Digital Quantum Estimation

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Quantum metrology calculates the ultimate precision of all estimation strategies, measuring what is their root-mean-square error (RMSE) and their Fisher information. Here, instead, we ask *how many bits* of the parameter we can recover; namely, we derive an information-theoretic quantum metrology. In this setting, we redefine "Heisenberg bound" and "standard quantum limit" (the usual benchmarks in the quantum estimation theory) and show that the former can be attained only by sequential strategies or parallel strategies that employ entanglement among probes, whereas parallel-separable strategies are limited by the latter. We highlight the differences between this setting and the RMSE-based one.

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The theory of quantum metrology [1–12] determines the ultimate precision in any estimation. The estimation of an unknown parameter generally requires a probe that interacts with the system to be sampled: The interaction encodes the parameter onto the probe, which is then measured. Clearly, if one uses N independent measurements, the root-mean-square error (RMSE) in the estimation scales as $1/\sqrt{N}$ (the standard quantum limit) as dictated by the central limit theorem. If one uses N parallel-entangled probes or one probe sequentially N times, the error can be reduced to $1/N$ (the Heisenberg bound) [4,13]. This precision can be attained without the use of entanglement at the measurement stage [4].

The RMSE is, however, ill suited for digital sensors, digital data processing, or even the digital archival of parameters, where the number of significant digits (bits) is a more useful figure of merit. Moreover, the techniques used in the conventional theory [e.g., the use of maximally path-entangled states of N photons (N00N) [14]] suffer from ambiguities in the typical case in which a phase is estimated [15,16], so that the reported RMSE does not typically refer to the true total error in the estimation [17–19]. Finally, this framework allows us to easily consider prior information on the parameter (but here we will consider only uniform prior), whereas in the RMSE case this is highly nontrivial [20,21].

In this Letter, we overcome these problems by replacing the RMSE (and Fisher information) with mutual information, which can be operationally seen as the number of bits of information that the quantum estimation strategy provides. Mutual information was used in quantum metrology [22–24], although always in connection to the RMSE. Here we rederive the theory from scratch, and we can use this connection to extend the RMSE bounds based on mutual information to more general estimation strategies. Our main result is a purely information-theoretic quantum metrology, obtaining also that (i) we redefine in a natural way the concepts of the Heisenberg bound (using the Holevo theorem) and of the standard quantum limit; (ii) for parallel estimation strategies, the Heisenberg bound can be attained but only in the presence of entanglement, as in the RMSE case; (iii) as expected, for parallel strategies without entanglement at the preparation, at most the standard quantum limit is achievable (and entanglement at the measurement stage is useless); (iv) instead, for sequential strategies (where one of the probes performs most of the samplings), the Heisenberg bound is attainable without using entanglement, as in the RMSE case; (v) increasing the Hilbert space dimension of the probe is helpful, in contrast to the RMSE case where a two-dimensional subspace is sufficient; (vi) the Heisenberg bound is achieved by the quantum phase estimation algorithm (QPEA) [25,26] and by the Pegg-Barnett phase states [27], in contrast to the RMSE case [17,18,28].

Heisenberg bound and standard quantum limit.—In quantum metrology, we estimate a parameter φ by first preparing one or more probes into an initial state ρ_0, then evolving them by applying N times the interaction U_φ that encodes the parameter onto the probe(s) and transforms the state into ρ_φ, and finally measuring ρ_φ. The aim is to find the ultimate precision attainable for the estimation strategy as a function of N. If the probe is finite dimensional, no estimation strategy can beat the Heisenberg bound $\propto 1/N$ for the RMSE.

