Exact Logarithmic Four-Point Functions in the Critical Two-Dimensional Ising Model

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(Received 17 April 2017; revised manuscript received 4 August 2017; published 6 November 2017)

Based on conformal symmetry we propose an exact formula for the four-point connectivities of Fortuin-Kasteleyn clusters in the critical Ising model when the four points are anchored to the boundary. The explicit solution we found displays logarithmic singularities. We check our prediction using Monte Carlo simulations on a triangular lattice, showing excellent agreement. Our findings could shed further light on the formidable task of the characterization of logarithmic conformal field theories and on their relevance in physics.

DOI: 10.1103/PhysRevLett.119.191601

Introduction.—Conformal symmetry in two dimensions [1] has been of extraordinary usefulness to study classical statistical mechanics models at criticality since the 1980s. It has notably found also extensive applications in the quantum realm, spanning from gapless one-dimensional systems [2], the quantum Hall effect [3], and entanglement [4]. Two dimensional conformal invariant quantum field theories (CFTs) and, in particular, Liouville theory are moreover the cornerstone of world-sheet geometry in string theory [5]. The simplest CFTs that capture the critical behavior of lattice models and quantum spin chains are *unitary*. Moreover, when their Hilbert space splits into a finite number of representations of the conformal (or some larger) symmetry are usually termed *rational*. CFTs are classified according to their central charge c; for instance, for a free massless boson $c = 1$.

However unitary and rational theories are far from exhausting the physically relevant conformally invariant theories. In the beginning of the 1990s the groundbreaking discovery [6] of an exact formula for the crossing probability in critical percolation forced to analyze theories violating these two assumptions. Percolation is a simple stochastic process where bonds or sites on a lattice can be occupied independently with a certain probability. Since the partition function is not affected by finite-size effects, the central charge of a putative CFT describing critical percolation is zero [7]. The existence of a nontrivial formula for the crossing probability makes it a prominent example of a nonunitary CFT (the only unitary CFT with $c = 0$ is trivial). For subsequent developments leading to the formulation of the stochastic Lowener evolution we refer to the reviews [8,9]. At the same time, it was recognized that as far as the conditions of unitarity and rationality are relaxed, CFT correlation functions can display striking logarithmic singularities that are actually the signatures of intricate realizations of the conformal symmetry [10,11]. The class of nonunitary and generally nonrational CFTs where these unconventional features show up, was christened logarithmic CFTs. Such theories were promptly argued to play a fundamental role in the characterization of disordered systems in two dimensions [12–14], for instance, by means of supersymmetry [15,16].

With these motivations, logarithmic CFTs were investigated in greater detail in the last decade, either from a purely algebraic point of view [17], either constructing lattice regularizations [16,18–20] or generalizing the study of crossing formulas in critical percolation [21–23]. Recently [24], it has also been suggested that an analytic nonunitary extension of Liouville theory [25] might describe the connectivity properties of critical bulk percolation and, more generally, the Q-state Potts model. The domain of applicability of Liouville theory in statistical mechanics is currently an important open problem; see Refs. [26,27] and also Ref. [28].

Despite these huge advances, a satisfactory understanding of logarithmic CFTs is still a long way off. Moreover, it is fair to say that few examples of explicit logarithmic singularities have been found in familiar statistical models: notably only in percolation [29,30] ($c = 0$), dense polymers [31–33]; see also Refs. [34,35] for applications to disordered systems. An exact Coulomb gas approach to CFT correlation functions [36–38] closely related to those considered in this Letter reveals an infinite number of logarithmic cases. These arise from operator mixing as is the case here. These results suggest the possibility of logarithmic behavior in multiple self-avoiding random walks (SAWs) [38].

