Universal Quantum Computation with Gapped Boundaries

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This Letter discusses topological quantum computation with gapped boundaries of two-dimensional topological phases. Systematic methods are presented to encode quantum information topologically using gapped boundaries, and to perform topologically protected operations on this encoding. In particular, we introduce a new and general computational primitive of topological charge measurement and present a symmetry-protected implementation of this primitive. Throughout the Letter, a concrete physical example, the $\mathbb{Z}_3$ toric code [$\mathfrak{D}(\mathbb{Z}_3)$], is discussed. For this example, we have a qutrit encoding and an abstract universal gate set. Physically, gapped boundaries of $\mathfrak{D}(\mathbb{Z}_3)$ can be realized in bilayer fractional quantum Hall $1/3$ systems. If a practical implementation is found for the required topological charge measurement, these boundaries will give rise to a direct physical realization of a universal quantum computer based on a purely Abelian topological phase.

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Introduction.—The quantum model of computation strikes a delicate balance between classical digital and analog computing models, as its stability lies closer to digital models, while its computational power is closer to analog ones. Still, a major obstacle to developing quantum computers lies in the susceptibility of qubits to decoherence. One elegant theoretical solution to this problem is topological quantum computation (TQC) [1–3]. TQC is a paradigm that information is encoded in topological degrees of freedom of certain quantum systems, thereby protected from local decoherence. While the standard implementation uses (non-Abelian) anyons in topological phases of matter, recent studies revealed that certain topological phases also support gapped boundaries. It is hence natural to study TQC with gapped boundaries [4–7].

Real samples of topological phases of matter such as fractional quantum Hall liquids and topological insulators have boundaries, which are usually conducting (gapless) even though the bulk are insulating (gapped). However, they can be modified to realize Dijkgraaf-Witten (DW) gauge theories, which are also given by Kitaev's quantum double Hamiltonian [3]. These theories support gapped boundaries in the sense that the extensions of the Hamiltonians to spaces (surfaces) with boundaries are still gapped; the Hamiltonian and algebraic frameworks are developed in Refs. [6,8]. These frameworks show that a gapped boundary effectively behaves as a non-Abelian anyon. However, while the existence of non-Abelian anyons is still uncertain, gapped boundaries of Abelian phases are much more routine and support topologically protected degeneracies even on the plane.

In this Letter, we apply our theory to a concrete physical example—the $\mathbb{Z}_3$ toric code $\mathfrak{D}(\mathbb{Z}_3)$—to obtain a universal gate set, which is a striking example of the extra computational power from gapped boundaries. This new direction opens up new vistas in both the theoretical study and experimental realization of TQC. We introduce a new computational primitive—topological charge measurement (TCM), which extends topological charge projection [4]. We propose a physical realization of symmetry-protected TCM in a fractional quantum spin Hall state, while leaving a fully topologically protected one to the future because which measurement is possible in gauge theory is an open fundamental question [9].

Our universal gate set for $\mathfrak{D}(\mathbb{Z}_3)$ is close to experimental technology in bilayer quantum Hall liquids. If a practical implementation is found for our TCM primitive, this gate set is a direct physical realization of a universal quantum computer.

Realization of $\mathbb{Z}_3$ toric code by bilayer $\nu = 1/3$ fractional quantum Hall liquids.—The $\mathbb{Z}_3$ toric code $\mathfrak{D}(\mathbb{Z}_3)$ can be realized in bilayer fractional quantum Hall (FQH) systems. Reference [10] considers an electron-hole bilayer FQH system, with a $1/3$ Laughlin state of opposite chirality in each layer. The topological order in this system is $\mathrm{SU}(3)_1 \times \overline{\mathrm{SU}(3)_1}$, which is equivalent to the $\mathbb{Z}_3$ toric code $\mathfrak{D}(\mathbb{Z}_3)$. (Together with physical electrons, $\overline{\mathrm{SU}(3)_1}$ is topologically equivalent to a $1/3$ Laughlin state.) Hence, we will recycle many of the results of Ref. [10].