A natural way to extend the Heisenberg bound to an information-theoretic setting is to use the Holevo theorem [29], which gives the maximum number of bits I attainable on a parameter φ encoded into a state ρ_φ, given the measurement results $\vec{m}$:

$$I(\vec{m}:\varphi) \leq S\left(\sum_\varphi p_\varphi \rho_\varphi\right) - \sum_\varphi p_\varphi S(\rho_\varphi), \qquad (1)$$

where $S(\rho) = -\mathrm{Tr}[\rho \log_2 \rho]$ is the von Neumann entropy and p_φ the prior probability of the parameter φ. Clearly, the accessible information is largest when ρ_φ are all pure states, and in this case the last sum in (1) is null and the Holevo bound is attainable if the states are orthogonal. We then define the info-theoretic Heisenberg bound as $S(\sum_\varphi p_\varphi \rho_\varphi)$. This quantity scales as $\log_2 N$, since we are applying N times the *same* transformation U_φ that encodes the unknown parameter φ (Supplemental Material [30]). So the Heisenberg bound is $I \simeq \log_2 N$, at least asymptotically for large N.

As regards the standard quantum limit, we note that any parameter affected by statistical noise can always be estimated as an average of some distribution: This underlies the fact that all statistical errors can be reduced by repeating the measurement and averaging the results. Then, the estimation of an average quantity of an arbitrary probability distribution through N data points is always limited by $I \leq \frac{1}{2}\log_2 N$, as a direct consequence of the central limit theorem [30]. Thus, we can use this bound as information-theoretic standard quantum limit. The above Heisenberg bound does not fall under this result, since in that case we are using a single-shot estimation and not an average: Each of the N outcomes is meaningless by itself in entangled strategies.

These definitions are consistent with the RMSE-based ones: An error $\Delta\varphi \simeq 1/N$ leads to the expectation that roughly $\log_2 N$ binary digits of the results are reliable, and similarly an error $\Delta\varphi \simeq 1/\sqrt{N}$ leads to the expectation that $\frac{1}{2}\log_2 N$ digits are reliable. Nonetheless, the RMSE and the mutual information capture different aspects of the estimation's quality, as shown below.

Below, we show which kinds of estimation strategies achieve these bounds. An example (the QPEA) shows that sequential and entangled-parallel strategies can achieve the info-theoretic Heisenberg bound. We then show that the optimal parallel-separable strategies can attain only the standard quantum limit. We finally discuss the role of the probe's dimensionality.

For the sake of simplicity, we will first restrict to two-dimensional probes (qubits), for which $U_\varphi = |0\rangle\langle 0| + e^{i2\pi\varphi}|1\rangle\langle 1|$ (with $|0\rangle$ and $|1\rangle$ the eigenstates of the generator of U_φ) and then separately analyze what happens in the (finite) d-dimension case. We use finite-dimensional probes and unitaries, so the parameter φ is periodic and we restrict to $\varphi \in [0, 1]$. As is customary in quantum metrology, we request no prior knowledge on the parameter to be estimated (uniform prior).

Sequential strategies.—In sequential strategies [4,26,32], the transformations U_φ act on a single probe sequentially, and ancillas may interact with the probe at any intermediate stage [Fig. 1(a)]. We consider the QPEA [25,26] as an example of the sequential strategy [Fig. 1(b)]: It needs $t = \log_2(N+1)$ qubits initialized in $|+\rangle \propto |0\rangle + |1\rangle$ states, where the zeroth qubit is subject to U_φ once, and the jth qubit is subject to U_φ 2^j times. The t qubits then undergo a quantum Fourier transform (QFT) and are measured in the computational basis, yielding a t-bit number m, from which φ can be estimated as $m/2^t$. One can see that the QPEA is a sequential