In this Letter we show how a logarithmic singularity due to operator mixing also arises in the context of arguably the best-known model of statistical mechanics: the two-dimensional Ising model; see Refs. [39,40] for a pedagogical introduction to the richness of the model. This is remarkable since it shows unambiguously that critical properties of Ising clusters are ruled by a logarithmic CFT. Moreover the four-point function we consider here, a four point connectivity in the Fortuin-Kasteleyn (FK) representation of the Ising model, is a natural observable that can be easily simulated with Monte Carlo (MC) methods. Logarithmic singularities in Ising connectivities were also argued to exist in Refs. [41,42], in this Letter we demonstrate it explicitly using CFT.

Finally, our findings could shed further light on the extremely challenging problem of the characterization of logarithmic CFTs and on their applications to physics.

Four points connectivities in the Ising model.—It is convenient to introduce the Ising FK clusters, starting from the ferromagnetic Q-state Potts model [43]; the Ising case is recovered by setting $Q = 2$. The model is defined on a finite simply connected domain $\mathcal{D}$ of the plane, see Fig. 1, and the choice of the underlying lattice is irrelevant at the critical point. To introduce the notion of cluster connectivities, we should first recall the FK representation [44]. The Q-state Potts model is defined in terms of spin variables $s(x)$ taking $1, \ldots, Q$ different values; its partition function can be expressed as a product over the lattice edges as

$$Z = \sum_{\{s(x)\}} \prod_{\langle x,y \rangle} [(1-p) + p\delta_{s(x),s(y)}], \tag{1}$$

where p is a parameter related to the temperature and the product in Eq. (1) extends only to next-neighboring sites. Suppose then to expand such a product: Each term in the expansion can be represented graphically by drawing a bond between x and y if the factor $p\delta_{s(x)s(y)}$ is selected, and leaving empty the bond if such a factor is absent. The set of nonempty bonds in each term of the expansion then defines a graph G on the underlying lattice that is called the FK graph. Such a graph might contain N_c different connected components (including isolated points), dubbed FK clusters. Moreover, on each cluster the spin values are constrained to be the same because of the Kronecker delta in Eq. (1). Summing over their possible Q values leads to the following rewriting of the Q-state Potts model partition function as a sum over graphs: $Z = \sum_G (1-p)^{\bar{n}_b} p^{n_b} Q^{N_c}$, where n_b and $\bar{n}_b$ are the number of occupied and empty bonds in the graph G. For arbitrary non-negative Q, the graph representation for Z is a generalized percolation problem known as random cluster model, where bonds occupied with probability p are not independent random variables. The fundamental observables in the random cluster model are the connectivities and they offer a purely geometrical interpretation of the magnetic Potts model phase transition. Connectivities represent the different probabilities with which n points of the plane can be partitioned into FK clusters. If the points are on the boundary of the domain $\mathcal{D}$, the total number of n-point connectivities is clearly equal to the number of noncrossing partitions of a set of n elements, i.e., the catalan number C_n; for example, if $n = 4$ there are $C_4 = 14$ of them. These functions are, however, not linearly independent, since they satisfy sum rules: for instance, the sum over all n-point connectivities has to be 1. Following Ref. [45], it is possible to show that a valid choice of n-point linearly independent connectivities is given by all the probabilities associated with configurations where no point is disconnected from all the others (non-singleton partitions). In the specific example of $n = 4$ and x_1, x_2, x_3, x_4 on the boundary of $\mathcal{D}$, see again Fig. 1, a possible choice of linearly independent connectivities is $P_{(1234)}$, $P_{(12)(34)}$, and $P_{(14)(23)}$. The function $P_{(1234)}$ denotes the probability that all the four points x_1, x_2, x_3, and x_4 are on the same FK cluster; $P_{(12)(34)}$ is instead the probability that x_1 and x_2 are in the same cluster, x_3 and x_4 are in the same cluster but these two are now different and analogously for $P_{(14)(23)}$. We also omitted for simplicity the explicit spacial dependence. Notice that when the points x_1, x_2, x_3, and x_4 are anchored to the boundary the function $P_{(13)(24)}$ does not appear since two clusters cannot cross.

