Giant Kovacs-Like Memory Effect for Active Particles

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Dynamical properties of self-propelled particles obeying a bounded confidence rule are investigated by means of kinetic theory and agent-based simulations. While memory effects are observed in disordered systems, we show that they also occur in active matter systems. In particular, we find that the system exhibits a giant Kovacs-like memory effect that is much larger than predicted by a generic linear theory. Based on a separation of time scales we develop a nonlinear theory to explain this effect. We apply this theory to driven granular gases and propose further applications to spin glasses.

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Collective behavior is a crucial aspect in the rapidly growing field of active matter [1–4]. Active particles under consideration can be animals like, e.g., insects, fish, or birds [5], interacting robots [6], or microscopic objects like, e.g., bacteria [7], nanodimers [8], or Janus particles [9]. There is shared belief, that, on a macroscopic level, active particle systems can be described by a minimal set of hydrodynamic fields [10]. There has been a large emphasis on deriving field equations using different approaches [10–24]. For polar particles with ferromagnetic alignment interactions the description by Toner and Tu's seminal equations [25,26] is well established. Steady states of homogeneous solutions have also been studied in detail [27–29]. However, memory effects and dynamical properties of active matter far from its steady state are largely unexplored.

To study nonstationary properties and possible history dependencies we consider the following prototype situation: imagine a substance that, at time $t = 0$, is in equilibrium with a heat bath of temperature T_1, and assume that the temperature of the heat bath could be changed instantaneously. Suppose we want to heat the substance to a higher temperature $T_f > T_1$. This could be achieved by simply adjusting the heat bath to the desired final temperature T_f. After a certain time, the substance will have relaxed to equilibrium at T_f. Trying to speed up this process one could initially set the heat bath to a higher temperature $T_2 > T_f$ and switch the temperature of the heat bath to T_f after a particular waiting time t_w. Because the amount of heat transferred to the system until time $t = t_w$ is increased by this protocol, we intuitively expect the system to reach the desired temperature faster. This procedure is related to a measuring protocol introduced by Kovacs *et al.* [30,31]. In addition to the aforementioned steps they chose the waiting time t_w in a particular way. Considering an observable Ψ, the waiting time was chosen such that at the moment of temperature switching t_w, the observable Ψ has the same value that it has in equilibrium at temperature T_f. Thus, $\Psi(t_w) = \Psi(t \to \infty)$. In the case of Refs. [30,31], Ψ is the volume and here it is the polar order parameter.

If the order parameter $\Psi(t)$ enslaved all other degrees of freedom, the system would be in steady state already at $t = t_w$ and remain unchanged for the rest of the experiment. If, however, at $t = t_w$ there are additional degrees of freedom that are not perfectly enslaved, they will couple to the observable $\Psi(t)$ such that it first departs from and later relaxes back towards its steady state value. This is known as the Kovacs effect. In recent years the Kovacs temperature protocol was applied, e.g., on glass systems [32–35], and molecular liquids [36], but also on nonequilibrium systems like driven granular gases [37–39]. In the latter case the role of the temperature is replaced by a driving force.

In this Letter, we provide one particular example of a polar active gas that exhibits a giant and long lasting Kovacs effect. To the best of our knowledge, the Kovacs effect has not yet been reported for a system of active particles. The effect is seen both in kinetic theory and in agent-based simulations, quantitatively agreeing with each other. The presence of such memory effects proves the existence and relevance of hidden degrees of freedom that are crucial for the dynamics of the system. Hence, one needs to be careful applying Toner-Tu theories [13,25,26,40–42], where usually only particle and momentum density are considered and all other degrees of freedom are assumed to be enslaved. Such an enslavement is clearly not occurring in the present system. The Kovacs hump is so large that it contradicts a linear theory on the Kovacs effect [43]. In fact, under the Kovacs protocol, the system relaxes more slowly towards its final steady state than it is doing by direct relaxation. This is highly counterintuitive because the intermediate step should bring the system closer to its final steady state.

As our central result we develop a nonlinear theory, given by Eq. (7), that agrees quantitatively with and explains the occurrence of the observed giant Kovacs effect. The applicability of the theory exceeds the present active matter system by far. We demonstrate this versatility

of the theory by applying it to a driven granular gas and propose further applications to disordered systems such as spin glasses.