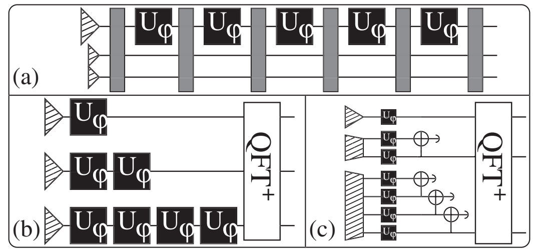

FIG. 1. Sequential and parallel-entangled strategies. (a) Sequential strategy, where a single probe (large triangle) samples N unitaries U_φ (black boxes) sequentially. Ancillary systems (small triangles) may interact through arbitrary intermediate unitaries (gray squares). (b) QPEA. To see that it is equivalent to a sequential strategy [26], where the last unitary is the inverse quantum Fourier transform (QFT†), use intermediate unitaries that swap the state of the ancillas with the state of the probe. The output (measured in the computational basis) is a t-bit digital estimate of the parameter φ with $t = \log_2(N+1)$. (c) Parallel QPEA, which uses entangled N00N states (dashed boxes) composed of $1, 2, 4, \ldots, 2^{t-1}$ qubits. The circles represent CNOT gates that remove the entanglement, and the cups represent the discarding of qubits in the state $|0\rangle$.

strategy by considering one of the qubits as the probe and the others as ancillas and inserting appropriate swap unitaries to swap the ancilla states and the probe state (the zeroth swap after a single U_φ action, the jth after 2^j actions) [26].

To evaluate how many of the bits of m are reliable, one needs to calculate the mutual information $I(m:\varphi)$, using the QPEA conditional probability

$$p(m|\varphi) = \frac{\sin^2[\pi(N+1)\varphi]}{(N+1)^2\sin^2\{\pi[\varphi - m/(N+1)]\}}. \quad (2)$$

The mutual information obtained from it has an asymptotic scaling in N given by [30]

$$I(m:\varphi) \to \log_2 N - 2 + 2\frac{\gamma + \ln(2) - 1}{\ln(2)} \simeq \log_2 N - 1.22, \quad (3)$$

where γ is the Euler-Mascheroni constant. Namely, it (quickly) asymptotically achieves the info-theoretic Heisenberg bound, apart from a small additive constant; see Fig. 2.

The QPEA is known to also achieve the best estimation in terms of a window function cost [26], but it cannot achieve the RMSE-based Heisenberg bound unless one repeats it a few times [17–19].

Parallel-entangled strategies.—The proof that parallel-entangled strategies can achieve the mutual-info Heisenberg bound is simple, since one can easily transform the sequential strategy detailed above into a parallel one by entangling the probes; see Fig. 1(c). This means that one uses N probes grouped in N00N states of increasing numbers of bits: $|0\rangle + |1\rangle$, $|00\rangle + |11\rangle, \ldots, |0\rangle^{2^j} + |1\rangle^{2^j}, \ldots$. When these

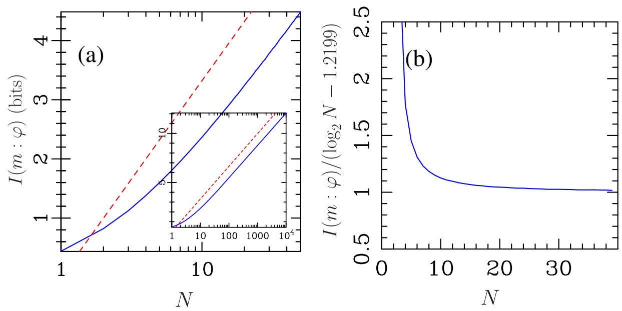

FIG. 2. Heisenberg bound of the QPEA. (a) Plot of the mutual information $I(m:\varphi)$ as a function of N (blue line) and of the function $\log_2 N$ (dashed red line). Note that I quickly acquires the same linear dependence in a log scale as the Heisenberg bound. The inset shows the same behavior for large N. (b) Ratio between the mutual information and $\log_2 N - 1.22$, showing the rapid onset of the asymptotic behavior to this quantity.

$\log_2(N+1)$ groups interact in parallel with the N transformations U_φ, the jth group acquires a phase of $2\pi 2^j\varphi$, the same as the corresponding probe in the QPEA strategy in Fig. 1(b). A simple network of controlled-NOT gates can transfer this phase to one of the probes in each group, and the other probes in the group are discarded. So the input to the final quantum Fourier transform is identical to the one of the conventional QPEA. Thus, both the output probability and the mutual information are the same as the ones calculated above: Also, the parallel-entangled strategy can achieve the Heisenberg bound (apart from a small additive constant).