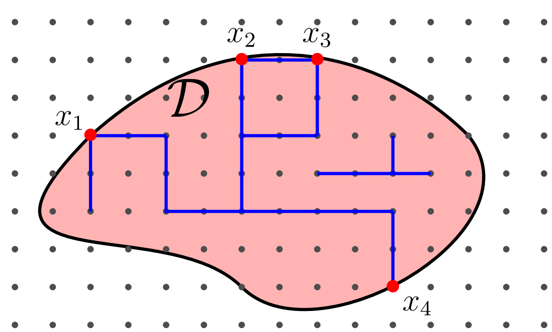

FIG. 1. The Q-state Potts model is defined on a finite domain $\mathcal{D}$ of the plane. FK graphs G on the underlying square lattice are drawn in blue. The figure shows a particular configuration where the four boundary points x_1, x_2, x_3, and x_4 are all connected by an FK cluster thus contributing to $P_{(1234)}$.

Exact solution.—We turn now to the exact determination of these three functions in the critical Ising model, using arguments inspired by the seminal work of Ref. [6]. At the critical point, conformal invariance allows one to map any simply connected domain $\mathcal{D}$ of the plane by the Riemann mapping theorem into the unit disk. Moreover, the points x_1, x_2, x_3, x_4 are mapped to points w_1, w_2, w_3, w_4 lying at the boundary (circumference) of such a disk. The three connectivities $P_{(1234)}$, $P_{(12)(34)}$, and $P_{(13)(24)}$ can be singled out by computing Potts partition functions with specific boundary conditions for the dual Potts spins [45]. As an example, let us suppose to fix the values of the dual Potts spins at the boundary of the disk to be 1, 2, 3, and 4 as in Fig. 2 left and to compute the Potts partition function in this case. Notice that this assignment will require at least four available colors, i.e., $Q \geq 4$, and it would be nonphysical for the Ising model. It has, however, certainly sense if we assume Q real and imagine to compute connectivities in the random cluster model at any values of Q and take eventually the limit $Q \to 2$. Configurations of dual FK clusters with such a particular choice of boundary conditions cannot contain clusters that cross from regions with boundary conditions α to regions with boundary conditions β if $\alpha \neq \beta$. Dual FK clusters are represented schematically by blue dashed curves in Fig. 2. Applying a duality transformation to the Potts model partition function [43],

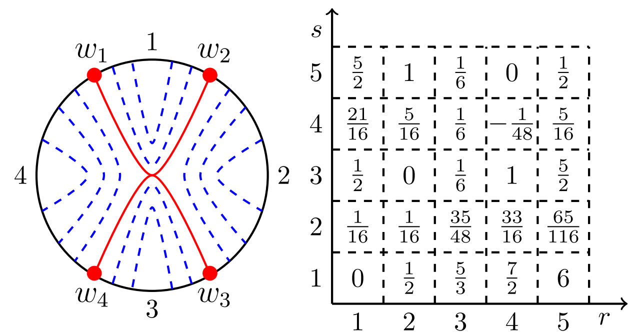

FIG. 2. On the left, schematic representation of allowed dual FK clusters (dashed blue curves) in the Potts model when boundary conditions that fix the values of the dual boundary spins to 1,2,3, and 4 are chosen. On the right, the Kac table, obtained from the scaling dimension $h_{r,s}$ in Eq. (2) for $Q = 2$ corresponding to the Ising model.

these configurations are in one-to-one correspondence with configurations where a single FK cluster, the continuous red curve in Fig. 2, connects the four boundary points. The reasoning above allows us to compute the connectivities as Potts partition functions with insertion of local operators $\phi_{(\alpha|\beta)}$ that switch the values of the dual spins at the boundary from $\alpha \to \beta$, $\alpha, \beta = 1, \ldots, Q$. In the jargon of CFT, the fields $\phi_{(\alpha|\beta)}$ are called boundary-condition-changing operators. In this way, we can argue, for example, that $P_{(1234)}$ has to be proportional to the correlation function of $\langle \phi_{(4|1)}(w_1)\phi_{(1|2)}(w_2)\phi_{(2|3)}(w_3)\phi_{(3|4)}(w_4)\rangle$.

