Loop Corrections to Supergravity on $AdS_5 \times S^5$

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We consider the four-point correlator of the stress-energy tensor multiplet in $\mathcal{N}=4$ super Yang-Mills theory. In the planar limit and at large 't Hooft coupling, such a correlator is given by the corresponding holographic correlation function in IIB supergravity on $AdS_5 \times S^5$. We consider subleading corrections in the number of colors, i.e., order $1/N^4$, at large 't Hooft coupling. This corresponds to loop corrections to the supergravity result. Consistency conditions, most notably, crossing symmetry, constrain the form of such corrections and lead to a complete determination of the spectrum of leading-twist intermediate operators.

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Introduction.—The AdS/CFT correspondence relates four-dimensional $\mathcal{N}=4$ Super Yang-Mills (SYM) theory to type IIB string theory on $AdS_5 \times S^5$ [1–3]. Even after 20 years of its formulation and in spite of tremendous progress in many directions, nonprotected quantities have only been explored in certain corners of the parameter space. One particularly interesting corner corresponds to planar SYM theory with large t'Hooft coupling $\lambda = g_{\mathrm{YM}}^2 N$, dual to classical supergravity on the bulk. In this regime, single-trace chiral primary operators (CPOs) of weight p, $\mathcal{O}_p$ map to supergravity fields with mass $m^2 = p(p-4)$, and their correlation functions can, in principle, be computed by tree-level Witten diagrams on AdS:

$$\langle \mathcal{O} \ldots \mathcal{O} \rangle_{\mathrm{conn}} \sim \frac{1}{N^2} \quad \text{(tree-level Witten diagram)}. \quad (1)$$

Three-point correlators of arbitrary CPOs, as well as the four-point correlator of the stress-tensor multiplet, were computed long ago [4,5]. Recently, an elegant algorithm based on symmetries and consistency conditions to determine the four-point correlator of arbitrary CPOs was proposed in Ref. [6]; see, also, Ref. [7].

In this Letter, we consider subleading corrections in $1/N$ to correlators in the large t' Hooft coupling regime. This corresponds to quantum corrections on the gravity side. Although some progress has been made for specific contributions (see Ref. [8]), loop diagrams in AdS are a largely unexplored subject, mostly due to technical difficulties that prohibit direct computations. The analytic bootstrap was initiated in Refs. [9,10] and developed into a powerful algebraic machinery in Refs. [11–13]. This algebraic formulation allowed a systematic study of loops in AdS started in Ref. [14] for sectors of CFTs. In this Letter, we would like to report the first complete results to order $1/N^4$ for a full-fledged CFT, $\mathcal{N}=4$ SYM theory. We will focus on the four-point correlator of the lowest component of the stress-tensor multiplet $\mathcal{O}_2$. In the planar limit and at large 't Hooft coupling, the intermediate operators consist of double-trace operators of spin ℓ and dimension

$$\Delta_{n,\ell} = 4 + 2n + \ell + \frac{\gamma_{n,\ell}^{(1)}}{N^2} + \frac{\gamma_{n,\ell}^{(2)}}{N^4} + \cdots,$$

where $n = 0, 1, \ldots$. The leading order correction $\gamma_{n,\ell}^{(1)}$ is given by the supergravity result. In this Letter, we study the consequences of superconformal symmetry, consistency of the operator product expansion (OPE), and crossing symmetry for the subleading corrections. This analysis is highly complicated by mixing among double-trace operators. Namely, there is more than one intermediate operator for a given twist and spin. From the bulk point of view, this corresponds to taking into account all Kaluza-Klein (KK) modes. In order to solve this problem, we study general correlators $\langle \mathcal{O}_p \mathcal{O}_p \mathcal{O}_q \mathcal{O}_q \rangle$ in the supergravity approximation. This allows us to disentangle the contribution from each KK mode and apply the methods of Refs. [13,14] to find $\gamma_{n,\ell}^{(2)}$.