Here, we consider a two-dimensional Vicsek-like model for self-propelled particles with bounded confidence interactions, introduced in Ref. [27]. Bounded confidence interactions have been studied first in the context of opinion formation models [44,45] to mimic the tendency to ignore others with opposite opinions. Another swarming model of this type was studied in Ref. [46]. One motivation for such interactions are experiments on the collective motion of *Bacillus subtilis* [47], where the investigators propose weaker interactions between cells with increasing orientation difference. In the Vicsek model, all particles move at constant speed in individual directions. For a period of unit time they evolve ballistically and afterwards they interact instantaneously. First, each particle adopts the mean direction of motion of all particles that are no further away than some interaction distance and that move in a direction that differs by no more than the angle α from the particle's own direction. Note that in that way each particle interacts at least with itself. Then, all particle directions are disturbed by a random deviation ξ that is drawn for each particle independently from the interval $[-\eta/2, \eta/2]$. The regular Vicsek model is recovered for $\alpha = \pi$. However, in this case the Kovacs effect appears to be negligibly small.

Considering only spatially homogeneous solutions, one obtains a time evolution equation for the angular distribution of the particle directions, which can be simplified further by assuming molecular chaos and a low particle density M. Employing the Fourier ansatz $p(\theta) = \sum_{k=0}^{\infty} x_k \cos(k\theta)$, the time evolution is given explicitly by a hierarchy of equations [27,29]

$$x_k(t+1) = \lambda_k \left(A_k x_k + \sum_{q=1}^{\infty} x_q [B_{kq} x_{|k-q|} + C_{kq} x_{k+q}] \right) \quad (1)$$

with $x_0 = (1/2\pi)$ at all times. The coefficients λ_k, A_k, B_{kq}, and C_{kq} can be found in the Supplemental Material [48]. For practical computations we have to truncate the Fourier series after finitely many terms, setting $x_k = 0$ for $k > n$. Here, we use $n = 100$ or $n = 200$. We denote the vector of Fourier modes $(x_1, \ldots, x_n)$ by $\mathbf{x}$. As an observable we consider the polar order parameter $\Psi := \langle \cos(\theta) \rangle = \pi x_1$. If the system is perfectly ordered, all particles move in the same direction and $\Psi = 1$. If, in contrast, the directions of all particles are completely random, we have $\Psi = 0$.

Linearizing in the change of temperature, which corresponds to a change in η in our case, the following relation between the Kovacs hump $\Psi(t)$ and the relaxation curve Ψ_{1f} (obtained after switching from η_1 to η_f) was derived in Ref. [43] by means of a master-equation approach:

$$\Psi(t) = \frac{1}{1-\gamma} \Psi_{1f}(t) - \frac{\gamma}{1-\gamma} \Psi_{1f}(t - t_w), \quad (2)$$

$$\gamma = \frac{\Psi_{1f}(t_w) - \Psi^*_{\eta_f}}{\Psi_{1f}(0) - \Psi^*_{\eta_f}}. \quad (3)$$

By $\Psi_{1f}(t)$ we denote the order parameter when the system is prepared at $t = 0$ in the steady state of noise strength η_1 and for $t > 0$ the noise strength is switched to η_f. The steady state value of the order parameter at noise strength η is denoted by Ψ^*_η. As long as the relaxation curve Ψ_{1f} is monotonic, it follows from Eq. (3) that $\gamma \in (0, 1)$, and hence

$$\Psi(t) \begin{cases} > \Psi_{1f}, & \text{if } \Psi_{1f} \text{ is increasing,} \\ < \Psi_{1f}, & \text{if } \Psi_{1f} \text{ is decreasing.} \end{cases} \quad (4)$$

We evaluated the time evolution (1) numerically and changed the noise strength according to the Kovacs protocol. The results are displayed in Fig. 1. In the same plot we also present results of agent-based simulations, where the positions z_j and directions θ_j of the jth particle are updated according to the ballistic motion,

$$z_j(t+1) = z_j(t) + v_0 \begin{pmatrix} \sin(\theta_j) \\ \cos(\theta_j) \end{pmatrix} \quad (5)$$

with particle speed v_0, and the collision rule

$$\tilde{\theta}_j = \Phi_j + \xi_j, \Phi_j = \arg \left[\sum_{l \in \{j\}} \exp(i\theta_l) \right], \quad (6)$$