Let us briefly recall that in the simplest case, the scaling fields $\phi_{r,s}$ of any CFT can be classified by two positive integers r, s such that their scaling dimensions are

$$h_{r,s} = \frac{[r(m+1) - sm]^2 - 1}{4m(m+1)}, \qquad m \in \mathbb{R}. \tag{2}$$

The parameter m is related to the central charge c of the CFT through $c(m) = 1 - [6/m(m+1)]$, and in turn for the Potts model $Q = 4\cos^2[\pi/(m+1)]$. The values $h_{r,s}$ can be represented into a lattice, dubbed the Kac table; for a CFT with $c = 1/2$ as the Ising model, the Kac table is represented in Fig. 2 on the right.

The boundary condition changing operator $\phi_{(\alpha|\beta)}$ was identified in Ref. [6] for any values of Q as the field $\phi_{1,3}$. Notice that at $c = 1/2$, the dimension of $\phi_{1,3}$ is $h_{1,3} = 1/2$ and coincides with the one of the Ising order parameter σ, when inserted at the boundary [46]; in this case the spin operator σ transforms as the field $\phi_{2,1}$. In the construction of the simplest conformal field theory describing the $\mathbb{Z}_2$ universality class these two fields can be actually identified and, consequently, the operator product algebra of $\{\phi_{1,1}, \phi_{2,1}, \phi_{1,2}\}$ closes. The self-consistent closure of the operator product algebra was used as a criterion in Ref. [1] to build the whole family of minimal conformal models, where only a finite numbers of Virasoro algebra representations should be considered and furthermore allows us to classify all the possible conformal boundary conditions [46]. However, when analyzing the connectivity properties of the Ising FK clusters, the identification of $\phi_{1,3}$ with $\phi_{2,1}$ is no longer possible. According to the general theory [1], see also Ref. [47], the four-point function of $\phi_{1,3}$ satisfies a linear differential equation of degree 3. If we map the unit disk to the upper half plane $\mathbb{H}$ and call $z_1, \ldots, z_4$ the images on the real axis of the boundary points $w_1, \ldots, w_4$ we have

$$\left\langle \prod_{i=1}^{4} \phi_{1,3}(z_i) \right\rangle_{\mathbb{H}} = \left(\frac{z_{42} z_{31}}{z_{21} z_{43} z_{32} z_{14}} \right)^{2h_{1,3}} F(\eta), \tag{3}$$

where $z_{ij} = z_i - z_j$ and $\eta = (z_{21} z_{43}/z_{42} z_{31})$ is the harmonic ratio ($0 < \eta < 1$). For the Ising model, the function $F(\eta)$ is the solution of the differential equation [48]