In addition, crossing symmetry allows the addition of solutions with finite support in the spin. From the bulk perspective, these ambiguities correspond to unknown coefficients in front of possible counterterms. For the present case, we expect such extra solutions to be absent for spin 2 and higher. In this case, for the leading-twist operators of spin 2 and 4, we obtain

$$\Delta_{0,2} = 6 - \frac{4}{N^2} - \frac{45}{N^4} + \cdots,$$
$$\Delta_{0,4} = 8 - \frac{48}{25}\frac{1}{N^2} - \frac{12768}{3125}\frac{1}{N^4} + \cdots.$$

Similar results can be obtained for any spin. In principle, our algorithm fixes also the OPE coefficients.

Stress-tensor correlator in $\mathcal{N}=4$ SYM theory.—In $\mathcal{N}=4$ SYM theory, the stress tensor sits in a half Bogomolny-Prasad-Sommerfeld (BPS) multiplet. The lowest component of this multiplet is a scalar operator $\mathcal{O}_2$ of dimension two in the [0,2,0] of the R-symmetry group $SU(4)$. Its four-point correlator reads

$$\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\rangle = \sum_{\mathcal{R}} \frac{\mathcal{G}^{(\mathcal{R})}(u,v)}{x_{12}^4 x_{34}^4},$$

where the sum runs over representations in the tensor product $[0,2,0]\times[0,2,0]$, and we have introduced the standard cross-ratios

$$u=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}, \qquad v=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}.$$

Superconformal symmetry allows us to write all contributions $\mathcal{G}^{(\mathcal{R})}(u,v)$ in terms of a single nonprotected function $\mathcal{G}(u,v)$ satisfying the following crossing relation

$$v^2\mathcal{G}(u,v)-u^2\mathcal{G}(v,u)+4(u^2-v^2)+\frac{4(u-v)}{c}=0,$$

where $c=[(N^2-1)/4]$ is the central charge. See Refs. [15,16] for a detailed discussion. This function can be decomposed into the contribution from operators in (semi) short multiplets and operators in long multiplets

$$\mathcal{G}(u,v)=\mathcal{G}^{\text{short}}(u,v)+\mathcal{H}(u,v),$$

where $\mathcal{G}^{\text{short}}(u,v)$ is independent of the coupling and can be found in Ref. [16], while $\mathcal{H}(u,v)$ admits a decomposition in superconformal blocks

$$\mathcal{H}(u,v)=\sum_{\tau,\ell} a_{\tau,\ell}u^{\tau/2}g_{\tau+4,\ell}(u,v),$$

where the sum runs over superconformal primary operators in long multiplets in the singlet of $SU(4)$ with twist (dimension minus the spin) τ and even spin ℓ.$a_{\tau,\ell}$ denotes the squared OPE coefficients. It is convenient to introduce cross-ratios $(z,\bar{z})$ with $z\bar{z}=u$, $(1-z)(1-\bar{z})=v$. In terms of these,

$$g_{\tau,\ell}(z,\bar{z})=\frac{z^{\ell+1}F_{\frac{\tau}{2}+\ell}(z)F_{\frac{\tau-2}{2}}(\bar{z})-\bar{z}^{\ell+1}F_{\frac{\tau}{2}+\ell}(\bar{z})F_{\frac{\tau-2}{2}}(z)}{z-\bar{z}},$$

where $F_\beta(z)={}_2F_1(\beta,\beta,2\beta;z)$ is the standard hypergeometric function. In the strict limit of infinite central charge, $\mathcal{H}(u,v)$ reduces to the generalized free fields result $\mathcal{H}^{(0)}(u,v)$, which agrees with the large c result in the Born approximation (free theory). The intermediate operators correspond to double-trace operators of twist $\tau_n=4+2n$ and OPE coefficients

$$a_{n,\ell}^{(0)}=\frac{\pi(\ell+1)(\ell+2n+6)\Gamma(n+3)\Gamma(\ell+n+4)}{2^{2\ell+4n+9}\Gamma(n+\frac{5}{2})\Gamma(\ell+n+\frac{7}{2})}.$$

