. Prosen, Phys. Rev. Lett. **106**, 217206 (2011).
[21] T. Prosen and E. Ilievski, Phys. Rev. Lett. **111**, 057203 (2013).
[22] T. Prosen, Nucl. Phys. **B886**, 1177 (2014).
[23] R. G. Pereira, V. Pasquier, J. Sirker, and I. Affleck, J. Stat. Mech. (2014) P09037.
[24] E. Ilievski, M. Medenjak, and T. Prosen, Phys. Rev. Lett. **115**, 120601 (2015).
[25] E. Ilievski, J. De Nardis, B. Wouters, J.-S. Caux, F. H. L. Essler, and T. Prosen, Phys. Rev. Lett. **115**, 157201 (2015).
[26] E. Ilievski and J. De Nardis, Phys. Rev. Lett. **119**, 020602 (2017).
[27] V. Dunjko, V. Lorent, and M. Olshanii, Phys. Rev. Lett. **86**, 5413 (2001).
[28] P. Ohberg and L. Santos, Phys. Rev. Lett. **89**, 240402 (2002).
[29] P. Pedri, L. Santos, P. Öhberg, and S. Stringari, Phys. Rev. A **68**, 043601 (2003).
[30] A. S. Campbell, D. M. Gangardt, and K. V. Kheruntsyan, Phys. Rev. Lett. **114**, 125302 (2015).
[31] I. Bouchoule, S. S. Szigeti, M. J. Davis, and K. V. Kheruntsyan, Phys. Rev. A **94**, 051602 (2016).
[32] G. De Rosi and S. Stringari, Phys. Rev. A **94**, 063605 (2016).
[33] S. R. White, Phys. Rev. Lett. **69**, 2863 (1992).
[34] U. Schollwoeck, Ann. Phys. (Amsterdam) **326**, 96 (2011), January 2011 Special Issue.
[35] C. Karrasch, J. H. Bardarson, and J. E. Moore, Phys. Rev. Lett. **108**, 227206 (2012).
[36] T. Barthel, New J. Phys. **15**, 073010 (2013).
[37] D. Kennes and C. Karrasch, Comput. Phys. Commun. **200**, 37 (2016).
[38] C. N. Yang and C. P. Yang, J. Math. Phys. **10**, 1115 (1969).
[39] M. Gaudin, Phys. Rev. Lett. **26**, 1301 (1971).
[40] This condition is equivalent to *local quasi-stationarity* as defined in Ref. [41], modulo the subtleties discussed in Ref. [42]. Thus the hydrodynamic approach is *a priori* inapplicable to initial states that do not admit such a description, such as Néel states.
[41] B. Bertini and M. Fagotti, Phys. Rev. Lett. **117**, 130402 (2016).
[42] B. Pozsgay, E. Vernier, and M. A. Werner, J. Stat. Mech. (2017) 093103.
[43] This property is preserved under time evolution from initial conditions with compact support in $x-k$ space (as are required for any numerical implementation) thanks to conservation of the mode occupancies, $\int dx\rho_k(x,t)$.
[44] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.119.220604 for a short review of the Thermodynamic Bethe Ansatz and the Bethe-Boltzmann equations, details of the numerical implementation, and remarks about the semi-Hamiltonian structure of Bethe-Boltzmann hydrodynamics.
[45] X. Zotos, Phys. Rev. Lett. **82**, 1764 (1999).
[46] W. Press, *Numerical Recipes: The Art of Scientific Computing* 3rd ed. (Cambridge University Press, Cambridge, England, 2007).

[47] R. LeVeque, *Numerical Methods for Conservation Laws*, Lectures in Mathematics ETH Zürich (Birkhäuser, Basel, 2013).
[48] In practice, the exact dimension corresponds to the TBA discretization, typically around 4000 in our implementation.
[49] P. D. Lax, Commun. Pure Appl. Math. **10**, 537 (1957).
[50] B. A. Dubrovin and S. P. Novikov, Akademiia Nauk SSSR Doklady **270**, 781 (1983).
[51] S. P. Tsarëv, Mathematics of the USSR Izvestija **37**, 397 (1991).
[52] D. Serre, Annales de l'Institut Fourier **39**, 953 (1989).
