# Maximum Relative Entropy of Coherence: An Operational Coherence Measure

Kaifeng Bu,[1,*] Uttam Singh,[2,3,†] Shao-Ming Fei,[4,5,‡] Arun Kumar Pati,[2,§] and Junde Wu[1,∥]

[1] *School of Mathematical Sciences, Zhejiang University, Hangzhou 310027, People's Republic of China*

[2] *Harish-Chandra Research Institute, Allahabad 211019, India*

[3] *Homi Bhabha National Institute, Training School Complex, Anushakti Nagar, Mumbai 400085, India*

[4] *School of Mathematical Sciences, Capital Normal University, Beijing 100048, People's Republic of China*

[5] *Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig, Germany*

(Received 16 August 2017; published 13 October 2017)

The operational characterization of quantum coherence is the cornerstone in the development of the resource theory of coherence. We introduce a new coherence quantifier based on maximum relative entropy. We prove that the maximum relative entropy of coherence is directly related to the maximum overlap with maximally coherent states under a particular class of operations, which provides an operational interpretation of the maximum relative entropy of coherence. Moreover, we show that, for any coherent state, there are examples of subchannel discrimination problems such that this coherent state allows for a higher probability of successfully discriminating subchannels than that of all incoherent states. This advantage of coherent states in subchannel discrimination can be exactly characterized by the maximum relative entropy of coherence. By introducing a suitable smooth maximum relative entropy of coherence, we prove that the smooth maximum relative entropy of coherence provides a lower bound of one-shot coherence cost, and the maximum relative entropy of coherence is equivalent to the relative entropy of coherence in the asymptotic limit. Similar to the maximum relative entropy of coherence, the minimum relative entropy of coherence has also been investigated. We show that the minimum relative entropy of coherence provides an upper bound of one-shot coherence distillation, and in the asymptotic limit the minimum relative entropy of coherence is equivalent to the relative entropy of coherence.

DOI: 10.1103/PhysRevLett.119.150405

*Introduction.*—Quantumness in a single system is characterized by quantum coherence, namely, the superposition of a state in a given reference basis. The coherence of a state may quantify the capacity of a system in many quantum manipulations, ranging from metrology [1] to thermodynamics [2,3]. Recently, various efforts have been made to develop a resource theory of coherence [4–10]. One of the earlier resource theories is that of quantum entanglement [11], which is a basic resource for various quantum information-processing protocols such as superdense coding [12], remote state preparation [13,14], and quantum teleportation [15]. Other notable examples include the resource theories of asymmetry [16–22], thermodynamics [23], and steering [24]. One of the main advantages that a resource theory offers is the lucid quantitative and operational description as well as the manipulation of the relevant resources at one's disposal; thus, the operational characterization of quantum coherence is required in the resource theory of coherence.

A resource theory is usually composed of two basic elements: free states and free operations. The set of allowed states (operations) under the given constraint is what we call the set of free states (operations). Given a fixed basis, say, $\{|i\rangle\}_{i=0}^{d-1}$ for a $d$-dimensional system, any quantum state which is diagonal in the reference basis is called an incoherent state and is a free state in the resource theory of coherence. The set of incoherent states is denoted by $\mathcal{I}$. Any quantum state can be mapped into an incoherent state by a full dephasing operation $\Delta$, where $\Delta(\rho) \coloneqq \sum_{i=0}^{d-1} \langle i|\rho|i\rangle |i\rangle\langle i|$. However, there is no general consensus on the set of free operations in the resource theory of coherence. We refer the following types of free operations in this work: maximally incoherent operations (MIO) [25], incoherent operations (IO) [4], dephasing-covariant operations (DIO) [25], and strictly incoherent operations (SIO) [10,25]. By MIO, we refer to the maximal set of quantum operations $\Phi$ which maps the incoherent states into incoherent states, i.e., $\Phi(\mathcal{I}) \subset \mathcal{I}$ [25]. IO is the set of all quantum operations $\Phi$ that admit a set of Kraus operators $\{K_i\}$ such that $\Phi(\cdot) = \sum_i K_i(\cdot)K_i^\dagger$ and $K_i \mathcal{I} K_i^\dagger \subset \mathcal{I}$ for any $i$ [4]. DIO are the quantum operations $\Phi$ with $[\Delta, \Phi] = 0$ [25]. SIO is the set of all quantum operations $\Phi$ admitting a set of Kraus operators $\{K_i\}$ such that $\Phi(\cdot) = \sum_i K_i(\cdot)K_i^\dagger$ and $\Delta(K_i \rho K_i^\dagger) = K_i \Delta(\rho) K_i^\dagger$ for any $i$ and any quantum state $\rho$. Both IO and DIO are subsets of MIO, and SIO is a subset of both IO and DIO [25]. However, IO and DIO are two different types of free operations, and there is no inclusion relationship between them (the operational gap between them can be seen in Ref. [26]).