$$[2\eta(1-\eta)]^2 F''' - 3(1 - \eta + \eta^2)F' + 3(2\eta - 1)F = 0. \tag{4}$$

Equation (4) has three linearly independent solutions $F_{1,1}(\eta)$, $F_{1,3}(\eta)$, and $F_{1,5}(\eta)$. The behavior for small η of each of the functions $F_{r,s}$ is of the form $\eta^{h_{r,s}}$ and the exponent $h_{r,s}$ coincides with the scaling dimension of the field $\phi_{r,s}$ that is produced in the operator product algebra [1]: $\phi_{1,3} \times \phi_{1,3} = \phi_{1,1} + \phi_{1,3} + \phi_{1,5}$. Although there is not a general procedure to solve the differential equation (4), we can proceed as follows. First, we observe that function $F_{1,1}(\eta)$ has to coincide apart from the prefactor in Eq. (3) with the four point function of the boundary spin σ and such a function [1] is the simple monodromy invariant [49] polynomial $1 - \eta + \eta^2$. It is also easy to understand what $F_{1,1}$ should be in terms of connectivities. Since the four-point function of the boundary spin operator can be fully defined in the minimal Ising model, it has to correspond to the unique partition function that requires only two colors to be constructed, namely, $\langle \phi_{(2|1)}(w_1)\phi_{(1|2)}(w_2)\phi_{(2|1)}(w_3) \times \phi_{(1|2)}(w_4)\rangle$. This, in turn, is proportional to the sum $P_t = P_{(1234)} + P_{(14)(23)} + P_{(12)(34)}$ of the three linearly independent connectivities. Using the known solution $F_{1,1}(\eta)$ we can reduce the degree of the differential equation (4) by substituting $F(\eta) = F_{1,1}(\eta) \int_0^\eta d\eta' G(\eta')$. The function $G(\eta)$ is finally obtained through a rational pull back of the Gauss hypergeometric function [50]; see also Ref. [48]. One gets two linearly independent solutions $G_{1,2}$ for $G(\eta)$, related by the transformation $\eta \to 1 - \eta$: $G_1(\eta) = f(\eta)$ and $G_2(\eta) = f(1-\eta)$. The function $f(\eta)$ is

$$f(\eta) = \frac{p(\eta)E(\eta) + q(\eta)K(\eta)}{\sqrt{(1-\eta)\eta}}, \tag{5}$$

with $K(\eta)$ and $E(\eta)$ the elliptic integrals of the first and second kind, respectively, and $p(\eta)$ and $q(\eta)$ rational functions of η [48]. The behavior for small η finally fixes, up to an overall constant, $F_{1,5}(\eta) = F_{1,1}(\eta)\int_0^\eta d\eta' f(\eta')$. The third linear independent solution to Eq. (4) can be chosen to be $F_{1,5}(1-\eta)$, which is actually a linear combination [51] of all the F's. Coming back to the connectivities we observe that in the limit $w_1 \to w_2$, $P_{(14)(23)}$ contains configurations where two FK clusters are separated by a dual line. These configurations are realized by the insertion of the operator $\phi_{1,5}$ [52] at the boundary and it was argued in Ref. [41] that in this case logarithmic singularities should arise. We conjecture then the following identification for the universal probability ratio, which we denote with $R(\eta)$

$$R(\eta) = \frac{P_{(14)(23)}(\eta)}{P_t(\eta)} = A\int_0^\eta d\eta' f(\eta'), \tag{6}$$

where the constant $A = [\int_0^1 d\eta f(\eta)]^{-1}$ is chosen to ensure that $R(1) = 1$. The conjecture (6) can be easily tested on an arbitrary geometry by applying a conformal mapping $z'(z)$. Since all the dimensionful parameters in Eq. (3) cancel when computing Eq. (6), one has only to express η in the new coordinates z'. Finally, we observe that denoting $1-\eta = \varepsilon$ one obtains [48] the small ε expansion for the ratio in Eq. (6):

$$R = 1 - \varepsilon^{1/2}[a_0 + a_1\varepsilon + a_2\varepsilon^2(1 + b\log\varepsilon) + O(\varepsilon^3)]. \tag{7}$$

The logarithmic singularity arises from the mixing of the level two descendants of $\phi_{1,3}$ with the field $\phi_{1,5}$ that have at $c = 1/2$ the same conformal dimension $h_{1,5} = 5/2$. This is the first example where a logarithmic singularity is explicitly calculated in the context of the critical Ising model. The logarithmic behavior in Eq. (7) has a completely different origin with respect to the well-known logarithmic divergence of the specific heat at the critical temperature [39,40]. It shows that the phenomenon of mixing of scaling fields and nondiagonalizability of the conformal dilation operator could arise potentially at any rational value of the central charge—a circumstance that was already recognized in Refs. [41,42] and Refs. [38,53–55], but for which in the Ising model no explicit result was available. In Refs. [56,57], a possible source of logarithmic behavior was also identified but appeared to be ruled out by numerical data.