[53] E. V. Ferapontov, Phys. Lett. A **158**, 112 (1991).
[54] E. B. Kolomeisky and J. P. Straley, Phys. Rev. B **46**, 11749 (1992).
[55] E. B. Kolomeisky, T. J. Newman, J. P. Straley, and X. Qi, Phys. Rev. Lett. **85**, 1146 (2000).
[56] K. V. Kheruntsyan, D. M. Gangardt, P. D. Drummond, and G. V. Shlyapnikov, Phys. Rev. Lett. **91**, 040403 (2003).
[57] Y. Castin and R. Dum, Phys. Rev. Lett. **77**, 5315 (1996).
[58] Y. Kagan, E. L. Surkov, and G. V. Shlyapnikov, Phys. Rev. A **55**, R18 (1997).
[59] B. Fang, G. Carleo, A. Johnson, and I. Bouchoule, Phys. Rev. Lett. **113**, 035301 (2014).
[60] Y. Y. Atas, D. M. Gangardt, I. Bouchoule, and K. V. Kheruntsyan, Phys. Rev. A **95**, 043622 (2017).
[61] D. M. Gangardt and G. V. Shlyapnikov, Phys. Rev. Lett. **90**, 010401 (2003).
[62] W. Xu and M. Rigol, Phys. Rev. A **95**, 033617 (2017).
[63] B. Doyon and T. Yoshimura, SciPost Phys. **2**, 014 (2017).
[64] B. Doyon and H. Spohn, arXiv:1705.08141.
[65] D. Gobert, C. Kollath, U. Schollwöck, and G. Schütz, Phys. Rev. E **71**, 036102 (2005).
[66] T. Kinoshita, T. Wenger, and D. Weiss, Nature (London) **440**, 900 (2006).
[67] S. Langer, F. Heidrich-Meisner, J. Gemmer, I. P. McCulloch, and U. Schollwöck, Phys. Rev. B **79**, 214409 (2009).
[68] S. Langer, M. Heyl, I. P. McCulloch, and F. Heidrich-Meisner, Phys. Rev. B **84**, 205115 (2011).
[69] J. P. Ronzheimer, M. Schreiber, S. Braun, S. S. Hodgman, S. Langer, I. P. McCulloch, F. Heidrich-Meisner, I. Bloch, and U. Schneider, Phys. Rev. Lett. **110**, 205301 (2013).
[70] C. Karrasch, J. E. Moore, and F. Heidrich-Meisner, Phys. Rev. B **89**, 075139 (2014).
[71] C. J. Bolech, F. Heidrich-Meisner, S. Langer, I. P. McCulloch, G. Orso, and M. Rigol, Phys. Rev. Lett. **109**, 110602 (2012).
[72] S. Langer, M. J. A. Schuetz, I. P. McCulloch, U. Schollwöck, and F. Heidrich-Meisner, Phys. Rev. A **85**, 043618 (2012).
[73] L. Vidmar, S. Langer, I. P. McCulloch, U. Schneider, U. Schollwöck, and F. Heidrich-Meisner, Phys. Rev. B **88**, 235117 (2013).
[74] R. Steinigeweg, F. Jin, D. Schmidtke, H. De Raedt, K. Michielsen, and J. Gemmer, Phys. Rev. B **95**, 035155 (2017).
[75] C. Karrasch, T. Prosen, and F. Heidrich-Meisner, Phys. Rev. B **95**, 060406 (2017).
[76] M. Rigol, V. Dunjko, and M. Olshanii, Nature (London) **452**, 854 (2008).
[77] T. Barthel and U. Schollwöck, Phys. Rev. Lett. **100**, 100601 (2008).
[78] E. Langmann, J. L. Lebowitz, V. Mastropietro, and P. Moosavi, Phys. Rev. B **95**, 235142 (2017).
[79] This follows by the dressing equations for energy and momentum, together with the fact that TBA kernels are difference kernels in the rapidity parameterization.
[80] J. Sirker, R. G. Pereira, and I. Affleck, Phys. Rev. Lett. **103**, 216602 (2009).
[81] J. Sirker, R. G. Pereira, and I. Affleck, Phys. Rev. B **83**, 035115 (2011).