Several operational coherence quantifiers have been introduced as candidate coherence measures, subjecting to physical requirements such as monotonicity under certain types of free operations in the resource theory of coherence. One canonical measure to quantify coherence is the relative entropy of coherence, which is defined as $C_r(\rho) = S[\Delta(\rho)] - S(\rho)$, where $S(\rho) = -\mathrm{Tr}[\rho \log \rho]$ is the von Neumann entropy [4]. The relative entropy of coherence plays an important role in the process of coherence distillation, in which it can be interpreted as the optimal rate to distill a maximally coherent state from a given state $\rho$ by IO in the asymptotic limit [7]. Besides, the $l_1$ norm of coherence [4], which is defined as $C_{l_1}(\rho) = \sum_{i\neq j}|\rho_{ij}|$ with $\rho_{ij} = \langle i|\rho|j\rangle$, has also attracted lots of discussions about its operational interpretation [27]. Recently, an operationally motivated coherence measure—robustness of coherence (ROC)—has been introduced, which quantifies the minimal mixing required to erase the coherence in a given quantum state [28,29]. There is growing concern about the operational characterization of quantum coherence, and further investigations are needed to provide an explicit and rigorous operational interpretation of coherence.

In this Letter, we introduce a new coherence measure based on maximum relative entropy and focus on its operational characterizations. Maximum and minimum relative entropies have been introduced and investigated in Refs. [30–33]. The well-known (conditional and unconditional) maximum and minimum entropies [34,35] can be obtained from these two quantities. It has been shown that maximum and minimum entropies are of operational significance in applications ranging from data compression [34,36] to state merging [37] and security of key [38,39]. Besides, maximum and minimum relative entropies have been used to define entanglement monotone, and their operational significance in the manipulation of entanglement has been provided in Refs. [30–33]. Here, we define maximum relative of coherence $C_{\max}$ based on maximum relative entropy and investigate the properties of $C_{\max}$. We prove that the maximum relative entropy of coherence for a given state $\rho$ is the maximum achievable overlap with maximally coherent states under DIO, IO, and SIO, which gives rise to an operational interpretation of $C_{\max}$ and shows the equivalence among DIO, IO, and SIO in an operational task. Besides, we show that the maximum relative entropy of coherence characterizes the role of quantum states in an operational task: subchannel discrimination. Subchannel discrimination is an important quantum information task which distinguishes the branches of a quantum evolution for a quantum system to undergo [40]. It has been shown that every entangled or steerable state is a resource in some instance of subchannel discrimination problems [40,41]. Here, we prove that every coherent state is useful in the subchannel discrimination of certain instruments, where the usefulness can be quantified by the maximum relative entropy of coherence of the given quantum state. By smoothing the maximum relative entropy of coherence, we introduce $\varepsilon$-smoothed maximum relative entropy of coherence $C^{\varepsilon}_{\max}$ for any fixed $\varepsilon > 0$ and show that the smooth maximum relative entropy gives an lower bound of coherence cost in a one-shot version. Moreover, we prove that, for any quantum state, the maximum relative entropy of coherence is equivalent to the relative entropy of coherence in the asymptotic limit.

Corresponding to the maximum relative entropy of coherence, we also introduce the minimum relative entropy of coherence $C_{\min}$ by minimum relative entropy, which is not a proper coherence measure as it may increase on average under IO. However, it gives an upper bound for the maximum overlap between the given states and the set of incoherent states. This implies that the minimum relative entropy of coherence also provides a lower bound of a well-known coherence measure, the geometry of coherence [6]. By smoothing the minimum relative entropy of coherence, we introduce $\varepsilon$-smoothed minimum relative entropy of coherence $C^{\varepsilon}_{\min}$ for any fixed $\varepsilon > 0$ and show that the smooth maximum relative entropy gives an upper bound of coherence distillation in a one-shot version. Furthermore, we show that the minimum relative of coherence is also equivalent to the distillation of coherence in the asymptotic limit. The relationship among $C_{\min}$, $C_{\max}$, and other coherence measures has also been investigated.