We briefly summarize the basic data for $\mathfrak{D}(\mathbb{Z}_3)$. Mathematically, a topological phase is described by a modular tensor category (MTC) $\mathcal{B}$ [11]. The anyon types are $e^a m^b$, $a, b = 0, 1, 2$, where e and m are $\mathbb{Z}_3$ unit gauge charge and flux, respectively (so $e^2 = \bar{e}$, $m^2 = \bar{m}$). [There are many terms in the literature referring to the same thing: a simple quasiparticle, an anyon, or a simple object of $\mathfrak{D}(G)$. An anyon type, topological charge, or superselection sector is an isomorphism class of the above.] The braiding statistics of the anyons is encoded in the modular $\mathcal{S} = [S_{ab}]$ and $\mathcal{T} = [T_{ab}]$ matrices [11],

$$S_{ab} = \omega^{-a_2 b_1 - a_1 b_2}, \qquad T_{ab} = \omega^{a_1 a_2} \delta_{ab}. \tag{1}$$

Here, $\omega = e^{2\pi i/3}$.

Gapped boundaries, degeneracy, and topological operations.—Let us first review the physics of gapped boundaries, as they will encode our topological qudits. A convenient physical description for a gapped boundary type is the consistent collection of (bosonic) anyons that can condense to vacuum to the boundary at no energy cost. Mathematically, this is given by a Lagrangian algebra $\mathcal{A}$ in the MTC $\mathcal{B}$ (see Refs. [6,8,12–14] and references therein for precise definitions), which can be represented as a direct sum of all condensed anyon types. For $\mathfrak{D}(\mathbb{Z}_3)$, there are two gapped boundary types: e boundaries and m boundaries, where e, e^2 and m, m^2 condense, respectively. The corresponding Lagrangian algebras are $1 + e + e^2$ and $1 + m + m^2$. In the bilayer $1/3$ Laughlin state description, the e- and m-boundary types correspond to holes with the two layers connected via electron pairing (i.e., superconducting) or tunneling.

Multiple gapped boundaries support a degenerate ground-state manifold. (The degeneracy is exponentially protected in all length scales, including distance between boundaries as well as lengths of the boundaries.) Consider a closed system with n gapped boundaries (Fig. 1). References [6,8] show that the ground state of the system is given by the different ways we can create n anyons out of vacuum, and condense all of them onto the boundaries as a fusion tree (Fig. 1). This fusion tree also specifies a choice of basis states for the ground state manifold. For example, if we have two e boundaries in a planar $\mathfrak{D}(\mathbb{Z}_3)$ theory, the ground state degeneracy is 3, labeled by $a_1 = \bar{c}$, $a_2 = c$, $c = \mathbf{1}, e, \bar{e}$. We denote the basis elements by $|c\rangle$ and encode our qutrit in this space.

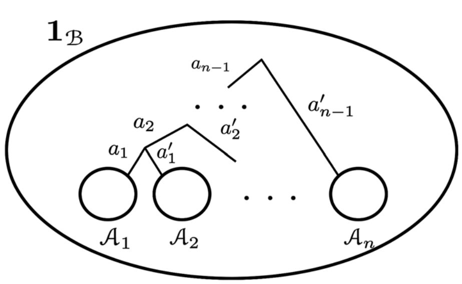

FIG. 1. Ground state for n gapped boundaries $\mathcal{A}_i$ on a plane and total charge vacuum. All edges are directed to point downward.

We now discuss the topological operations on gapped boundaries, which induce unitary transformations in the degenerate subspace. We focus on the $\mathfrak{D}(\mathbb{Z}_3)$ example and leave the general results to the Supplemental Material [15].

Tunnel-a operations.—The first topological operation is to tunnel an anyon a from one gapped boundary ($\mathcal{A}_1$) to another ($\mathcal{A}_2$), where a ($\bar{a}$) condenses on $\mathcal{A}_1$ ($\mathcal{A}_2$). Physically, this corresponds to applying the a string operator [3] along a path γ connecting the two gapped boundaries. This operation, known as a Wilson line operator, is denoted by $W_a(\gamma)$. For the $\mathfrak{D}(\mathbb{Z}_3)$ theory, it can be represented as follows:

$$W_a(\gamma)|b\rangle = |a \times b\rangle. \tag{2}$$

Expressing $W_a(\gamma)$ as a matrix that acts on the ground state subspace, we see that $W_e(\gamma)$ implements the single-qutrit Pauli-X gate σ_3^x.