The four-point correlator admits an expansion around large central charge c:

$$\mathcal{H}(u,v)=\mathcal{H}^{(0)}(u,v)+\frac{1}{c}\mathcal{H}^{(1)}(u,v)+\frac{1}{c^2}\mathcal{H}^{(2)}(u,v)+\cdots.$$

Accordingly,

$$\tau_{n,\ell}=4+2n+\frac{1}{c}\gamma_{n,\ell}^{(1)}+\frac{1}{c^2}\gamma_{n,\ell}^{(2)}+\cdots,$$
$$a_{n,\ell}=a_{n,\ell}^{(0)}+\frac{1}{c}a_{n,\ell}^{(1)}+\frac{1}{c^2}a_{n,\ell}^{(2)}+\cdots. \tag{2}$$

In this Letter, we will focus in the limit of large 't Hooft coupling λ. In this regime, there are no new operators appearing in the OPE at this order, and $\mathcal{H}^{(1)}(u,v)$ can be computed from the classical supergravity result [6,7]. This leads to the following correction for the spectrum and OPE coefficients [11,17,18],

$$\gamma_{n,\ell}^{(1)}=-\frac{\kappa_n}{(1+\ell)(6+\ell+2n)},$$
$$a_{n,\ell}^{(1)}=\frac{1}{2}\partial_n(a_{n,\ell}^{(0)}\gamma_{n,\ell}^{(1)}),$$

where $\kappa_n=(n+1)(n+2)(n+3)(n+4)$. For a given n and ℓ there is more than one superconformal primary in the singlet of $SU(4)$, except for $n=0$. The above corrections should then be interpreted as (weighted) averages. This will be important below.

From leading to subleading corrections.—Our aim is to compute $\gamma_{n,\ell}^{(2)}$ and $a_{n,\ell}^{(2)}$ from crossing symmetry. Since $\mathcal{G}^{\text{short}}(u,v)$ receives contributions only up to order $1/c$, the crossing equation for $\mathcal{H}^{(2)}(u,v)$ reads

$$v^2\mathcal{H}^{(2)}(u,v)=u^2\mathcal{H}^{(2)}(v,u). \tag{3}$$

We will follow the strategy in Ref. [14] and determine the piece proportional to $\log^2 u$ in $\mathcal{H}^{(2)}(u,v)$ from the CFT data at order $1/c$. By crossing symmetry, this leads to a precise divergence proportional to $\log^2 v$. Matching this divergence fixes $\gamma_{n,\ell}^{(2)}$ and $a_{n,\ell}^{(2)}$ to all orders in $1/\ell$. Plugging Eq. (2) into the conformal block decomposition and expanding up to order $1/c^2$, we find

$$\mathcal{H}^{(2)}(u,v)=\sum_{n,\ell}\bigg(a_{n,\ell}^{(2)}+\frac{1}{2}a_{n,\ell}^{(0)}\gamma_{n,\ell}^{(2)}\partial_n+\frac{1}{2}a_{n,\ell}^{(1)}\gamma_{n,\ell}^{(1)}\partial_n$$
$$+\frac{1}{8}a_{n,\ell}^{(0)}(\gamma_{n,\ell}^{(1)})^2\partial_n^2\bigg)u^{2+n}g_{n,\ell}(u,v), \tag{4}$$

where $g_{n,\ell}(u,v)\equiv g_{\tau_n^{(0)}+4,\ell}(u,v)$. The piece proportional to $\log^2 u$ is

$$\mathcal{H}^{(2)}(u,v)|_{\log^2 u} = \sum_{n,\ell} \frac{1}{8} a^{(0)}_{n,\ell} (\gamma^{(1)}_{n,\ell})^2 u^{2+n} g_{n,\ell}(u,v). \quad (5)$$

A serious obstacle to compute this is the mixing among double-trace operators $[\mathcal{O}_2,\mathcal{O}_2]_{n,\ell}, [\mathcal{O}_3,\mathcal{O}_3]_{n-1,\ell},\ldots$. They have the same twist and spin at zeroth order and transform under the same representation of $SU(4)$. Hence, the sum in Eq. (5) should contain an extra index I, which we leave implicit, to account for degenerate operators at tree level. For the same reason, the quantities above should be interpreted as averages weighted by their respective OPE coefficient at zeroth order. Therefore, the weighted average $\langle(\gamma^{(1)}_{n,\ell})^2\rangle$ does not follow from the leading order result, except for $n=0$, for which there is a unique state. This problem can be solved by considering the complete family of four-point correlators $\langle\mathcal{O}_p\mathcal{O}_p\mathcal{O}_q\mathcal{O}_q\rangle$ in the supergravity approximation. Let us show how this works in detail.