*Main results.*—Let $\mathcal{H}$ be a $d$-dimensional Hilbert space and $\mathcal{D}(\mathcal{H})$ be the set of density operators acting on $\mathcal{H}$. Given two operators $\rho$ and $\sigma$ with $\rho \geq 0$, $\mathrm{Tr}[\rho] \leq 1$, and $\sigma \geq 0$, the maximum relative entropy of $\rho$ with respect to $\sigma$ is defined by [30,31]

$$D_{\max}(\rho||\sigma) \coloneqq \min\{\lambda : \rho \leq 2^{\lambda}\sigma\}. \tag{1}$$

We introduce a new coherence quantifier by maximum relative entropy: maximum relative entropy of coherence $C_{\max}$,

$$C_{\max}(\rho) \coloneqq \min_{\sigma\in\mathcal{I}} D_{\max}(\rho||\sigma), \tag{2}$$

where $\mathcal{I}$ is the set of incoherent states in $\mathcal{D}(\mathcal{H})$.

We now show that $C_{\max}$ satisfies the conditions a coherence measure needs to fulfil. First, it is obvious that $C_{\max}(\rho) \geq 0$. And since $D_{\max}(\rho||\sigma) = 0$ iff $\rho = \sigma$ [30], we have $C_{\max}(\rho) = 0$ if and only if $\rho \in \mathcal{I}$. Besides, as $D_{\max}$ is monotone under completely positive and trace preserving (CPTP) maps [30], we have $C_{\max}[\Phi(\rho)] \leq C_{\max}(\rho)$ for any incoherent operation $\Phi$. Moreover, $C_{\max}$ is nonincreasing on average under incoherent operations; that is, for any incoherent operation $\Phi(\cdot) = \sum_i K_i(\cdot)K_i^{\dagger}$ with $K_i\mathcal{I}K_i^{\dagger} \subset \mathcal{I}$, $\sum_i p_i C_{\max}(\tilde{\rho}_i) \leq C_{\max}(\rho)$, where $p_i = \mathrm{Tr}[K_i\rho K_i^{\dagger}]$ and $\tilde{\rho}_i = K_i\rho K_i^{\dagger}/p_i$; see the proof in Supplemental Material [42].

*Remark.*—We have proven that the maximum relative entropy of coherence $C_{\max}$ is a bona fide measure of coherence. Since $D_{\max}$ is not jointly convex, we may not expect that $C_{\max}$ has convexity, which is a desirable (although not a fundamental) property for a coherence quantifier. However, we can prove that, for $\rho = \sum_i^n p_i\rho_i$, $C_{\max}(\rho) \le \max_i C_{\max}(\rho_i)$. Suppose that $C_{\max}(\rho_i) = D_{\max}(\rho_i||\sigma_i^*)$ for some $\sigma_i^*$; then from the fact that $D_{\max}(\sum_i p_i\rho_i||\sum_i p_i\sigma_i) \le \max_i D_{\max}(\rho_i||\sigma_i)$ [30], we have $C_{\max}(\rho) \le D_{\max}(\sum_i p_i\rho_i||\sum_i p_i\sigma_i^*) \le \max_i D_{\max}(\rho_i||\sigma_i^*) = \max_i C_{\max}(\rho_i)$. Besides, although $C_{\max}$ is not convex, we can obtain a proper coherence measure with convexity from $C_{\max}$ by the approach of a convex roof extension; see Supplemental Material [42].

In the following, we concentrate on the operational characterization of the maximum relative entropy of coherence and provide operational interpretations of $C_{\max}$.

*Maximum overlap with maximally coherent states.*—At first we show that $2^{C_{\max}}$ is equal to the maximum overlap with the maximally coherent state that can be achieved by DIO, IO, and SIO.

*Theorem 1.*—Given a quantum state $\rho \in \mathcal{D}(\mathcal{H})$, we have

$$2^{C_{\max}(\rho)} = d\max_{\mathcal{E},|\Psi\rangle} F[\mathcal{E}(\rho), |\Psi\rangle\langle\Psi|]^2, \tag{3}$$

where $F(\rho,\sigma) = \mathrm{Tr}[|\sqrt{\rho}\sqrt{\sigma}|]$ is the fidelity between states $\rho$ and $\sigma$ [54], $|\Psi\rangle \in \mathcal{M}$, $\mathcal{M}$ is the set of maximally coherent states in $\mathcal{D}(\mathcal{H})$, and $\mathcal{E}$ belongs to either DIO, IO, or SIO.

(See the proof in Supplemental Material [42].)

