Infrared Quantum Information

Daniel Carney, Laurent Chaurette, Dominik Neuenfeld, and Gordon Walter Semenoff
Department of Physics and Astronomy University of British Columbia, 6224 Agricultural Road, Vancouver, British Columbia V6T 1Z1, Canada
(Received 14 June 2017; published 30 October 2017)

We discuss information-theoretic properties of low-energy photons and gravitons in the S matrix. Given an incoming n-particle momentum eigenstate, we demonstrate that unobserved soft photons decohere nearly all outgoing momentum superpositions of charged particles, while the universality of gravity implies that soft gravitons decohere nearly all outgoing momentum superpositions of *all* the hard particles. Using this decoherence, we compute the entanglement entropy of the soft bosons and show that it is infrared-finite when the leading divergences are resummed in the manner of Bloch and Nordsieck.

DOI: 10.1103/PhysRevLett.119.180502

Introduction.—The massless nature of photons and gravitons leads to an infrared catastrophe, in which the S matrix becomes ill defined due to divergences coming from low-energy virtual bosons. The usual solution to this problem, originally given by Bloch and Nordsieck in electrodynamics [1] and extended to gravity by Weinberg [2], is to argue that an infinite number of low-energy bosons are radiated away during a scattering event; this leads to divergences which cancel the divergences from the virtual states, and physical predictions in terms of infrared-finite inclusive transition probabilities.

In this Letter, we study quantum information-theoretic aspects of this proposal. Since each photon and graviton has two polarization states and three momentum degrees of freedom, one might suspect that the low-energy radiation produced during scattering could carry a huge amount of information. Here we demonstrate that, according to the methodology of [1–3], if the initial state is an incoming n-particle momentum eigenstate, the "soft" bosonic divergences can lead to complete decoherence of the momentum state of the outgoing "hard" particles. This decoherence is avoided only for superpositions of pairs of outgoing states for which an infinite set of angle-dependent currents match; see Eq. (11). In simple examples like QED, this will be enough to get complete decoherence of all momentum superpositions. In less simple cases, one is still left with an extremely sparse density matrix dominated by its diagonal elements.

Having traced the radiation in this fashion, we obtain an infrared-finite, mixed reduced density matrix for the hard particles. In the simple cases when we get a completely diagonal matrix, we compute the entanglement entropy carried by the soft gauge bosons. The answer is finite and scales like the logarithm of the energy resolution E of a hypothetical soft boson detector.

While the tracing out of the soft radiation can be viewed as a physical statement about the energy resolution of a real detector, in this formalism, the trace is also forced on us by mathematical consistency: it is the only way to get well-defined transition probabilities from the infrared-divergent S matrix. There is an alternative approach to the infrared catastrophe, in which one constructs an IR-finite S matrix of transition amplitudes between "dressed" matter states [4–7]. In such an approach, there are no divergences and so one is not forced to trace over any soft radiation. Whether the two formalisms lead to the same physical picture is an interesting question, and we leave a detailed comparison to future work.

Recently, the infrared structure of gauge theories has become a topic of much interest due to the proposal that soft radiation may encode information about the history of formation of a black hole [8–10]. We hope that our work can make this discussion more quantitatively grounded; we comment on black holes at the end of this Letter. More generally, it is of interest to understand the information-theoretic nature of the infrared sector of quantum field theories, and our paper is intended to make some first steps in this direction.

Decoherence of the hard particles.—Fix a single-particle energy resolution E. We define soft bosons as those with energy less than E, and hard particles as anything else. Consider an incoming state $|\alpha\rangle_{\rm in}$ consisting of hard particles, charged or otherwise, of definite momenta [11]. The S matrix evolves this into a coherent superposition of states with hard particles β and soft bosons $b = \gamma, h$ (photons γ and gravitons h),

$$|\alpha\rangle_{\rm in} = \sum_{\beta b} S_{\beta b,\alpha} |\beta b\rangle_{\rm out}. \tag{1}$$

Hereafter we drop the subscript on kets, which will always be out-states. Tracing out the bosons $|b\rangle$, the reduced density matrix for the outgoing hard particles is

$$\rho = \sum_{\beta\beta' b} S_{\beta b,\alpha} S^*_{\beta' b,\alpha} |\beta\rangle\langle\beta'|. \tag{2}$$