For a given twist $\tau_n = 4+2n$, all double-trace operators $[\mathcal{O}_2,\mathcal{O}_2]_n, [\mathcal{O}_3,\mathcal{O}_3]_{n-1},\cdots$ mix, and the eigenfunctions of the Hamiltonian are certain combinations of those

$$\Sigma_i = \alpha_i^2 [O_2,O_2]_n + \cdots + \alpha_i^n [O_{2+n},O_{2+n}]_0, \quad (6)$$

where the dependence on the spin is implicit. We will consider this problem at leading order in $1/c$. We choose the double-trace operators $[\mathcal{O}_k,\mathcal{O}_k]$ to be canonically normalized; hence, the coefficients α_i^p form an orthonormal matrix in this basis. In order to solve the mixing problem, consider the correlators $\langle\mathcal{O}_p\mathcal{O}_p\mathcal{O}_q\mathcal{O}_q\rangle$. At zeroth order, the operators (6) appear with OPE coefficient

$$\sum_i c_{pp\Sigma_i} c_{qq\Sigma_i} = \eta_p \eta_q \sum_i \alpha_i^p \alpha_i^q,$$

where η_p, η_q could also depend on the spin and the twist. At order $1/c$, these operators acquire an anomalous dimension γ_i. From the explicit result for the correlator $\langle\mathcal{O}_p\mathcal{O}_p\mathcal{O}_q\mathcal{O}_q\rangle$ in the supergravity approximation, we read off

$$\sum c_{pp\Sigma_i} c_{qq\Sigma_i} \gamma_i = \eta_p \eta_q \sum_i \alpha_i^p \alpha_i^q \gamma_i \equiv \eta_p \eta_q \langle\gamma\rangle_{pq}.$$

The averages $\langle\gamma\rangle_{pq}$ for a given twist can be conveniently packed in a mixing matrix $M^{(n)}$. For instance, for $n=1$, after analyzing the correlators with p, $q=2$, 3 found in Ref. [19], we find following mixing matrix,

$$M^{(1)} = \kappa_1 \begin{pmatrix} -\frac{1}{J^2-12} & -\frac{6}{(J^2-12)\sqrt{J^2-6}} \\ -\frac{6}{(J^2-12)\sqrt{J^2-6}} & -\frac{J^2+24}{J^4-18J^2+72} \end{pmatrix},$$

where $J^2 = (\ell+4)(\ell+5)$ for $n=1$. We have analyzed this problem for several values p, q, using the explicit supergravity results in Refs. [19–21]. The averages $\langle\gamma^2\rangle_{pq}$ at order $1/c^2$ are given by the elements of $M^{(n)}M^{(n)}$. We are interested in $\langle\gamma^2\rangle_{22}$. For instance, for the case $n=1$, we obtain

$$\langle\gamma^2_{n=1,\ell}\rangle_{22} = \kappa_1^2 \left(\frac{7}{(J^2-12)^2} + \frac{1}{J^2-6} - \frac{1}{J^2-12} \right).$$

In general, we find the following remarkable structure,

$$\langle(\gamma^{(1)}_{n,\ell})^2\rangle = \frac{\kappa_n^3(5+2n)}{120(J^2-(n+2)(n+3))^2} + \sum_{j=2}^{n+2} \frac{\beta_{n,j}}{J^2-j(j+1)}, \quad (7)$$

where $J^2 = (\ell+n+3)(\ell+n+4)$. The coefficients $\beta_{n,j}$ have been computed explicitly up to twist 10. We will see, however, that they can be fixed for any value of the twist by resorting to crossing symmetry.