Here, although IO, DIO, and SIO are different types of free operations in the resource theory of coherence [25,26], they have the same behavior in the maximum overlap with the maximally coherent states. From the view of coherence distillation [7], the maximum overlap with maximally coherent states can be regarded as the distillation of coherence from given states under IO, DIO, and SIO. As fidelity can be used to define certain distances, thus $C_{\max}(\rho)$ can also be viewed as the distance between the set of a maximally coherent state and the set of $\{\mathcal{E}(\rho)\}_{\mathcal{E}\in\theta}$, where $\theta =$ DIO, IO, or SIO.

Besides distillation of coherence, another kind of coherence manipulation is the coherence cost [7]. Now we study the one-shot version of coherence cost under MIO based on the smooth maximum relative entropy of coherence. We define the one-shot coherence cost of a quantum state $\rho$ under MIO as

$$C_{C,\mathrm{MIO}}^{(1),\varepsilon}(\rho) \coloneqq \min_{\substack{\mathcal{E}\in\mathrm{MIO}\\ M\in\mathbb{Z}}}\{\log M : F[\rho, \mathcal{E}(|\Psi_+^M\rangle\langle\Psi_+^M|)]^2 \ge 1-\varepsilon\},$$

where $|\Psi_+^M\rangle = (1/\sqrt{M})\sum_{i=1}^M |i\rangle$, $\mathbb{Z}$ is the set of integer, and $\varepsilon > 0$. The $\varepsilon$-smoothed maximum relative entropy of coherence of a quantum state $\rho$ is defined by

$$C_{\max}^{\varepsilon}(\rho) \coloneqq \min_{\rho'\in B_\varepsilon(\rho)} C_{\max}(\rho'), \tag{4}$$

where $B_\varepsilon(\rho) \coloneqq \{\rho' \ge 0 : \|\rho' - \rho\|_1 \le \varepsilon, \mathrm{Tr}[\rho'] \le \mathrm{Tr}[\rho]$. We find that the smooth maximum relative entropy of coherence gives a lower bound of the one-shot coherence cost. Given a quantum state $\rho \in \mathcal{D}(\mathcal{H})$, for any $\varepsilon > 0$,

$$C_{\max}^{\varepsilon'}(\rho) \le C_{C,\mathrm{MIO}}^{(1),\varepsilon}(\rho), \tag{5}$$

where $\varepsilon' = 2\sqrt{\varepsilon}$; see the proof in Supplemental Material [42].

Besides, in view of the smooth maximum relative entropy of coherence, we can obtain the equivalence between the maximum relative entropy of coherence and the relative entropy of coherence in the asymptotic limit. Since the relative entropy of coherence is the optimal rate to distill a maximally coherent state from a given state under certain free operations in the asymptotic limit [7], the smooth maximum relative entropy of coherence in the asymptotic limit is just the distillation of coherence. That is, given a quantum state $\rho \in \mathcal{D}(\mathcal{H})$, we have

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}C_{\max}^{\varepsilon}(\rho^{\otimes n}) = C_r(\rho). \tag{6}$$

(The proof is presented in Supplemental Material [42].)

*Maximum advantage achievable in subchannel discrimination.*—Now, we investigate another quantum information-processing task: subchannel discrimination, which can also provide an operational interpretation of $C_{\max}$. Subchannel discrimination is an important quantum information task which is used to identify the branch of a quantum evolution to undergo. We consider some special instance of the subchannel discrimination problem to show the advantage of coherent states.

A linear completely positive and trace nonincreasing map $\mathcal{E}$ is called a subchannel. If a subchannel $\mathcal{E}$ is trace preserving, then $\mathcal{E}$ is called a channel. An instrument $\mathfrak{I} = \{\mathcal{E}_a\}_a$ for a channel $\mathcal{E}$ is a collection of subchannels $\mathcal{E}_a$ with $\mathcal{E} = \sum_a\mathcal{E}_a$, and every instrument has its physical realization [40]. A dephasing covariant instrument $\mathfrak{I}^D$ for a DIO $\mathcal{E}$ is a collection of subchannels $\{\mathcal{E}_a\}_a$ such that $\mathcal{E} = \sum_a\mathcal{E}_a$. Similarly, we can define incoherent instrument $\mathfrak{I}^I$ and strictly incoherent instrument $\mathfrak{I}^S$ for channel $\mathcal{E} \in$ IO and $\mathcal{E} \in$ SIO, respectively.