Note that the use of controlled-NOT gates after the action of the transformations U_φ imply that this procedure requires an entangled detection strategy (in contrast, the QFT does not require entanglement among probes [33]). It is still an open question whether a parallel-entangled strategy can achieve the info-theoretic Heisenberg bound with a separable detection, as is the case for the RMSE bound. The Heisenberg bound is *not* achieved [30] if one uses the same detection strategy as in the RMSE case (namely, projecting each probe onto the $|\pm\rangle \propto |0\rangle \pm |1\rangle$ states) or if one employs the single-qubit optimal strategy according to Davies's theorem (see below).

Parallel-separable strategies.—To prove that without entanglement the parallel strategies cannot achieve the Heisenberg bound, one needs to analyze the optimal strategy and show that it can achieve only the standard quantum limit (whereas to prove that the sequential and entangled strategies can achieve the Heisenberg bound, we merely had to exhibit an example, the QPEA above).

In the separable case, the optimal input state for each qubit probe is an equatorial state, such as $(|0\rangle + |1\rangle)/\sqrt{2}$, which is evolved by U_φ into $|\varphi\rangle = (|0\rangle + e^{i2\pi\varphi}|1\rangle)/\sqrt{2}$. Indeed, equatorial states maximize the distinguishability between the input and output. The N parallel probes after the U_φ evolutions emerge in a joint state

$$|\varphi\rangle^{\otimes N} = \sum_{j=0}^{N} \sqrt{\frac{1}{2^N}\binom{N}{j}}\, e^{i2\pi j\varphi}|S_j\rangle, \tag{4}$$

where $|S_j\rangle$ is the normalized symmetric state obtained by summing over all possible permutations with j ones; e.g., for $N=4$, $|S_1\rangle \propto |0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle$, and $|S_2\rangle \propto |0011\rangle + |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle$.

To obtain the positive operator-valued measure (POVM) that maximizes the mutual information on this state, we use Davies's theorem [34]: If the input is covariant with respect to a group that admits an irreducible unitary representation U_φ, then there exists a unit vector $|r\rangle$ such that the mutual information is maximized by the POVM

$$\Pi_\phi = \frac{d}{|G|} U_\phi |r\rangle\langle r| U_\phi^\dagger, \tag{5}$$

where d is the dimension of the system Hilbert space and $|G|$ is the number of elements in the group [34]. Davies's theorem can be extended to continuous parameters φ by requiring the compactness of the group [35] and to unitary representations that are irreducible only on equatorial states [36].

Since the state $|\varphi\rangle^{\otimes N}$ spans only the $(N+1)$-dimensional symmetric subspace of the N-qubit space, we can limit ourselves to it. So the optimal POVM is given by (5) with $d = N+1$, $|G| = 1$, and $|r\rangle$ a state in the symmetric subspace: $|r\rangle = \sum_j \alpha_j |S_j\rangle$. Apart from an irrelevant phase factor, this state is uniquely determined by the POVM's normalization condition $\int d\phi \Pi_\phi = \mathbb{1}$ (see [37]). Indeed, this condition is satisfied only if $|\alpha_j| = 1/\sqrt{N+1}$ for all j. Hence, an optimal POVM is

$$\Pi_\phi = (N+1)|\phi\rangle\langle\phi|, \quad \text{with} \quad |\phi\rangle \equiv \frac{1}{\sqrt{N+1}} \sum_{n=0}^{N} e^{i2\pi n\phi}|S_n\rangle. \tag{6}$$

Then, the conditional probability of finding the result ϕ (which is our estimate of the unknown parameter) when the true value is φ is

$$p(\phi|\varphi) = (\langle\varphi|^{\otimes N})\Pi_\phi(|\varphi\rangle^{\otimes N}) \tag{7}$$

$$= \sum_{n,n'=0}^{N} \frac{1}{2^N} \sqrt{\binom{N}{n}\binom{N}{n'}}\, e^{i2\pi(\phi-\varphi)(n-n')}, \tag{8}$$

whence one can calculate the mutual information $I(\phi:\varphi)$. Its asymptotic scaling [30] is

$$I(\phi:\varphi) \to \frac{1}{2}\log_2 N + \frac{1}{2}\log_2\frac{2\pi}{e} \simeq \frac{1}{2}\log_2 N + 0.6, \tag{9}$$

namely, the standard quantum limit for the mutual information (apart from a small additive constant). The explicit evaluation of $I(m:\varphi)$ shows that it quickly attains the asymptotic expression [Fig. 3(a)]. This proves that separable probes can achieve at most the standard quantum