Using the usual soft factorization theorems [2,3,12], we can write the amplitudes in terms of the amplitudes for $\alpha \to \beta$ multiplied by soft factors, one for each boson:

$$S_{\beta b,\alpha} = S_{\beta,\alpha} F_{\beta,\alpha}(\gamma) G_{\beta,\alpha}(h), \tag{3}$$

where the soft factors F, G are

$$F_{\beta,\alpha}(\gamma) = \sum_{n\in\alpha,\beta} \sum_{\pm} \prod_{i\in\gamma} \frac{e_n \eta_n}{(2\pi)^{3/2} |\mathbf{k}_i|^{1/2}} \frac{p_n^\mu \epsilon^*_{\mu,\pm}(\mathbf{k}_i)}{p_n \cdot k_i - i\eta_n \epsilon},$$
$$G_{\beta,\alpha}(h) = \sum_{n\in\alpha,\beta} \sum_{\pm} \prod_{i\in h} \frac{M_p^{-1} \eta_n}{(2\pi)^{3/2} |\mathbf{k}_i|^{1/2}} \frac{p_n^\mu p_n^\nu \epsilon^*_{\mu\nu,\pm}(\mathbf{k}_i)}{p_n \cdot k_i - i\eta_n \epsilon}. \tag{4}$$

Here the index n runs over all the incoming and outgoing hard particles, i runs over the outgoing soft bosons; $\eta_n = -1$ for an incoming and $+1$ for an outgoing hard particle. The e_n are electric charges and $M_p = (8\pi G_N)^{-1/2}$ is the Planck mass, and the ϵ's are polarization vectors or tensors for outgoing soft photons and gravitons, respectively. By an argument identical to the one employed by Weinberg [2], and assuming we can neglect the total lost energy E_T compared to the energy of the hard particles, we can use this factorization to perform the sum over soft bosons in (2), and we find that

$$\sum_b S_{\beta b,\alpha} S^*_{\beta' b,\alpha} = S_{\beta,\alpha} S^*_{\beta',\alpha} \left(\frac{E}{\lambda}\right)^{\tilde{A}_{\beta\beta',\alpha}} \left(\frac{E}{\lambda}\right)^{\tilde{B}_{\beta\beta',\alpha}} \times f\left(\frac{E}{E_T}, \tilde{A}_{\beta\beta',\alpha}\right) f\left(\frac{E}{E_T}, \tilde{B}_{\beta\beta',\alpha}\right). \tag{5}$$

Here $\lambda \ll E$ is an infrared regulator used to cut off momentum integrals which we will send to zero later; one can think of λ as a mass for the photon and graviton. The exponents are

$$\tilde{A}_{\beta\beta',\alpha} = -\sum_{\substack{n\in\alpha,\beta \\ n'\in\alpha,\beta'}} \frac{e_n e_{n'} \eta_n \eta_{n'}}{8\pi^2} \beta_{nn'}^{-1} \ln\left(\frac{1+\beta_{nn'}}{1-\beta_{nn'}}\right),$$
$$\tilde{B}_{\beta\beta',\alpha} = \sum_{\substack{n\in\alpha,\beta \\ n'\in\alpha,\beta'}} \frac{m_n m_{n'} \eta_n \eta_{n'}}{16\pi^2 M_p^2} \frac{1+\beta_{nn'}^2}{\beta_{nn'}\sqrt{1-\beta_{nn'}^2}} \ln\left(\frac{1+\beta_{nn'}}{1-\beta_{nn'}}\right), \tag{6}$$

and f is a complicated function which can be found in [13]; for $E/E_T = \mathcal{O}(1)$ and for small A, f may be approximated as $f(1, A) \approx 1 - \pi^2 A^2/12 + \mathcal{O}(A^4)$. In these formulas, $\beta_{nn'}$ is the relative velocity between particles n and n',

$$\beta_{nn'} = \sqrt{1 - \frac{m_n^2 m_{n'}^2}{(p_n \cdot p_{n'})^2}}.$$

For future use, we note that $0 \le \beta \le 1$, and both of the dimensionless functions of β appearing in (6) run over $[2, \infty)$ as β runs from 0 to 1. We have $\beta_{nm} = 0$ if and only if $p_n = p_m$.