Given an instrument $\mathfrak{I} = \{\mathcal{E}_a\}_a$ for a quantum channel $\mathcal{E}$, let us consider a positive operator valued measurement (POVM) $\{M_b\}_b$ with $\sum_b M_b = \mathbb{I}$. The probability of successfully discriminating the subchannels in instrument $\mathfrak{I}$ by POVM $\{M_b\}_b$ for input state $\rho$ is given by

$$p_{\mathrm{succ}}(\mathfrak{I}, \{M_b\}_b, \rho) = \sum_a \mathrm{Tr}[\mathcal{E}_a(\rho)M_a]. \tag{7}$$

The optimal probability of success in subchannel discrimination of $\mathfrak{I}$ over all POVMs is given by

$$p_{\mathrm{succ}}(\mathfrak{I}, \rho) = \max_{\{M_b\}_b} p_{\mathrm{succ}}(\mathfrak{I}, \{M_b\}_b, \rho). \tag{8}$$

If we restrict the input states to be incoherent ones, then the optimal probability of success among all incoherent states is given by

$$p_{\mathrm{succ}}^{\mathrm{ICO}}(\mathfrak{I}) = \max_{\sigma\in\mathcal{I}} p_{\mathrm{succ}}(\mathfrak{I}, \sigma). \tag{9}$$

We have the following theorem.

*Theorem 2.*—Given a quantum state $\rho$, $2^{C_{\max}(\rho)}$ is the maximal advantage achievable by $\rho$ compared with incoherent states in all subchannel discrimination problems of dephasing-covariant, incoherent, and strictly incoherent instruments:

$$2^{C_{\max}(\rho)} = \max_{\mathfrak{J}} \frac{p_{\mathrm{succ}}(\mathfrak{J},\rho)}{p_{\mathrm{succ}}^{\mathrm{ICO}}(\mathfrak{J})}, \tag{10}$$

where $\mathfrak{J}$ is either $\mathfrak{J}^D$, $\mathfrak{J}^I$, or $\mathfrak{J}^S$, denoting the dephasing-covariant, incoherent, and strictly incoherent instrument, respectively.

The proof of Theorem 2 is presented in Supplemental Material [42]. This result shows that the advantage of coherent states in certain instances of subchannel discrimination problems can be exactly captured by $C_{\max}$, which provides another operational interpretation of $C_{\max}$ and also shows the equivalence among DIO, IO, and SIO in the information-processing task of subchannel discrimination.

*Minimum relative entropy of coherence $C_{\min}(\rho)$.*—Given two operators $\rho$ and $\sigma$ with $\rho \geq 0$, $\mathrm{Tr}[\rho] \leq 1$, and $\sigma \geq 0$, the maximum and minimum relative entropy of $\rho$ relative to $\sigma$ are defined as

$$D_{\min}(\rho||\sigma) \coloneqq -\log \mathrm{Tr}[\Pi_\rho \sigma], \tag{11}$$

where $\Pi_\rho$ denotes the projector onto supp$\rho$, the support of $\rho$. Corresponding to $C_{\max}(\rho)$ defined in (2), we can similarly introduce a quantity defined by the minimum relative entropy:

$$C_{\min}(\rho) \coloneqq \min_{\sigma\in\mathcal{I}} D_{\min}(\rho||\sigma). \tag{12}$$

Since $D_{\min}(\rho||\sigma) = 0$ if supp$\rho$ = supp$\sigma$ [30], we have $\rho \in \mathcal{I} \Rightarrow C_{\min}(\rho) = 0$. However, the converse direction may not be true; for example, let $\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|+\rangle\langle +|$ with $|+\rangle = 1/\sqrt{2}(|1\rangle + |2\rangle)$, and then $\rho$ is coherent but $C_{\min}(\rho) = 0$. Besides, as $D_{\min}$ is monotone under CPTP maps [30], we have $C_{\min}[\Phi(\rho)] \leq C_{\min}(\rho)$ for any $\Phi \in$ IO. However, $C_{\min}$ may increase on average under IO (see Supplemental Material [42]). Thus, $C_{\min}$ is not a proper coherence measure as $C_{\max}$.

Although $C_{\min}$ is not a good coherence quantifier, it still has some interesting properties in the manipulation of coherence. First, $C_{\min}$ gives an upper bound of the maximum overlap with the set of incoherent states for any given quantum state $\rho \in \mathcal{D}(\mathcal{H})$:

$$2^{-C_{\min}(\rho)} \geq \max_{\sigma\in\mathcal{I}} F(\rho,\sigma)^2. \tag{13}$$

Moreover, if $\rho$ is pure state $|\psi\rangle$, then the above equality holds; that is,

$$2^{-C_{\min}(\psi)} = \max_{\sigma\in\mathcal{I}} F(\psi,\sigma)^2. \tag{14}$$

See the proof in Supplemental Material [42].