Numerical results.—Simulations have been carried out on the Ising model at the exactly known critical temperature on a triangular lattice in triangles of sides of lengths $L = 9$, 17, 33, 65, 129, and 257 with open boundary conditions collecting a number of samples up 10^{10}. The random number generator employed is given in Ref. [58]. We implemented the efficient Swendsen-Wang algorithm [59] that provides direct access to the FK clusters [44].

In order to use our results for the upper half plane (6) in the triangle geometry a Schwarz-Christoffel is in order. Given a z in $\mathbb{H}$ and a z' belonging to the interior of an equilateral triangle with vertices $(-1, 1, i\sqrt{3})$ the mapping

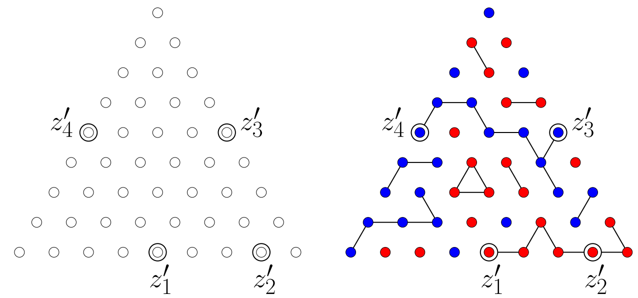

FIG. 3. (Left) Triangular lattice ($L = 9$) with the four points z_1', z_2', z_3', and z_4' highlighted. (Right) A realization of FK clusters contributing to the probability $P_{(12)(34)}$.

reads $z' = [2z\Gamma(\frac{5}{6})_2F_1(\frac{1}{2},\frac{2}{3};\frac{3}{2};z^2)/\sqrt{\pi}\Gamma(\frac{1}{3})]$, $_2F_1$ being the Gauss hypergeometric function. In the simulations the three points z_1', z_3', and z_4' have been fixed in the midpoint of each side, while the point z_2' takes any position on the boundary between z_1' and z_3'. Since the problem is symmetric under rotation of $2\pi/3$ and $4\pi/3$ around the center of the triangle, also the configurations obtained with these rotations have been measured to enhance the statistics. An example of the simulated system together with a realization of FK clusters is presented in Fig. 3. The ratios $P_{(12)(34)}/P_t$, $P_{(14)(23)}/P_t$, $P_{(1234)}/P_t$, because of the symmetry $P_{(12)(34)}(\eta) = P_{(14)(23)}(1-\eta)$ are not independent and only one function suffices to specify all of them, that is $R(\eta)$ as defined in Eq. (6). In Fig. 4 we show the simulation results together with the CFT prediction for R for the four largest lattices. In the inset of Fig. 4 we show the deviations from the exact result.

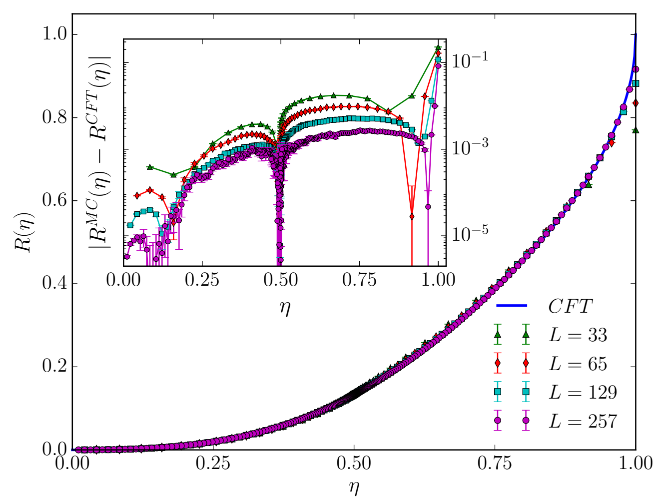

FIG. 4. Universal ratio $R(\eta)$ (6) for the lattice sizes $L = 33$, 65, 129, and 257 denoted by triangles, diamonds, squares, and circles, respectively. Errors are smaller than the symbol size. The CFT prediction is plotted with the continuous line. In the inset deviations of MC data from the theory are shown with the same symbols used in the main figure, lines are just guides to the eyes.