In order to understand this, we introduce the following sequence of functions denoted twist conformal blocks (TCBs):

$$H_n^{(m)}(z,\bar{z}) = \sum_\ell \frac{a^{(0)}_{n,\ell}}{J^{2m}} u^n g_{n,\ell}(z,\bar{z}).$$

For instance, the zeroth order correlator can be expressed in terms of TCB $H_n^{(0)}(z,\bar{z})$:

$$\mathcal{H}^{(0)}(z,\bar{z}) = \sum_n H_n^{(0)}(z,\bar{z}). \quad (8)$$

The explicit form of (super) conformal blocks leads to the following structure:

$$H_n^{(0)}(z,\bar{z}) = \frac{(z\bar{z})^{\tau_n/2}}{z-\bar{z}} (F_{\frac{\tau+2}{2}}(\bar{z}) h_n^{(0)}(z) - F_{\frac{\tau+2}{2}}(z) h_n^{(0)}(\bar{z})). \quad (9)$$

By plugging this into Eq. (8) and expanding around $z=0$, the functions $h_n^{(0)}(z)$, $h_n^{(0)}(\bar{z})$ can be found,

$$h_n^{(0)}(z) = \rho_n[2(3+n)\bar{z}\,{}_2F_1(3+n,4+n,2(3+n);z) + (3+n)(\bar{z}-2)\,{}_2F_1(4+n,4+n,2(3+n);z)],$$

where $\rho_n = -\{[\pi\Gamma(n+3)^2]/[4^{2n+5}\Gamma(n+\frac{5}{2})\Gamma(n+\frac{7}{2})]\}$. In order to find the functions $H_n^{(m)}(z,\bar{z})$ for $m\neq 0$, we note that there exists a quadratic Casimir operator with eigenfunction $u^n g_{n,\ell}(z,\bar{z})$ and eigenvalue J^2. This implies that $H_n^{(m)}(z,\bar{z})$ admits the same factorization as in Eq. (9) with $h_n^{(0)}(z) \to h_n^{(m)}(z)$, where the functions $h_n^{(m)}(z)$ satisfy a recursion relation. More precisely, we obtain

$$\mathcal{D}_{su} h_n^{(m+1)}(z) = h_n^{(m)}(z), \qquad \mathcal{D}_{su} = z^{-n-3} D z^{n+3},$$

and $D = (1-z)z^2\partial^2 - z^2\partial$. This recursion relation together with $h_n^{(0)}(z)$ allows us to find $H_n^{(m)}(z,\bar{z})$ for the first few values of m and also as various expansions. The divergent behavior as $\bar{z}\to 1$ for $m=2,3,\ldots$ will be important for us. From the explicit answer,

$$h_n^{(0)}(\bar{z})|_{\rm div} = \frac{a_n}{(1-\bar{z})^2} + \frac{b_n}{1-\bar{z}}.$$

It can be then seen that

$$h_n^{(m)}(\bar{z}) = q_n^{(m)}(\bar{z})\log^2(1-\bar{z}), \qquad m=2,3,\ldots$$

with $q_n^{(m)}(\bar{z}) \sim (1-\bar{z})^{m-2}$ as $\bar{z}\to 1$ and

$$\mathcal{D}_{su} q_n^{(m+1)}(\bar{z}) = q_n^{(m)}(\bar{z}).$$

The functions $q_n^{(m)}(\bar{z})$ can be build recursively to any desired order.