The divergences as $\lambda \to 0$ in (5) come from summing over an infinite number of radiated, on-shell bosons. There are also infrared divergences inherent to the transition amplitude $S_{\beta,\alpha}$ itself coming from virtual bosons. Again following Weinberg, we can add these divergences up, and we have that

$$S_{\beta,\alpha} = S^\Lambda_{\beta,\alpha} \left(\frac{\lambda}{\Lambda}\right)^{A_{\beta,\alpha}/2} \left(\frac{\lambda}{\Lambda}\right)^{B_{\beta,\alpha}/2}, \tag{7}$$

where now $S^\Lambda_{\beta,\alpha}$ means the amplitude computed using only virtual bosons of energy above Λ, and

$$A_{\beta,\alpha} = -\sum_{n,m\in\alpha,\beta} \frac{e_n e_m \eta_n \eta_m}{8\pi^2} \beta_{nm}^{-1} \ln\left[\frac{1+\beta_{nm}}{1-\beta_{nm}}\right],$$
$$B_{\beta,\alpha} = \sum_{n,m\in\alpha,\beta} \frac{m_n m_m \eta_n \eta_m}{16\pi^2 M_p^2} \frac{1+\beta_{nm}^2}{\beta_{nm}\sqrt{1-\beta_{nm}^2}} \ln\left[\frac{1+\beta_{nm}}{1-\beta_{nm}}\right]. \tag{8}$$

An infrared-divergent "Coulomb" phase is suppressed in (7). We will see shortly that this phase cancels out of all the relevant density matrix elements.

Putting the above results together, we find that the reduced density matrix coefficient for $|\beta\rangle\langle\beta'|$ is given by

$$\rho_{\beta\beta'} = S^\Lambda_{\beta,\alpha} S^{\Lambda *}_{\beta',\alpha} \left(\frac{E}{\lambda}\right)^{\tilde{A}_{\beta\beta',\alpha}} \left(\frac{\lambda}{\Lambda}\right)^{A_{\beta,\alpha}/2 + A_{\beta',\alpha}/2} \times \left(\frac{E}{\lambda}\right)^{\tilde{B}_{\beta\beta',\alpha}} \left(\frac{\lambda}{\Lambda}\right)^{B_{\beta,\alpha}/2 + B_{\beta',\alpha}/2} \times f\left(\frac{E}{E_T}, \tilde{A}_{\beta\beta',\alpha}\right) f\left(\frac{E}{E_T}, \tilde{B}_{\beta\beta',\alpha}\right). \tag{9}$$

The question is how this behaves in the limit that the infrared regulator $\lambda \to 0$. The coefficient scales as $\lambda^{\Delta A + \Delta B}$, where

$$\Delta A_{\beta\beta',\alpha} = \frac{A_{\beta,\alpha}}{2} + \frac{A_{\beta',\alpha}}{2} - \tilde{A}_{\beta\beta',\alpha},$$
$$\Delta B_{\beta\beta',\alpha} = \frac{B_{\beta,\alpha}}{2} + \frac{B_{\beta',\alpha}}{2} - \tilde{B}_{\beta\beta',\alpha}. \tag{10}$$

In the Supplemental Material [14], we prove that both of these exponents are positive-definite, $\Delta A_{\beta\beta',\alpha} \ge 0$ and $\Delta B_{\beta\beta',\alpha} \ge 0$. The density matrix components (9) which survive as the regulator $\lambda \to 0$ are those for which $\Delta A = \Delta B = 0$; all other density matrix elements will vanish.

To give necessary and sufficient conditions for $\Delta A = \Delta B = 0$, we define two currents for each spatial velocity vector $\mathbf{v}$. We assume for simplicity that only massive particles carry electric charge. For massive particles, there are electromagnetic and gravitational currents defined as

$$j_{\mathbf{v}}^{\mathrm{EM}} = \sum_i e^i a^{i\dagger}_{\mathbf{p}_i(\mathbf{v})} a^i_{\mathbf{p}_i(\mathbf{v})},$$
$$j_{\mathbf{v}}^{GR} = \sum_i E_i(\mathbf{v}) a^{i\dagger}_{\mathbf{p}_i(\mathbf{v})} a^i_{\mathbf{p}_i(\mathbf{v})}. \qquad (11)$$

Here i labels particle species, e^i their charges, and m^i their masses; the kinematic quantities $\mathbf{p}_i(\mathbf{v}) = m_i \mathbf{v}/\sqrt{1-\mathbf{v}^2}$ and $E_i(\mathbf{v}) = m_i/\sqrt{1-\mathbf{v}^2}$ are the momentum and energy of species i when it has velocity $\mathbf{v}$. For lightlike particles we have to separately define the gravitational current, since a velocity and species does not uniquely determine a momentum:

$$j_{\mathbf{v}}^{GR,m=0} = \sum_i \int_0^\infty d\omega \omega a^{i\dagger}_{\omega \mathbf{v}} a^i_{\omega \mathbf{v}}. \qquad (12)$$

Momentum eigenstates of any number of particles are obviously eigenstates of these currents and we denote their eigenvalues $j_{\mathbf{v}}|\alpha\rangle = j_{\mathbf{v}}(\alpha)|\alpha\rangle$. These currents are presumably related to the family of charges defined in [15], but the detailed relation is left to future work.