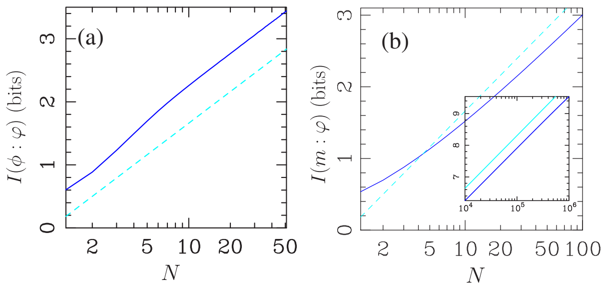

FIG. 3. Standard quantum limit for unentangled probes. (a) Plot of the mutual information $I(\phi:\varphi)$ relative to the optimal POVM (6), which uses entangled measurements, as a function of N (blue line) and of the standard quantum limit $\log_2(N)/2$ (cyan dashed line). The fluctuations are due to the Monte Carlo integration used here. (b) Plot of the mutual information $I(\vec{m}:\varphi)$ of (12), relative to the separable POVM that projects onto $|\pm\rangle$ each probe, as a function of N (blue line) and of the standard quantum limit $\log_2(N)/2$ (cyan dashed line). The inset shows the large-N scaling. Both cases asymptotically scale at the standard quantum limit (apart from a small additive constant).

limit: The Davies's theorem gives the best possible estimation strategy, and we have optimized the input states only among the separable ones.

The above strategy uses separable input states but an entangled POVM Π_ϕ (the states $|\phi\rangle$ are entangled). We now show that the standard quantum limit can be achieved also by a strategy separable both at the input and at the measurement. Indeed, consider the strategy in which we measure the separable state $|\varphi\rangle^{\otimes N}$ with a projective POVM which projects onto the states $|\pm\rangle \propto |0\rangle \pm |1\rangle$ each of the N qubits separately. The outcome will be a string $\vec{m}$ of N zero-one results corresponding to outcome "+" or "−" at each qubit, respectively. The probability of each outcome is $p(+|\varphi) = \cos^2(\pi\varphi)$ and $p(-|\varphi) = \sin^2(\pi\varphi)$, so the probability of obtaining the whole string $\vec{m}$ is

$$p(\vec{m}|\varphi) = \sin^{2\kappa}(\pi\varphi)\cos^{2(N-\kappa)}(\pi\varphi), \tag{10}$$

where κ is the number of ones in the string $\vec{m}$ (its Hamming weight). The unknown parameter is easily estimated from the vector $\vec{m}$ as κ/N. The marginal probability of the string $\vec{m}$ is then

$$p(\vec{m}) = \int_0^1 d\varphi\, p(\vec{m}|\varphi) = \frac{[2(N-\kappa)]!(2\kappa)!}{2^{2N}(N-\kappa)!\kappa!N!}, \tag{11}$$

whence mutual information is [30]

$$I(\vec{m}:\varphi) = N/\ln 2 + \sum_{\kappa=0}^{N} \frac{[2(N-\kappa)]!(2\kappa)!}{4^N[(N-\kappa)!]^2(\kappa!)^2}\log_2\frac{(N-\kappa)!\kappa!N!}{[2(N-\kappa)]!(2\kappa)!}. \tag{12}$$

The asymptotic scaling of $I(\vec{m}:\varphi)$ of Eq. (12) for large N was numerically checked [Fig. 3(b)] and goes as $\simeq \log(N)/2 - 0.395$ (the constant was evaluated numerically), as expected from the standard quantum limit.