Returning to the problem of $\langle(\gamma_{n,\ell}^{(1)})^2\rangle$, we then propose the following expansion:

$$\langle(\gamma_{n,\ell}^{(1)})^2\rangle = \sum_m \frac{c_m^n}{J^{2m}}.$$

This allows us to write the piece proportional to $\log^2 u$ in $\mathcal{H}^{(2)}(u,v)$ in terms of TCB and the coefficients c_m^n. Furthermore, crossing plus consistency with the conformal partial wave (CPW) expansion, e.g., absence of $\log^3 v$, fixes the range of m to be $m=2,3,\cdots$. From the explicit expression for TCB found above, we can extract the contribution proportional to $\log^2 v$,

$$\mathcal{H}^{(2)}(u,v)|_{\log^2 u \log^2 v} = \frac{1}{8}\sum_{m,n} c_m^n \frac{u^{n+2}}{\bar{z}-z} F_{n+3}(z) q_n^{(m)}(\bar{z}),$$

where $q_n^{(m)}(\bar{z})$ is defined above. This contribution should be crossing symmetric by itself. This imposes a set of linear constraints on the coefficients c_m^n. The expansion (7) is consistent with this set of constraints, and, furthermore, the constraints fix uniquely the coefficients $\beta_{n,j}$ for all twists. Up to twist 10, they agree with the ones found by explicit computations.

Having found $\langle(\gamma_{n,\ell}^{(1)})^2\rangle$, we now turn into the sums

$$S_n(z,\bar{z}) \equiv \sum_\ell a_{n,\ell}^{(0)}(\gamma_{n,\ell}^{(1)})^2 u^n g_{n,\ell}(z,\bar{z}). \tag{10}$$

Given Eq. (7), these sums can be solved as follows. Denoting by $\mathcal{C}$ the quadratic Casimir with eigenfunction $u^n g_{n,\ell}(z,\bar{z})$ and eigenvalue J^2, we obtain

$$(\mathcal{C} - j(j+1))\left(\sum_\ell \frac{a_{n,\ell}^{(0)}}{J^2 - j(j+1)} u^n g_{n,\ell}(z,\bar{z})\right) = H_n^{(0)}(z,\bar{z}),$$

which gives a differential equation for the components of the sums (10). This can be easily solved case by case. The sums have the following structure:

$$S_n(z,\bar{z}) = \frac{u^n}{z-\bar{z}}(F_{n+3}(\bar{z})s_n(z) - F_{n+3}(z)s_n(\bar{z})). \tag{11}$$

For instance, for the first case,

$$s_0(z) = \frac{48\log(1-z)((z^2-6z+6)\log(1-z) - 3z^2+6z)}{z^5}.$$

Spectrum at order $1/c^2$.—We will now consider the crossing equation (3) at order $1/c^2$. Our strategy will be to expand it around $z=0$, $\bar{z}=1$ and focus in terms proportional to different powers of $\log z$, $\log(1-\bar{z})$. The $\log z$ dependence in Eq. (4) only arises when the derivative hits u^{2+n}. On the other hand, the behavior around $\bar{z}=1$ is more subtle, and one needs to perform the sum over the spin. The piece proportional to $\log^2 z \log^2(1-\bar{z})$ has already been discussed. The relation proportional to $\log z \log^2(1-\bar{z})$ leads to

$$\sum_{n,\ell} u^n \left(\frac{1}{2}(a_{n,\ell}^{(0)}\gamma_{n,\ell}^{(2)} + a_{n,\ell}^{(1)}\gamma_{n,\ell}^{(1)}) + a_{n,\ell}^{(0)}(\gamma_{n,\ell}^{(1)})^2 \frac{\partial_n}{4}\right) g_{n,\ell}(z,\bar{z}) + \sum_n \frac{\log\bar{z}}{4} S_n(z,\bar{z})|_{\log^2(1-\bar{z})} = \frac{1}{8}\sum\nolimits_n S_n(1-\bar{z},1-z)|_{\log z}.$$