The photonic exponent $\Delta A_{\beta\beta',\alpha}$ is zero if and only if the charged currents in β are the same as those in β'; the gravitational exponent $\Delta B_{\beta\beta',\alpha}$ is zero if and only if *all* the hard gravitational currents in β are the same as those in β'. This is demonstrated in detail in the Supplemental Material [14]. For any such pair of outgoing states $|\beta\rangle$, $|\beta'\rangle$, (9) becomes independent of the IR regulator λ and is thus finite as $\lambda \to 0$,

$$\rho_{\beta\beta'} = S^{\Lambda *}_{\beta'\alpha} S^{\Lambda}_{\beta\alpha} \mathcal{F}_{\beta\beta',\alpha}(E, E_T, \Lambda), \qquad (13)$$

where

$$\mathcal{F}_{\beta\beta',\alpha} = f\left(\frac{E}{E_T}, \tilde{A}_{\beta\beta',\alpha}\right) f\left(\frac{E}{E_T}, \tilde{B}_{\beta\beta',\alpha}\right) \left(\frac{E}{\Lambda}\right)^{\tilde{A}_{\beta\beta',\alpha}+\tilde{B}_{\beta\beta',\alpha}}. \qquad (14)$$

As explained by Weinberg, rescaling Λ simply renormalizes $\rho_{\beta\beta'}$ [2]. For diagonal density matrix elements $\beta = \beta'$, we obtain the standard transition probabilities

$$\rho_{\beta\beta} = |S^{\Lambda}_{\beta\alpha}|^2 \mathcal{F}_{\beta\alpha}(E, E_T, \Lambda) \qquad (15)$$

with $\mathcal{F}_{\beta\alpha} = f(A_{\beta,\alpha}) f(B_{\beta,\alpha}) (E/\Lambda)^{A_{\beta,\alpha}+B_{\beta,\alpha}}$ [2]. On the other hand, if there is even a single $\mathbf{v}$ for which one of the currents (11) or (12) does not have the same eigenvalue in $|\beta\rangle$ and $|\beta'\rangle$, then the density matrix coefficient decays as $\lambda^{\Delta A + \Delta B} \to 0$ as the regulator $\lambda \to 0$. We see that the unobserved soft bosons have almost completely decohered the momentum state of the hard particles. Only a very sparse subset of superpositions in which all the $j_{\mathbf{v}}(\beta) = j_{\mathbf{v}}(\beta')$ survive.

Examples.—To get a feel for the results presented in the previous section, we consider a few examples. First, consider any scattering with a single incoming and outgoing charged particle, like potential or single Compton scattering. Let the incoming momentum be $\alpha = p$ and the outgoing momenta of the two branches $\beta = q$, $\beta' = q'$. We have either directly from the definition (10) or the theorem (1) in the supplement that

$$\Delta A_{qq',p} = -\frac{e^2}{8\pi^2}[2 - \gamma_{qq'}], \qquad (16)$$

where $\gamma_{qq'} = \beta_{qq'}^{-1} \ln(1+\beta_{qq'})/(1-\beta_{qq'})$. This ΔA is easily seen to equal zero if and only if $q = q'$. Thus other than the spin degree of freedom, the resulting density matrix for the charge is exactly diagonal in momentum space.

To see an example where the current-matching condition is nontrivially fulfilled, consider a theory with two charged particle species of charge e and $e/2$ and the same mass. Then we can get an outgoing superposition of a state $\beta = (e, q)$ and one with two half-charges $\beta' = (e/2, q_1') + (e/2, q_2')$. The differential exponent for such a superposition is

$$\Delta A_{\beta\beta',p} = -\frac{e^2}{8\pi^2}\left[3 + \frac{1}{2}\gamma_{q_1 q_2} - \gamma_{q q_1} - \gamma_{q q_2}\right], \qquad (17)$$

which is zero if $q = q_1 = q_2$. In other words, the currents (11) cannot distinguish between a full charge of momentum q and two half-charges of the same momentum.

Entropy of the soft bosons.—We have seen that the reduced density matrix for the outgoing hard particles is very nearly diagonal in the momentum basis. In a simple example like a theory with various scalar fields ϕ_i of different, nonzero masses m_i, the soft graviton emission causes *complete* decoherence into a diagonal momentum-space reduced density matrix for the hard particles. More generally, we may have a sparse set of superpositions, and in any case spin and other internal degrees of freedom are unaffected by the soft emission.