Loop-a operations.—Analogously, one can create a pair of anyons $\bar{a}$ in the bulk, loop one of them around a gapped boundary, and annihilate the pair. When we loop a counterclockwise around the boundary, this is known as the Wilson loop operator $W_a(\alpha_i)$ where α_i is the loop encircling boundary $\mathcal{A}_i$. The Supplemental Material [15] shows that

$$W_a(\alpha_2)|b\rangle = \frac{S_{ab}}{d_b}|b\rangle. \tag{3}$$

Braiding gapped boundaries.—Another topological operation is to braid gapped boundaries around each other. This gives multiple-qudit operations that can produce entangling gates. Physically, braiding corresponds to moving gapped boundaries around each other, e.g., by tuning the Hamiltonian H_{GB} of Refs. [6,8] adiabatically.

We may arbitrarily braid n gapped boundaries with total charge vacuum around each other to obtain a unitary transformation on the ground state, so long as we return each boundary to its original position. Mathematically, this means that the braiding matrices form a representation of the (spherical) n-strand pure braid group P_n [16]. They can be computed using the diagrammatic rules of anyon models and the basis states of gapped boundaries. For most purposes of quantum computation, it is sufficient to consider 2-qudit encodings, where $n = 4$ (Fig. 2). In general, one must compute all six generators of P_4. As an example, we derive the formula for the generator σ_2^2 in the Supplemental Material [15].

Topological charge measurement.—For a DW theory, the gapped boundary braidings only generate a finite group [16]. Inspired by the results of Ref. [4], we introduce topological charge measurement based on the Wilson operators. Before we discuss the general case, recall that topological charge projection can detect the total charge of a collection of quasiparticles inside a certain region, e.g., by sending probe particle along a path enclosing the region and performing interferometric measurement. As a

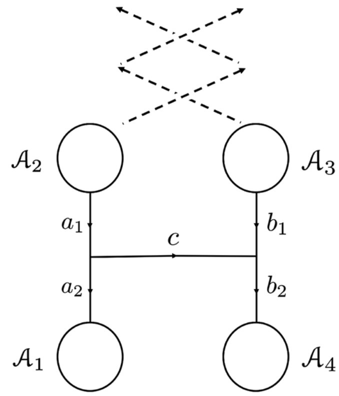

FIG. 2. Braiding of two gapped boundaries (σ_2^2). Solid lines indicate tunneling operators from the basis vectors (i.e., not motion of the holes), while dotted lines indicate how the holes move in the braiding process.

generalization, we can use similar methods to perform measurement of topological charge through any loop, not just contractible ones, possibly on a higher-genus surface [4].

Recall that $\mathfrak{D}(\mathbb{Z}_3)$ splits into two theories $\mathcal{B} = \mathcal{C} \boxtimes \bar{\mathcal{C}}$ with $\mathcal{C} = \mathrm{SU}(3)_1$ which do not interact in the bulk, but are "stuck together" at the original boundaries of $\mathcal{B}$. The planar region Y also splits into two mirror layers, $S_+(Y)$ and $S_-(Y)$, which are completely disjoint in the bulk but "stuck together" at the boundaries of Y. This way, we can view the system as a single layer of $\mathcal{C}$ on a higher-genus surface. Similarly, each loop α in Y becomes a loop l_α in $S_+(Y)$ or $S_-(Y)$, while an arc γ connecting two boundaries lifts to a loop l_γ going around both layers. Let β be one of these loops. Figure 3 illustrates this for $n = 2$.

Define $\mathcal{O}_x(\beta) = W_x(\alpha_i)$ (tunneling operator in $\mathcal{C}$) if β is the lifting of the line α_i, and $\mathcal{O}_x(\beta) = W_{x\bar{x}}(\gamma_i)$ (loop operator in $\mathcal{B}$) if β is the lifting of the loop γ_i. By Ref. [4], the projection measuring topological charge a through β can be expressed as

$$P_\beta^{(a)} = \sum_{x \in \mathcal{C}} S_{0a} S_{xa}^* \mathcal{O}_x(\beta). \tag{4}$$

The sum runs over the anyon labels x of $\mathcal{C}$, and S_{ab} is the modular $\mathcal{S}$ matrix of $\mathcal{C}$. The Wilson operators $W_x(\alpha_i)$ and $W_{x\bar{x}}(\gamma_i)$ are computed using the formulas (2) and (3) with the data of $\mathcal{C}$ and $\mathcal{B}$, respectively. As shown in [4], topological charge projections generate all mapping class group representations $V_\mathcal{C}(Y)$ of a closed surface Y from the anyon theory $\mathcal{C}$.