In this Letter, we will restrict ourselves to corrections to the spectrum of leading-twist operators $\gamma_{0,\ell}^{(2)}$. This amounts to considering the small z limit of the relation above. On the lhs, only the term $n=0$ will survive, while on the rhs, all terms will contribute in this limit. The sum over derivatives of conformal blocks with the insertion $(\gamma_{n,\ell}^{(1)})^2$ can be computed with some effort. This result together with the derivative relation for $a_{n,\ell}^{(1)}$ leads to

$$\sum_\ell \frac{1}{2} a_{0,\ell}^{(0)}\hat{\gamma}_{0,\ell}^{(2)} g_{0,\ell}^{\rm coll}(\bar{z}) + \frac{\partial_n}{4}(\kappa_n^2\rho_n F_{3+n}(\bar{z})\log(1-\bar{z}))|_{n=0} + \frac{\log\bar{z}}{4} S_0(z,\bar{z})|_{z^0\log^2(1-\bar{z})} = \frac{1}{8}\sum\nolimits_n S_n(1-\bar{z},1-z)|_{z^0\log z}, \tag{12}$$

where $g_{0,\ell}^{\rm coll}(\bar{z})$ is the small z limit of $g_{0,\ell}(z,\bar{z})$ and $\hat{\gamma}_{n,\ell}^{(2)} = \gamma_{n,\ell}^{(2)} - 1/2\gamma_{n,\ell}^{(1)}\partial_n\gamma_{n,\ell}^{(1)}$. All terms in the above relation except the first one are exactly computable. Crossing symmetry implies that $\hat{\gamma}^{(2)}_{0,\ell}$ should be such that its insertion produces a $\log^2(1-\bar{z})$ divergence times a fully fixed expansion in powers of $(1-\bar{z})$. This problem can be solved by proposing

$$\hat{\gamma}^{(2)}_{0,\ell} = \sum_m \frac{b_m}{J^{2m}}.$$

Hence, the first term in Eq. (12) can be written in terms of TCB $H^{(m)}_n(z,\bar{z})$ at $z=0$. As before, crossing symmetry plus consistency with the CPW expansion fixes the range $m=2,3,\ldots$. From the procedure outlined above, one can compute the contribution proportional to $\log^2(1-\bar{z})$ for $h^{(m)}_n(\bar{z})$ for $m=2,3,\ldots$. This allows us to determine all coefficients b_m and, hence, $\hat{\gamma}^{(2)}_{0,\ell}$ to all orders in $1/\ell$. The result can be organized as to make manifest the contribution from each KK mode. We start by representing $\langle(\gamma^{(1)}_{n,\ell})^2\rangle$ as follows:

$$\langle(\gamma^{(1)}_{n,\ell})^2\rangle = \sum_{p=2}^{\infty} \frac{\alpha_p \kappa_n^2}{(J^2-(n+2)(n+3))^2} \times \prod_{k=2}^{p-1} \frac{(n-k+2)(n+k+3)}{(J^2-k(k+1))},$$

where $\alpha_p = p^2(p^2-1)/12$. Each term inside the sum represents the contribution from the pth KK mode, or more precisely, the intermediate double-trace operators $[\mathcal{O}_p,\mathcal{O}_p]$. We can then compute the contribution to $\hat{\gamma}^{(2)}_{0,\ell}$ from each KK mode. From the bulk point of view, this has the interpretation of an expansion into KK modes running along the loop. Following the steps outlined above, we can compute $\hat{\gamma}^{(2)}_{0,\ell}$ to all orders in $1/\ell$. Remarkably, the resulting series can be resummed exactly. For the massless KK modes, one obtains

$$\hat{\gamma}^{(2)}_{0,\ell}|_{p=2} = -\frac{144(3J^4-2J^2+4)}{(J^2-6)^2(J^2-2)J^2}.$$

Taking into account only the massless KK mode should be equivalent to doing the bulk computation in $5d$ supergravity. In this case, the answer is convergent and finite for all values of the spin. This is consistent with the fact that $5d$ supergravity is free of divergences at one loop. For $p=3,4,\ldots$, the results have the following structure,

$$\hat{\gamma}^{(2)}_{0,\ell}|_p = \frac{P^{(2p+6)}(\ell)}{J^2(J^2-2)(J^2-6)^2} + \frac{Q^{(p+1)}(J^2)}{J^2-6}\psi^{(2)}(\ell+1),$$

where P and Q are polynomials such that this contribution starts at order J^{-2p} at large J. An important comment is in order. Even though the contribution of each KK mode leads to an asymptotic series in $1/J$, the sum of all of them leads to a convergent series. This is in tune with Ref. [22]. Let us consider the contribution of a generic KK mode for finite or small values of the spins. The general structure is