In a simple example with a purely diagonal reduced density matrix, it is straightforward to compute the entanglement entropy of the soft emitted bosons. The total hard + soft system is in a bipartite pure state, with the partition being between the hard particles and soft bosons, so the entanglement entropy of the bosons is the same as that of the hard particles. Following the calculation in [16–18], we can simply write down the entropy:

$$S = -\sum_\beta |S^{\Lambda}_{\beta\alpha}|^2 \mathcal{F}_{\beta\alpha} \ln[|S^{\Lambda}_{\beta\alpha}|^2 \mathcal{F}_{\beta\alpha}]. \qquad (18)$$

This sum is infrared-finite; again, $\mathcal{F}$ is given in (14), and the superscript Λ means the naive S matrix computed with virtual bosons only of energies greater than Λ. Given the explicit form of $\mathcal{F}$, we see that the entropy scales like the log of the infrared detector resolution E.

Discussion.—According to the solution of the infrared catastrophe advocated in [1–3], an infinite number of very low-energy photons and gravitons are produced during scattering events. We have shown that if taken seriously, considering this radiation as lost to the environment completely decoheres almost any momentum state of the outgoing hard particles. The basic idea is simple: the radiation is essentially classical, so any two scattering events are easy to distinguish by their radiation.

The physical content of this result is somewhat unclear. A conservative view is that the methodology of [1–3] is ill suited to finding outgoing density matrices. As remarked earlier, in this formalism, one *must* trace the radiation to get well-defined transition probabilities. An alternative would be to use the infrared-finite S-matrix program [4–7], in which no trace over radiation is needed at all. But then we need to understand where the physical low-energy radiation is within that formalism—since after all, a photon that is lost to the environment certainly does decohere the system.

The decoherence found here is for the momentum states of the particles: at lowest order in their momenta, soft bosons do not lead to decoherence of spin degrees of freedom. However, the subleading soft theorems [19–21] do involve the spin of the hard particles, so going to the next order in the soft particles would be interesting [22]. We would also like to understand to what extent our answers depend on the infinite-time approximation used in the S-matrix approach. Our calculation shows that the pointer basis for a scattering experiment in QED or gravity is the eigenbasis of the current operators (11). This is consistent with the fact that in theories with long-range forces, the asymptotic dynamics is not free, but rather controlled by classical currents [6]. We leave a detailed study of this issue to future work.

To end, we comment on potential applications to the black hole information paradox. The idea advocated in [8,9] is that correlations between the hard and soft particles mean that information about the black hole state can be encoded into soft radiation. In [10,23,24], the dressed-state formalism and soft factorization has been used to argue that the soft particles simply factor out of the S matrix and thus contain no such information. In the approach used here, it is manifest that the outgoing hard state and outgoing soft state are highly correlated, leading to the decoherence of the hard state. The outgoing density matrix for the hard particles, while not completely thermal, has been mixed in momentum as much as possible while retaining consistency with standard QED and perturbative gravity predictions.

We can make a crude order-of-magnitude estimate of the entropy carried by soft photons during the black hole formation and evaporation process. Consider an incoming pure state which has high enough center-of-mass energy $\sqrt{s}$ and small enough impact parameter that black hole formation is very likely. The black holes are only intermediate states; if we wait long enough, they will have evaporated into (hard) Hawking radiation. According to our results, then, tracing the final soft radiation will leave a final state $\rho_{\rm hard} = \sum_B P_B \mathcal{F}_{\mathcal{B}} |\mathcal{B}\rangle\langle\mathcal{B}| + \sum_{\mathcal{R}} \mathcal{P}_{\mathcal{R}} \mathcal{F}_{\mathcal{R}} |\mathcal{R}\rangle\langle\mathcal{R}|$. Here B means a state of Hawking radiation coming from a particular intermediate black hole state, R represents a branch where no black hole was ever formed, and the $\mathcal{F}$ are the soft factors (14). There is evidence that at high center-of-mass energies, black holes should have production cross sections given by their geometric areas $P_B \sim \pi r_h^2(\sqrt{s})$ and dominate the outoging states [25]. Using this in (18) and neglecting branches R without black holes, one obtains a hard-soft entanglement entropy scaling like the black hole area times logarithmic soft factors. In this sense one might view the soft radiation as containing a significant fraction of the black hole entropy [26].

We thank Scott Aaronson, Tim Cox, Colby Delisle, Bart Horn, Raphael Flauger, John Preskill, Philip Stamp, Bill Unruh, and Jordan Wilson for discussions. All of us are grateful for support from NSERC, and D. C. from the Templeton Foundation and the Pacific Institute of Theoretical Physics.

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[26] A similar proposal was given by Strominger [20]. Strominger's paper and discussions with a referee inspired the argument here.