Moreover, for the geometry of coherence defined by $C_g(\rho) = 1 - \max_{\sigma\in\mathcal{I}} F(\rho,\sigma)^2$ [6], $C_{\min}$ also provides a lower bound for $C_g$ as follows:

$$C_g(\rho) \geq 1 - 2^{-C_{\min}(\rho)}. \tag{15}$$

Now let us consider again the one-shot version of distillable coherence under MIO by modifying and smoothing the minimum relative entropy of coherence $C_{\min}$. We define the one-shot distillable coherence of a quantum state $\rho$ under MIO as

$$C_{D,\mathrm{MIO}}^{(1),\varepsilon}(\rho) \coloneqq \max_{\substack{\mathcal{E}\in\mathrm{MIO}\\ M\in\mathbb{Z}}} \{\log M : F[\mathcal{E}(\rho), |\Psi_+^M\rangle\langle\Psi_+^M|]^2 \geq 1-\varepsilon\},$$

where $|\Psi_+^M\rangle = (1/\sqrt{M})\sum_{i=1}^M |i\rangle$ and $\varepsilon > 0$.

For any $\varepsilon > 0$, we define the smooth minimum relative entropy of coherence of a quantum state $\rho$ as follows:

$$C_{\min}^{\varepsilon}(\rho) \coloneqq \max_{\substack{0\leq A\leq \mathbb{I}\\ \mathrm{Tr}[A\rho]\geq 1-\varepsilon}} \min_{\sigma\in\mathcal{I}} -\log \mathrm{Tr}[A\sigma], \tag{16}$$

where $\mathbb{I}$ denotes the identity. It can be shown that $C_{\min}^{\varepsilon}$ is an upper bound of one-shot distillable coherence,

$$C_{D,\mathrm{MIO}}^{(1),\varepsilon}(\rho) \leq C_{\min}^{\varepsilon}(\rho) \tag{17}$$

for any $\varepsilon > 0$; see the proof in Supplemental Material [42].

The distillation of coherence in the asymptotic limit can be expressed as

$$C_{D,\mathrm{MIO}} = \lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{n} C_{D,\mathrm{MIO}}^{(1),\varepsilon}(\rho).$$

It has been proven that $C_{D,\mathrm{MIO}}(\rho) = C_r(\rho)$ [7]. Here we show that the equality in inequality (18) holds in the asymptotic limit as the $C_{\min}$ is equivalent to $C_r$ in the asymptotic limit. Given a quantum state $\rho \in \mathcal{D}(\mathcal{H})$, then

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{n} C_{\min}^{\varepsilon}(\rho^{\otimes n}) = C_r(\rho). \tag{18}$$

(The proof is presented in Supplemental Material [42].)

We have shown that $C_{\min}$ gives rise to the bounds for maximum overlap with the incoherent states and for one-shot distillable coherence. Indeed, the exact expression of $C_{\min}$ for some special class of quantum states can be calculated. For pure state $|\psi\rangle = \sum_{i=1}^d \psi_i|i\rangle$ with $\sum_{i=1}^d |\psi_i|^2 = 1$, we have $C_{\min}(\psi) = -\log \max_i |\psi_i|^2$. For maximally coherent state $|\Psi\rangle = (1/\sqrt{d})\sum_{j=1}^d e^{i\theta_j}|j\rangle$, we have $C_{\min}(\Psi) = \log d$, which is the maximum value for $C_{\min}$ in $d$-dimensional space.

*Relationship between $C_{\max}$ and other coherence measures.*—First, we investigate the relationship among $C_{\max}$, $C_{\min}$, and $C_r$. Since $D_{\min}(\rho||\sigma) \leq S(\rho||\sigma) \leq D_{\max}(\rho||\sigma)$ for any quantum states $\rho$ and $\sigma$ [30], one has

$$C_{\min}(\rho) \leq C_r(\rho) \leq C_{\max}(\rho). \tag{19}$$

Moreover, as mentioned before, these quantities are all equal in the asymptotic limit.

Above all, $C_{\max}$ is equal to the logarithm of robustness of coherence, as $\mathrm{ROC}(\rho) = \min_{\sigma\in\mathcal{I}}\{s \geq 0 | \rho \leq (1+s)\sigma\}$ and $C_{\max}(\rho) = \min_{\sigma\in\mathcal{I}} \min\{\lambda : \rho \leq 2^{\lambda}\sigma\}$ [25]; that is,