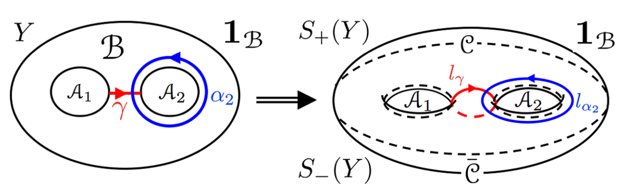

FIG. 3. Topological charge projection ($n = 2$).

For our purpose, we generalize these projections to TCMs which are the complements of topological charge projections (the more general definition is in the Supplemental Material [15]). Specifically, given an anyon label a and the lifting β of a Wilson line or loop as above, we consider the projection $1 - P_\beta^{(a)}$. Physically, this can be implemented by adding such nonlocal operators to the effective Hamiltonian of the ground state subspace,

$$H' = -t W_a(\beta) + \mathrm{H.c.} \tag{5}$$

Here, t is the (complex) tunneling amplitude. This effective Hamiltonian then projects the system to the desired state space.

Universal gate set with $\mathfrak{D}(\mathbb{Z}_3)$ gapped boundaries.—Let us now specialize to $\mathfrak{D}(\mathbb{Z}_3)$, or the bilayer $\nu = 1/3$ FQH. Reference [10] proposed to use superconducting $(1 + e + \bar{e})$ boundaries to encode qutrits, so the readout can be done with electric charge measurement. We follow this scheme, and occasionally use the other (m-boundary) encoding as an ancilla.

By Ref. [17], one universal qutrit gate set is the metaplectic gate set: 1. The single-qutrit Hadamard gate H_3. 2. The two-qutrit entangling gate SUM_3. 3. The single-qutrit generalized phase gate $Q_3 = \mathrm{diag}(1, 1, \omega)$. 4. Any nontrivial single-qutrit classical (i.e., Clifford) gate not equal to H_3^2. 5. A projection M of a state in the qutrit space $\mathbb{C}^3$ to $\mathrm{Span}\{|0\rangle\}$ and its orthogonal complement $\mathrm{Span}\{|1\rangle, |2\rangle\}$, so that the resulting state is coherent if projected into $\mathrm{Span}\{|1\rangle, |2\rangle\}$.

We now discuss how each of these gates can be implemented from the aforementioned topological operations. First, we discuss the implementation of 1–4: 1. H_3 is equal to the modular $\mathcal{S}$ matrix of the anyon theory $\mathrm{SU}(3)_1$, so it is in the representation of mapping class group of the torus and can be implemented via a sequence of topological charge projections. 2. For SUM_3, consider braiding one hole of a e-boundary target qutrit with another hole of a m-boundary control qutrit (i.e., apply σ_2^2, as shown in Fig. 4). This gives

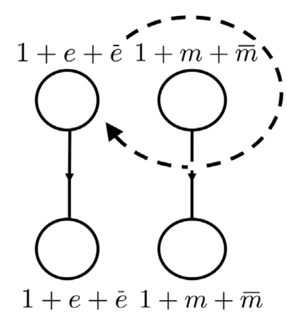

FIG. 4. Braid for the $\wedge \sigma_3^z$ gate.

$$\sigma_2^2 = \mathrm{diag}(1,1,1,1,\omega,\omega^2,1,\omega^2,\omega) = \wedge\, \sigma_3^z. \tag{6}$$

Because we implemented the Hadamard and $\wedge\, \sigma_3^z = (I_3 \otimes H_3)\mathrm{SUM}_3(I_3 \otimes H_3)$, conjugating the target qutrit by Hadamards gives the SUM gate between a $1+e+\bar{e}$ qutrit and a $1+m+\bar{m}$ qutrit. We then have a short circuit (Fig. 5) using these SUM gates to implement a SUM gate between two $1+e+\bar{e}$ qutrits. After the circuit, one must interpret the measurement outcome of the ancilla qutrit. If we measure $|m^j\rangle$, we must apply $(\sigma_3^x)^j$ to the control-out [e.g., by applying $W_{e^j}(\gamma)$]. 3. By Ref. [4], topological charge projections can be used to implement $\mathrm{diag}(1,\omega,\omega)$, the Dehn twist of the $\mathrm{SU}(3)_1$ theory. We follow this by a generalized Pauli-Z gate to obtain Q_3. 4. By Eq. (2), the tunneling operator $W_e(\gamma)$ implements the single-qutrit Pauli-X gate σ_3^x.