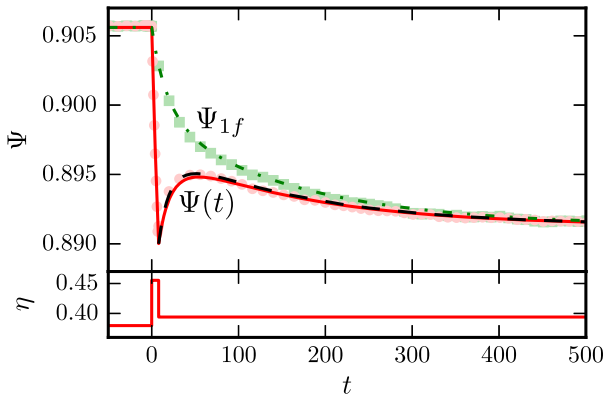

FIG. 1. The relaxation curve $\Psi_{1f}(t)$ (dash-dotted line) and the Kovacs hump $\Psi(t)$ (solid red line) obtained from the kinetic theory (1) are in good agreement with agent-based simulations (light green squares and light red circles, respectively). The linear theory (2) (dashed black line) describes the Kovacs effect well for short $t_w = 8$. System parameters are $\alpha = 0.47\pi$, $M = 0.2$, $\eta_1 = 0.3797$, $\eta_2 = 0.4553$, $\eta_f = 0.3940$ [49]. The lower part shows the noise strength according to the Kovacs protocol.

where $\tilde{\theta}_j$ denotes the postcollisional direction of the jth particle and ξ_j are independent random variables that are drawn uniformly from the interval $[-\eta/2, \eta/2]$. The set $\{j\}$ contains the indices of all particles that interact with the jth particle according to the bounded confidence rule. There is good agreement between agent-based simulations and kinetic theory (1). We clearly find that the Kovacs effect is present in this system. In Fig. 1 where $\alpha = 0.47\pi$ and the waiting time $t_w = 8$ is small, the linear theory (2) agrees well with the data.

However, for a different parameter set, in particular, a smaller $\alpha = 0.35\pi$, and for the inverse Kovacs protocol with $\eta_1 > \eta_f > \eta_2$, displayed in Fig. 2, we recognize that the Kovacs hump $\Psi(t)$ and the relaxation curve $\Psi_{1f}(t)$ intersect. That means the relaxation towards the final steady state under the Kovacs protocol is slower than the direct relaxation from η_1 to η_f. This surprising result clearly violates condition (4), which is a consequence of the linear theory (2). The Kovacs hump is giant compared to the predictions of the linear theory (dashed black line in Fig. 2). In the Supplemental Material [48] we rederive Eq. (2) for time discrete dynamical systems. In contrast to Ref. [43], in our derivation we do not need to assume that the steady state is Boltzmann distributed. We argue that this linearized theory is only applicable if the waiting time is short. Thus, in the present case, the linear theory is not sufficient to describe the system, and nonlinear effects are crucial.

Under assumptions specified below, we develop a nonlinear theory finding that the vector of Fourier modes $\mathbf{x}(t)$ obtained from the Kovacs protocol for $t \geq t_w$ is related to both relaxation curves $\mathbf{x}_{1f}$ (switch from η_1 to η_f) and $\mathbf{x}_{2f}$ (switch from η_2 to η_f) by the following central result

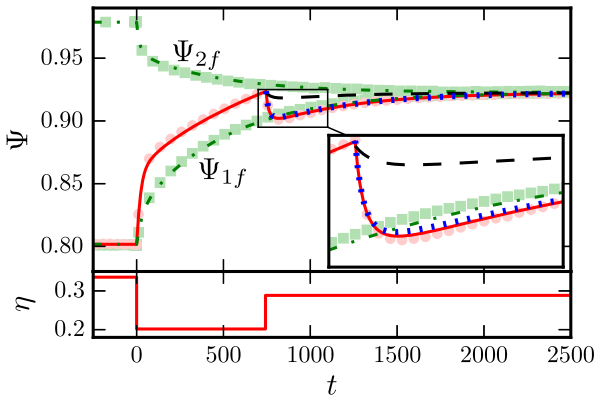

FIG. 2. The relaxation curves $\Psi_{1f}(t)$, $\Psi_{2f}(t)$ (dash-dotted line) and the Kovacs hump $\Psi(t)$ (solid red line) obtained from the kinetic theory (1) are in good agreement with agent-based simulations (light green squares and light red circles, respectively). The linear theory (2) (dashed black line) fails to describe the Kovacs hump whereas the nonlinear theory (7) with time shift $\hat{t} = 332$ (dotted blue line) agrees well. Parameters are $\alpha = 0.35\pi$, $M = 0.2, t_w = 745, \eta_1 = 0.3354, \eta_2 = 0.2017, \eta_f = 0.2884$ [49].

$$\mathbf{x}(t) = \mathbf{x}_{2f}(t - t_w) + \mathbf{x}_{1f}(t - \hat{t}) - \mathbf{x}_{2f}(t - \hat{t}). \quad (7)$$