$2^{C_{\max}(\rho)} = 1 + \mathrm{ROC}(\rho)$. Thus, the operational interpretations of $C_{\max}$ in terms of maximum overlap with maximally coherent states and subchannel discrimination can also be viewed as the operational interpretations of ROC. It is known that robustness of coherence plays an important role in a phase discrimination task, which provides an operational interpretation for robustness of coherence [28]. This phase discrimination task investigated in Ref. [28] is just a special case of the subchannel discrimination in dephasing-covariant instruments. Because of the relationship between $C_{\max}$ and ROC, we can obtain the closed form of $C_{\max}$ for some special class of quantum states. As an example, let us consider a pure state $|\psi\rangle = \sum_{i=1}^{d} \psi_i |i\rangle$. Then $C_{\max}(\psi) = \log[(\sum_{i=1}^{d} |\psi_i|)^2] = 2\log(\sum_{i=1}^{d} |\psi_i|)$. Thus, for maximally coherent state $|\Psi\rangle = (1/\sqrt{d}) \sum_{j=1}^{d} e^{i\theta_j} |j\rangle$, we have $C_{\max}(\Psi) = \log d$, which is the maximum value for $C_{\max}$ in $d$-dimensional space.

Since $\mathrm{ROC}(\rho) \le C_{l_1}(\rho)$ [28] and $1 + \mathrm{ROC}(\rho) = 2^{C_{\max}(\rho)}$, then $C_{\max}(\rho) \le \log[1 + C_{l_1}(\rho)]$. We have the relationship among these coherence measures,

$$\begin{aligned} C_{\min}(\rho) \le C_r(\rho) \le C_{\max}(\rho) &= \log[1 + \mathrm{ROC}(\rho)] \\ &\le \log[1 + C_{l_1}(\rho)], \end{aligned}$$

which implies that $2^{C_r(\rho)} \le 1 + C_{l_1}(\rho)$ (see also [27]).

*Conclusion.*—We have investigated the properties of the maximum and minimum relative entropy of coherence, especially the operational interpretation of the maximum relative entropy of coherence. It has been found that the maximum relative entropy of coherence characterizes the maximum overlap with the maximally coherent states under DIO, IO, and SIO, as well as the maximum advantage achievable by coherent states compared with all incoherent states in subchannel discrimination problems of all dephasing-covariant, incoherent, and strictly incoherent instruments, which also provides new operational interpretations of robustness of coherence and illustrates the equivalence of DIO, IO, and SIO in these two operational tasks. The study of $C_{\max}$ and $C_{\min}$ also makes the relationship between the operational coherence measures (e.g., $C_r$ and $C_{l_1}$) more clear. These results may highlight the understanding of the operational resource theory of coherence.

Besides, the relationships among the smooth maximum and minimum relative entropy of coherence and one-shot coherence cost and distillation have been investigated explicitly. As both the smooth maximum and minimum relative entropy of coherence are equal to the relative entropy of coherence in the asymptotic limit and because of the significance of the relative entropy of coherence in the distillation of coherence, further studies are desired on the one-shot coherence cost and distillation.

This work is supported by the Natural Science Foundation of China (Grants No. 11171301, No. 10771191, No. 11571307, and No. 11675113) and the Doctoral Programs Foundation of the Ministry of Education of China (Grant No. J20130061).

*bkf@zju.edu.cn
†uttamsingh@hri.res.in
‡feishm@cnu.edu.cn
§akpati@hri.res.in
∥wjd@zju.edu.cn