The implementation of the coherent projection M is the most challenging part of the proposal. First, we relate M to a TCM. A planar $\mathfrak{D}(\mathbb{Z}_3)$ with two $1+e+\bar{e}$ boundaries can be viewed as double layers of $\mathrm{SU}(3)_1$ connected via two handles; the curve γ connecting the two boundaries lifts to a loop in this perspective. By Eq. (4), projecting to vacuum within this loop gives

$$P_\gamma^{(1)} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \tag{7}$$

The eigenvalues and eigenspaces of $P_\gamma^{(1)}$ are

$$\lambda = 0\colon \mathrm{Span}\left\{\begin{bmatrix} 1 \\ \omega \\ \bar{\omega} \end{bmatrix}, \begin{bmatrix} 1 \\ \bar{\omega} \\ \omega \end{bmatrix}\right\} \quad \lambda = 1\colon \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}. \tag{8}$$

One then obtains the coherent projection M by conjugating the orthogonal projector $1 - P_\gamma^{(1)}$ with the Hadamard, i.e., $H_3^\dagger(1 - P_\gamma^{(1)})H_3$. While $P_\gamma^{(1)}$ is a topological charge projection as in Ref. [4], $1 - P_\gamma^{(1)}$ is a general TCM.

We now have universal quantum computation using gapped boundaries of $\mathfrak{D}(\mathbb{Z}_3)$. This is very significant, as we achieve universal quantum computation using only an Abelian topological quantum field theory [all anyon braidings in $\mathfrak{D}(\mathbb{Z}_3)$ are projectively trivial], without using state injection, as in Ref. [7].

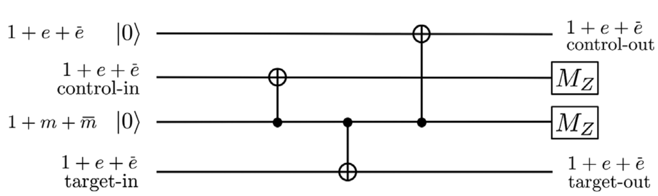

FIG. 5. Short circuit (generalizing Ref. [7]) to use three SUM gates between $1+m+\bar{m}$ and $1+e+\bar{e}$ qutrits to implement a SUM gate between $1+e+\bar{e}$ qutrits. All entangling gates drawn are SUM_3.

Symmetry-protected realization.—In physical realizations such as bilayer FQH, the TCM can be implemented as follows: we tune the system such that the quasiparticle tunneling along the desired loop is enhanced, so that the system has the projected charge state as the ground state. This can be achieved by, e.g., using gate configurations to diminish the energy gap. We consider $1 - P_\gamma^{(a)}$ as a concrete example. The desired term in the Hamiltonian we would like to create is $H' = -tW_\gamma(e) + \mathrm{H.c.}$, where t is the (complex) tunneling amplitude and $W_\gamma(e)$ is the Wilson tunneling operator. W_γ has eigenvalues $1, \omega, \bar{\omega}$. The coherent projection requires that the eigenvalues of H' split into two sets, one of which has two degenerate eigenvalues. This puts a stringent constraint on the complex phase of t. The simplest choice is that t is real.

The requirement that t is real is beyond topological protection. Physically, such condition can be met in a fractional quantum spin Hall state [18,19], an interacting analog of quantum spin Hall insulator enriched by time-reversal symmetry. Topologically, this phase is identical to bilayer $\nu = 1/3$ Laughlin state, if the layer index is actually identified as the electron spin up and down. In such a state, the time-reversal symmetry exchanges the two layers. The e anyon in this physical realization is the bound state of the spin up or down quasiholes. Therefore, the tunneling amplitude of e has to be real since e is time-reversal invariant, and the TCM is symmetry protected.

Conclusions.—Gapped boundaries provide the missing $\pi/8$-gate for a universal gate set from the doubled Ising theory [4]. In this Letter, we use our symmetry-protected TCM to obtain a coherent projection, which augments the topological operations from Ref. [10] for the $\mathbb{Z}_3$ toric code to a universal gate set for a qutrit computational model. The $\mathbb{Z}_3$ toric code is realized by bilayer fractional quantum Hall liquids [10], whereas it is not yet clear how to physically realize the doubled Ising theory. The challenge for a realistic implementation of our universal gate set now lies in a practical realization of the coherent projection.

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