We derive this equation not by linearizing the time evolution (1) itself but only the change of the system's state after t_w. Furthermore, we do not linearize in the change of the noise strength but we take a strong nonlinear dependence of the relaxation speed on the noise strength into account, see the Supplemental Material [48] for details. The first vector component of Eq. (7) yields the corresponding relation for the order parameter $\Psi = \pi x_1$ but Eq. (7) is more general, yielding a relation for all Fourier modes. The time shift $\hat{t}$ in Eq. (7) depends on the change of the relaxation speed when η is switched (see Ref. [48] for details). The right-hand side of Eq. (7) is displayed as the blue dotted line in Fig. 2. It coincides well with the Kovacs hump. We discuss a plot of the second Fourier mode of the same process in Ref. [48].

With Eq. (7) we can understand why the Kovacs effect is so large. We find that the derivative of $\Psi(t)$ at $t = t_w$ is given by $\Psi'_{2f}(0) + \Psi'_{1f}(t_w - \hat{t}) - \Psi'_{2f}(t_w - \hat{t})$. Here, the relaxation curve Ψ_{2f} is decaying very fast in the beginning and both relaxation curves are decaying much more slowly at later times, cf. Fig. 2. Since $|\Psi'_{1f}(t)|$ and $|\Psi'_{2f}(t)|$ are of comparable size, the term $\Psi'_{2f}(0)$ is dominant. That means a fast relaxation of Ψ_{2f} at $t = 0$ leads to a fast and therefore strong change of $\Psi(t)$ in a short period after $t = t_w$. Thus, whenever $|\Psi'_{1f}| \sim |\Psi'_{2f}|$ and Ψ_{2f} is relaxing fast in the beginning and much more slowly at later times we expect a giant Kovacs effect.

The derivation of Eq. (7) is not system specific but it is valid for a wide class of systems with the following two properties. (i) In the relaxation dynamics there must be a separation of time scales. In particular, there has to be one mode that relaxes much more slowly than all others such that at t_w we can assume that all modes but one are already completely relaxed. (ii) Furthermore, we need to assume that this slow mode is not too sensitive to changes in η and t such that the slowest relaxation mode for η_2 is approximately equal to the one at η_f.

For the present system these properties are numerically verified in Ref. [48]. They can also be understood intuitively. Assume that a large population of particles moves into direction $\theta = 0$. Then, they interact only with others that move in a direction from the interval $[-\alpha, \alpha]$. For $\alpha = 0.35\pi$ this interval is a little larger than $2\pi/3$. Particles that have directions outside this interval can not interact with the first population. Therefore, it is possible that there is a second, relatively large, population of particles that moves into the opposite direction $\theta = \pi$. Then, the interaction intervals of both populations are disjoint and all particles can interact only with either the first or the second population. Particles can be driven from one population to the other only by noise. For small noise strength this process is very slow, in particular, because the interaction of particles within the

same population acts against this noise driven mechanism. In contrast, the concentration of particles that belong to one population due to aligning interactions is a much faster process.

Thus, we identified a slow relaxation process in the system: the noise driven reorientation of a group of particles moving in the opposite direction from that of the majority. This relaxation process is robust against moderate changes in noise strength. Although the relaxation speed depends strongly on the noise strength, the mechanism remains the slowest dynamics as long as the noise is not too strong. In the derivation of Eq. (7) we neither use that the initial configuration is a steady state nor that the order parameter $\Psi(t_w) = \Psi(t \to \infty)$ (see Ref. [48]). Thus, Eq. (7) is applicable to more general protocols than is the Kovacs experiment and it might be valid for completely different systems with slow dynamics like, e.g., driven granular gases, spin glasses, or other disordered systems [50]. Furthermore, it might be applied to bounded confidence opinion formation models with random influence, as there is also a separation of time scales between the fast formation of opinion clusters and their slow unification towards consensus, see, e.g., Refs. [51–53]. The applicability can be verified theoretically or experimentally by measuring the Kovacs hump as well as the relaxation curves Ψ_{1f} and Ψ_{2f} in different types of systems. The time shift $\hat{t}$ might be obtained from theoretical calculations specific to the system or simply from a fit of experimental or simulation data.