[1] V. Giovannetti, S. Lloyd, and L. Maccone, Nat. Photonics **5**, 222 (2011).
[2] M. Lostaglio, K. Korzekwa, D. Jennings, and T. Rudolph, Phys. Rev. X **5**, 021001 (2015).
[3] M. Lostaglio, D. Jennings, and T. Rudolph, Nat. Commun. **6**, 6383 (2015).
[4] T. Baumgratz, M. Cramer, and M. B. Plenio, Phys. Rev. Lett. **113**, 140401 (2014).
[5] D. Girolami, Phys. Rev. Lett. **113**, 170401 (2014).
[6] A. Streltsov, U. Singh, H. S. Dhar, M. N. Bera, and G. Adesso, Phys. Rev. Lett. **115**, 020403 (2015).
[7] A. Winter and D. Yang, Phys. Rev. Lett. **116**, 120404 (2016).
[8] N. Killoran, F. E. S. Steinhoff, and M. B. Plenio, Phys. Rev. Lett. **116**, 080402 (2016).
[9] E. Chitambar, A. Streltsov, S. Rana, M. N. Bera, G. Adesso, and M. Lewenstein, Phys. Rev. Lett. **116**, 070402 (2016).
[10] E. Chitambar and G. Gour, Phys. Rev. Lett. **117**, 030401 (2016).
[11] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[12] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).
[13] A. K. Pati, Phys. Rev. A **63**, 014302 (2000).
[14] C. H. Bennett, D. P. DiVincenzo, P. W. Shor, J. A. Smolin, B. M. Terhal, and W. K. Wootters, Phys. Rev. Lett. **87**, 077902 (2001).
[15] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[16] S. D. Bartlett, T. Rudolph, and R. W. Spekkens, Rev. Mod. Phys. **79**, 555 (2007).
[17] G. Gour and R. W. Spekkens, New J. Phys. **10**, 033023 (2008).
[18] G. Gour, I. Marvian, and R. W. Spekkens, Phys. Rev. A **80**, 012307 (2009).
[19] I. Marvian, Ph.D. thesis, University of Waterloo, 2012.
[20] I. Marvian and R. W. Spekkens, New J. Phys. **15**, 033001 (2013).
[21] I. Marvian and R. W. Spekkens, Phys. Rev. A **90**, 062110 (2014).
[22] I. Marvian and R. W. Spekkens, Nat. Commun. **5**, 3821 (2014).
[23] F. G. S. L. Brandão, M. Horodecki, J. Oppenheim, J. M. Renes, and R. W. Spekkens, Phys. Rev. Lett. **111**, 250404 (2013).
[24] R. Gallego and L. Aolita, Phys. Rev. X **5**, 041008 (2015).
[25] E. Chitambar and G. Gour, Phys. Rev. A **94**, 052336 (2016).
[26] K. Bu and C. Xiong, arXiv:1604.06524.

[27] S. Rana, P. Parashar, A. Winter, and M. Lewenstein, arXiv:1612.09234.
[28] C. Napoli, T. R. Bromley, M. Cianciaruso, M. Piani, N. Johnston, and G. Adesso, Phys. Rev. Lett. **116**, 150502 (2016).
[29] M. Piani, M. Cianciaruso, T. R. Bromley, C. Napoli, N. Johnston, and G. Adesso, Phys. Rev. A **93**, 042107 (2016).
[30] N. Datta, IEEE Trans. Inf. Theory **55**, 2816 (2009).
[31] N. Datta, Int. J. Quantum. Inform. **07**, 475 (2009).
[32] F. G. S. L. Brandao and N. Datta, IEEE Trans. Inf. Theory **57**, 1754 (2011).
[33] F. Buscemi and N. Datta, IEEE Trans. Inf. Theory **56**, 1447 (2010).
[34] R. Renner and S. Wolf, in *Proceedings of the International Symposium on Information Theory* (IEEE, Chicago, IL, 2004), p. 233.
[35] R. Renner, *Security of Quantum Key Distribution* (ETH Zurich, 2005).
[36] J. M. Renes and R. Renner, IEEE Trans. Inf. Theory **58**, 1985 (2012).
[37] M. Horodecki, J. Oppenheim, and A. Winter, Commun. Math. Phys. **269**, 107 (2007).
[38] H. Buhrman, M. Christandl, P. Hayden, H.-K. Lo, and S. Wehner, Phys. Rev. Lett. **97**, 250501 (2006).
[39] R. Konig, R. Renner, and C. Schaffner, IEEE Trans. Inf. Theory **55**, 4337 (2009).
[40] M. Piani and J. Watrous, Phys. Rev. Lett. **114**, 060404 (2015).
[41] M. Piani and J. Watrous, Phys. Rev. Lett. **102**, 250501 (2009).
[42] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.119.150405 for the details of the proof, which includes Refs. [43–53].
[43] V. Vedral and M. B. Plenio, Phys. Rev. A **57**, 1619 (1998).
[44] L. Vandenberghe and S. Boyd, SIAM Rev. **38**, 49 (1996).
[45] J. Watrous, Theory Comput. **5**, 217 (2009).
[46] F. G. Brandao and K. Svore, arXiv:1609.05537.
[47] Y. Peng, Y. Jiang, and H. Fan, Phys. Rev. A **93**, 032326 (2016).
[48] K. M. R. Audenaert, J. Phys. A **40**, 8127 (2007).
[49] T. Ogawa and H. Nagaoka, IEEE Trans. Inf. Theory **46**, 2428 (2000).
[50] S. Du, Z. Bai, and X. Qi, arXiv:1504.02862.
[51] M. Hayashi, *Quantum Information: An Introduction* (Springer, New York, 2006).
[52] F. G. S. L. Brandão and M. B. Plenio, Commun. Math. Phys. **295**, 791 (2010).
[53] F. G. S. L. Brandão and G. Gour, Phys. Rev. Lett. **115**, 070503 (2015).
[54] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2010).