Conclusion.—In this Letter we have calculated the four point connectivities of FK clusters in the critical Ising model and show that they can display logarithmic branch cuts. This is a first explicit example where such a type of singularities are determined exactly for a theory that also has a nontrivial sector belonging to the series of unitary minimal models. Previous exact CFT studies focused on percolation and SAW [30,36–38]. Our findings are fully corroborated by numerical simulations, showing excellent agreement. Similar structures, expected in many other important two-dimensional models including critical percolation, fully deserve the attention of future investigations. It would be also of clear interest to analyze whether logarithmic singularities could be found in higher dimensions [60,61], for instance, in the three dimensional critical Ising model.

G. G. thanks the International Institute of Physics of Natal for the kind hospitality. J. V. is grateful to J. Dubail, G. Mussardo, and R. Santachiara for a critical reading of the manuscript and many important and interesting observations. We thank also D. Bernard and A. Cappelli for correspondence and, in particular, J. Cardy for pointing out his result in Ref. [41] about logarithmic observables in the Ising model.

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[1] A. Belavin, A. Polyakov, and A. Zamolodchikov, Nucl. Phys. **B241**, 333 (1984).
[2] F. D. H. Haldane, J. Phys. C **14**, 2585 (1981).
[3] G. Moore and N. Read, Nucl. Phys. **B360**, 362 (1991).
[4] C. Holzhey, F. Larsen, and F. Wilczek, Nucl. Phys. **B424**, 443 (1994); P. Calabrese and J. Cardy, J. Stat. Mech. (2004) P06002.
[5] A. Polyakov, Phys. Lett. **103B**, 207 (1981).
[6] J. Cardy, J. Phys. A **25**, L201 (1992).
[7] H. W. J. Blote, J. L. Cardy, and M. P. Nightingale, Phys. Rev. Lett. **56**, 742 (1986); I. Affleck, Phys. Rev. Lett. **56**, 746 (1986).
[8] M. Bauer and D. Bernard, Phys. Rep. **432**, 115 (2006).
[9] J. Cardy, Ann. Phys. (Amsterdam) **318**, 81 (2005).
[10] V. Gurarie, Nucl. Phys. **B410**, 535 (1993).
[11] L. Rozansky and H. Saleur, Nucl. Phys. **B376**, 461 (1992).
[12] J. S. Caux, I. Kogan, and A. Tsvelik, Nucl. Phys. **B466**, 444 (1996).
[13] J. Cardy, arXiv:cond-mat/9911024.
[14] V. Gurarie and A. Ludwig, J. Phys. A **35**, L377 (2002).
[15] I. Gruzberg, A. Ludwig, and N. Read, Phys. Rev. Lett. **82**, 4524 (1999).
[16] N. Read and H. Saleur, Nucl. Phys. **B613**, 409 (2001).
[17] T. Creutzig and D. Ridout, J. Phys. A **46**, 494006 (2013).
[18] P. Pearce, J. Rasmussen, and J-B. Zuber, J. Stat. Mech. (2006) P11017.
[19] N. Read and H. Saleur, Nucl. Phys. **B777**, 263 (2007); **B777**, 316 (2007).
[20] J. Dubail, J. Jacobsen, and H. Saleur, Nucl. Phys. **B834**, 399 (2010).
[21] G. Watts, J. Phys. A **29**, L363 (1996).
[22] J. Simmons, J. Phys. A **46**, 494015 (2013).
[23] S. Flores, J. Simmons, P. Kleban, and R. Ziff, J. Phys. A **50**, 064005 (2017).