Beyond qubits.—We now drop the assumption of two-dimensional probes (qubits) and consider the effect of a d-dimensional Hilbert space of the probes. In this case, we must consider the transformation $U_\varphi = \sum_{n=0}^{d-1} e^{i2\pi n\varphi}|n\rangle\langle n|$, where $|n\rangle$ are eigenstates with eigenvalue n of the generator H of U_φ. Intuitively, one expects that a two-dimensional probe will give outcomes in bits (base-2 numbers) and that a d-dimensional probe will give outcomes in base-d numbers. We will see that this intuition is correct: One will asymptotically gain the factor $\log_2 d$ of a change of basis in the logarithms in the mutual information definition. We can prove this result using a d-dimensional extension of the QPEA for the sequential and entangled protocols and using the Pegg-Barnett states for the separable protocol (also shown in Ref. [28]).

The QPEA for d-dimensional systems [38] is a straightforward extension of the QPEA. Its output is a number m composed of t base-d digits, whence the parameter φ can be estimated as m/d^t. The conditional probability of obtaining m given φ is

$$p(m|\varphi) = \frac{\sin^2(\pi\varphi d^t)}{d^{2t}\sin^2[\pi(\varphi - m/d^t)]}, \tag{13}$$

analogous to (2). The mutual information is then

$$\begin{aligned} I(m:\varphi) &= t\log_2 d + \int_0^1 d\varphi \sum_m p(m|\varphi)\log_2 p(m|\varphi) \\ &\to t\log_2 d - 1.22, \end{aligned} \tag{14}$$

where the asymptotic scaling is derived in the same way as for (3). The Heisenberg bound is still achieved, since in (14) $t \simeq \log_2 N/\log_2 d + 1$. The form of $I(m:\varphi)$ as a function of d is the same as the one shown in Fig. 2 if one replaces $N+1$ with d^t: Compare (13) with (2). Hence, as in the previous case, the asymptotic scaling (14) kicks in very rapidly.

In the separable case, we can find a similar Heisenberg bound and $\log_2 d$ factor by preparing each d-dimensional probe in the Pegg-Barnett state $\sum_n |n\rangle/\sqrt{d}$ [27], which is evolved by U_φ into the state $|\varphi, d\rangle \equiv \sum_n e^{2\pi i n\varphi}|n\rangle/\sqrt{d}$. A measurement that extracts information from the probe asymptotically approaching $\log_2 d$ bits is a projective POVM onto the states $|\varphi_j, d\rangle$ with $\varphi_j = j/d$ (with integer $j \le d-1$) [28]. This is equivalent to the above d-dimensional QPEA for a single probe $t = 1$, so the mutual information of this Pegg-Barnett procedure is given by Eq. (14) with $t = 1$, where again we find a $\log_2 d$ factor. Hence, also in the separable case, an increase in the probes dimension leads to a $\log_2 d$ increase in precision.

Note that, also in the RMSE case, an increase in the probe dimension increases the precision, because we can access larger eigenvalues of the generator of U_φ. However, in that case, one can always restrict the probes to a two-dimensional subspace, spanned by the eigenvectors $|0\rangle$ and $|d-1\rangle$ relative to the minimum and maximum eigenvalues of the generator H [4]. In the mutual-info case, this is not true anymore: The above $\log_2 d$ increase in precision is absent if we limit the probe states to the subspace spanned by these two states [30]. Interestingly, the Pegg-Barnett states are known to be useless in achieving the RMSE-based Heisenberg scaling in the dimension d [39], in contrast to the above $\log_2 d$ scaling result. These two facts emphasize that, although RMSE and mutual info give consistent indications on the measurement precision, they really capture different aspects of it.

Conclusions.—We have given an information-theoretic quantum metrology, leading to the main results of ordinary RMSE-based quantum metrology but highlighting some peculiar differences from it. We did not consider the effect of noise and experimental imperfections here, leaving it to future work, since this substantially complicates the theory, as happens in the RMSE case, e.g., [8,10,40–43].

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