For one specific granular gas we verify Eq. (7) explicitly. In Refs. [37,39] the dynamics of a driven granular gas is given by differential equations, Eqs. (6a) and (6b) in Ref. [37]. Evaluating these equations numerically we obtain the Kovacs hump $\Psi(t)$ as well as the relaxation curves $\Psi_{1f}(t)$ and $\Psi_{2f}(t)$, such that we can test the validity of Eq. (7). In order to satisfy conditions (i) and (ii) we must choose intermediate and final noise intensities close to each other. In that way we assure that the waiting time t_w is long and that the slowest relaxation mode can be approximated as constant. In Fig. 3 we show the anomalous Kovacs effect for the driven granular gas of Refs. [37,39]. We find that also the anomalous effect is described well by Eq. (7) whereas the linear theory (2) fails completely. We can understand why the Kovacs hump has the opposite sign by investigating the derivative of Eq. (7). Here, $|\Psi'_{1f}| \gg |\Psi'_{2f}|$ and hence the term $\Psi'_{1f}(t - \hat{t})$ is dominant, causing the anomalous effect.

For spin glasses, property (i) is not realistic but it can be replaced by an alternative assumption. Consider the coherence length l of a spin glass such that length scales smaller than l are equilibrated and length scales larger than l are frozen. The coherence length depends on the age of the system. Turning it around, one finds that the age of the system typically grows faster than exponentially with the coherence length [32,54]. Translating this picture into

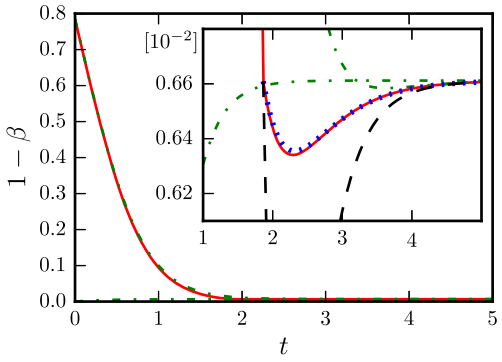

FIG. 3. Anomalous Kovacs effect for a driven granular gas. The order parameter β is related to the granular temperature. Displayed are relaxation curves (green dash-dotted lines) and the Kovacs hump (red solid line). The Kovacs effect is very small and shown in the inset in units of 10^{-2}. The nonlinear theory (blue dotted line), Eq. (7) with $\hat{t} = 0$, describes the anomalous effect well; the linear theory (black dashed line), Eq. (2), fails. Details on the data are given in Ref. [48].

the framework of relaxing eigenmodes we assume the following property: (i*) after the waiting time t_w some eigenmodes have relaxed completely, some others have not relaxed at all, and there is a single mode that is partly relaxed. If property (i*) holds instead of (i), the derivation of Eq. (7) remains valid. The validity of Eq. (7) for spin glasses could be verified in the future. However, steady states might be difficult to prepare both in experiments and in simulations. Fortunately, as argued above, Eq. (7) also holds when the initial state is not a steady state. Thus, $\mathbf{x}_{1f}$ can be the relaxation curve from any (possibly nonstationary) state towards the steady state of η_f. Note, that the measurement of the relaxation curve $\mathbf{x}_{2f}$ requires the preparation of the steady state belonging to $\eta = \eta_2$. However, Eq. (7) yields still insights on the system even if $\mathbf{x}_{2f}$ can not be measured.

The time shift $\hat{t}$ in Eq. (7) does not depend on the initial configuration but only on the intermediate and final noise strength and the waiting time t_w. Hence, rewriting Eq. (7) as

$$\mathbf{x}(t) - \mathbf{x}_{1f}(t - \hat{t}) = \mathbf{x}_{2f}(t - t_w) - \mathbf{x}_{2f}(t - \hat{t}) \qquad (8)$$

we find that the right-hand side is independent of the initial configuration. Therefore, also the left-hand side must be identical for different initial configurations. Even if steady states cannot be prepared in an experimental or simulation setup, the quantities on the left-hand side of Eq. (8) can still be measured. Comparing them for different initial configurations Eq. (8) can be verified and one could extract information on the relaxation from one steady state to another without preparing steady states.

In summary, we observe a giant, long lasting Kovacs memory effect in a bounded confidence active matter system, both in agent-based simulations and in kinetic theory. The effect is unexpected as it contradicts a well-known linear

theory. The giant memory effect proves the presence and relevance of hidden degrees of freedom that are not incorporated in the Toner-Tu equations. We develop a quantitative nonlinear theory that connects the Kovacs hump with two relaxation curves with applicability beyond active matter. We apply the nonlinear theory to a driven granular gas where it succeeds to quantitatively describe the anomalous Kovacs effect. We further propose experimental validations and applications to spin glasses and other disordered systems, as well as opinion formation models.

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