[24] G. Delfino and J. Viti, J. Phys. A **44**, 032001 (2011); R. Santachiara, M. Picco, J. Viti, and G. Delfino, Nucl. Phys. **B875**, 719 (2013).
[25] Al. Zamolodchikov, Theor. Math. Phys. **142**, 183 (2005).
[26] Y. Ikhlef, J. L. Jacobsen, and H. Saleur, Phys. Rev. Lett. **116**, 130601 (2016).
[27] M. Picco, S. Ribault, and R. Santachiara, SciPost Phys. **1**, 009 (2016).
[28] S. Ribault and R. Santachiara, J. High Energy Phys. 08 (2015) 109.
[29] R. Vasseur, J. Jacobsen, and H. Saleur, J. Stat. Mech. (2012) L07001.
[30] S. Flores, P. Kleban, and J. Simmons, arXiv:1505.07756.
[31] H. Saluer, Nucl. Phys. **B382**, 486 (1992).
[32] M. Gaberdiel and H. G. Kausch, Nucl. Phys. **B538**, 631 (1999).
[33] H. Kausch, Nucl. Phys. **B583**, 513 (2000).
[34] J. Jacobsen, P. Le Doussal, M. Picco, R. Santachiara, and K. Wise, Phys. Rev. Lett. **102**, 070601 (2009).
[35] R. Vasseur, Phys. Rev. B **92**, 014205 (2015).
[36] S. Flores and P. Kleban, Commun. Math. Phys. **333**, 389 (2015); **333**, 435 (2015).
[37] S. Flores and P. Kleban, Commun. Math. Phys. **333**, 597 (2015).
[38] S. Flores and P. Kleban, Commun. Math. Phys. **333**, 669 (2015).
[39] G. Delfino, J. Phys. A **37**, R45 (2004).
[40] G. Mussardo, *Statistical Field Theory* (Oxford University Press, New York, 2010), ISBN: .
[41] J. Cardy, J. Phys. A **46**, 494001 (2013).
[42] R. Vasseur and J. Jacobsen, Nucl. Phys. **B880**, 435 (2014).
[43] F. Y. Wu, Rev. Mod. Phys. **54**, 235 (1982).
[44] P. W. Kasteleyn and C. M. Fortuin, J. Phys. Soc. Jpn. **26**, 11 (1969); C. M. Fortuin and P. W. Kasteleyn, Physica (Amsterdam) **57**, 536 (1972).
[45] G. Delfino and J. Viti, Nucl. Phys. **B852**, 149 (2011).
[46] J. Cardy, Nucl. Phys. **B324**, 581 (1989).
[47] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory* (Springer, New York, 1997), ISBN: .
[48] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.119.191601 for more information on the solution of the differential equation (4).
[49] Invariant under the exchange of η with $1-\eta$, that corresponds to exchange the points x_1 with x_3.
[50] E. Imamoglu and M. van Hoeij, arXiv:1606.01576.
[51] According to Frobenius ODE theory, the function $F_{1,3}(\eta)$ is logarithmic, therefore, $F_{1,5}(1-\eta)$ also has a logarithmic singularity when $\eta \to 0$ as we will see.
[52] J. Cardy, J. Phys. A **31**, 5 (1997).
[53] R. Vasseur, J. Jacobsen, and H. Saleur, Nucl. Phys. **B851**, 314 (2011).
[54] A. Gainutdinov and R. Vassuer, Nucl. Phys. **B868**, 223 (2013).
[55] R. Santachiara and J. Viti, Nucl. Phys. **B882**, 229 (2014).

[56] E. Laplame and Y. Saint-Aubin, J. Phys. A **34**, 1825 (2001).

[57] L-P. Arguin and Y. Saint-Aubin, Phys. Lett. B **541**, 384 (2002).

[58] M. Matsumoto and T. Nishimura, ACM Trans. Model. Comput. Simul. **8**, 3 (1998).

[59] R. H. Swendsen and J.-S. Wang, Phys. Rev. Lett. **58**, 86 (1987).

[60] R. Rattazzi, V. Rychkov, E. Tonni, and A. Vichi, J. High Energy Phys. 12 (2008) 031.

[61] M. Hogervorst, M. Paulos, and A. Vichi, arXiv:1605.03959.