$$\hat{\gamma}^{(2)}_{0,\ell}|_p = \alpha_p \frac{P^{(14+2\ell)}(p)}{(p^2-4)(p^2-1)p} + \alpha_p (p^2-4)(p^2-1)p^3 Q^{(4+2\ell)}(p)\psi^{(2)}(p)$$

for some polynomials P, Q. At large p, we find

$$\hat{\gamma}^{(2)}_{0,\ell}|_p \sim \frac{\alpha_p}{p^{3+2\ell}}.$$

Since $\alpha_p \sim p^4$, this implies the sum over p is actually divergent for spin zero. This agrees with the presence of a quadratic divergence in the $10d$ supergravity computation [23]. For spin 2 and higher, we get a convergent sum. For instance,

$$\sum_{p=3} \hat{\gamma}^{(2)}_{0,2}|_p = -\frac{4523}{1680},$$

$$\sum_{p=3} \hat{\gamma}^{(2)}_{0,4}|_p = -\frac{3832}{21875},$$

which leads to the results quoted in the introduction. Similar results are obtained for arbitrary spin [24].

Discussion.—We have reported the first complete results for the CFT data of unprotected operators in $\mathcal{N}=4$ SYM theory to order $1/N^4$ and at large 't Hooft coupling. A more detailed exposition will appear in Ref. [24]. There are several open questions that would be nice to address.

It would be interesting to compute explicitly $\gamma^{(2)}_{n,\ell}$ for $n>0$. Once this is found, it would be interesting to study its large n behavior and compare it to the expectations from the bulk perspective. It would be important to understand if solutions with finite support in the spin are present. Preliminary results show that crossing symmetry does not require nonanalytical corrections at finite spin. On the other hand, crossing symmetry allows the addition of any of the truncated solutions constructed in Refs. [11,25]. From the bulk perspective, these solutions correspond to counterterms. The $10d$ supergravity computation contains a quadratic divergence proportional to $\lambda^{1/2}\mathcal{R}^4$; see Ref. [26] Eq. (4.2). This leads to a contribution which becomes large for large λ but has support only for spin zero. Indeed, this divergence is visible in our computation when summing over KK modes. We expect other extra solutions are not present. Note that this ambiguity is already present at leading order in $1/c$. In this case, all truncated solutions are forbidden by requiring consistency with the flat space limit; see, e.g., Ref. [6]. Presumably, consistency with the flat space amplitude to order $1/c^2$ will also forbid most extra solutions. There are several results in the literature that bound the behavior of $\gamma^{(1)}_{n,\ell}$ for large n (see, e.g., Refs. [27–31]), and it would be interesting to extend these results to the order we are considering.

Leading order corrections in $1/\lambda$ are, in principle, possible to consider. At leading order, they correspond to the addition of the first truncated solution with a known coefficient [32]. At order $1/N^4$, one would have to "square" the supergravity contribution plus this contribution. Since the latter is truncated in the spin, the extra sums involved are very simple. This computation is also expected to lead to divergences, since the first truncated solution grows much faster with n than supergravity. One could also consider the exchange of a finite number of single-trace operators, combining the results of Ref. [33] with the methods of this Letter.

It would be interesting to study the full four-point correlator in spacetime. In this Letter, we have computed explicitly the piece proportional to $\log^2 u$, which should encode the full physical information about the correlator [22]. For instance, from this piece, through crossing, the CFT data follow to all orders in $1/\ell$, and from this, the four-point correlator can be reconstructed up to pieces which contribute only for finite values of the spin. It would be interesting to study this problem in Mellin space. This would be the first step to extend the results of Ref. [6] to include loops.

The expansion in $1/N$ for nonprotected quantities in the context of AdS/CFT duality is a largely unexplored subject. Our result opens a window to study this problem systematically and quantitatively.

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Note added.—Recently, our paper appeared in arXiv, and an independent computation was presented [34]. Our results are in full agreement.

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