Investigating Anisotropic Quantum Hall States with Bimetric Geometry

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We construct a low energy effective theory of anisotropic fractional quantum Hall (FQH) states. We develop a formalism similar to that used in the bimetric approach to massive gravity, and apply it to describe Abelian anisotropic FQH states in the presence of external electromagnetic and geometric backgrounds. We derive a relationship between the shift, the Hall viscosity, and a new quantized coupling to anisotropy, which we term anisospin. We verify this relationship by numerically computing the Hall viscosity for a variety of anisotropic quantum Hall states using the density matrix renormalization group. Finally, we apply these techniques to the problem of nematic order and clarify certain disagreements that exist in the literature about the meaning of the coefficient of the Berry phase term in the nematic effective action.

Introduction.—In recent years there have been a plethora of new advancements in the physics of the fractional quantum Hall (FQH) effect. Notably, several related developments involving the interplay of quantum Hall physics and geometry have emerged. First, it was recently understood that the response of a quantum Hall state to variations of the background spatial geometry reveals universal properties of the state that go beyond topological effective theory [1–17]. A particularly interesting quantity is the Hall viscosity [18–20], which in rotationally invariant systems is related to the shift [1,19,21], and is given by a Berry phase accumulated by the quantum Hall wave function on a torus under adiabatic changes of the aspect ratio. When rotational invariance is broken, the Hall viscosity becomes a multicomponent tensor [22]; however, its properties and relation to Berry phases have not yet been understood.

Additionally, there has been a flurry of recent experimental [23–26] and theoretical [27–31] interest in quantum Hall states with spontaneously broken rotational symmetry, i.e., nematic quantum Hall states [32,33]. Because the nematic order parameter is described by a symmetric matrix, it couples to the microscopic degrees of freedom in a way similar to the background spatial metric [29,30]. In the isotropic phase, fluctuations of the nematic order parameter are massive and describe the dynamics of the angular momentum two gapped Girvin-Macdonald-Platzman (GMP) [34] magnetoroton mode [29]. In the symmetry broken phase the fluctuations of the order parameter are gapless (up to lattice and boundary effects).

In this Letter, we develop a unifying formalism that bridges the chasm between these new areas of quantum Hall physics. We explain how to construct a low energy effective theory of quantum Hall states with quadrupolar anisotropy, coupled to perturbations of both the electromagnetic field and spatial geometry. Our construction is reminiscent of a bimetric theory of massive gravity [35]. The first metric is determined by the geometry of

space, while the second metric is determined by the anisotropy. Note, however, that we treat both metrics as nondynamical background fields. We use this formalism to derive (the nondissipative parts of) linear response coefficients in the presence of anisotropy. We also introduce a new response function that probes the coupling of a quantum Hall state to anisotropy. In order to verify our model, we numerically compute the Hall viscosity for a variety of anisotropic quantum Hall states. Our construction is also well suited to describe the nematic quantum Hall states in the isotropic phase and with a quenched configuration of the nematic order parameter as discussed in Supplemental Material [36].

Geometry.—We start with a brief review of the geometry relevant in quantum Hall physics. Spatial geometry is described by a set of vielbeins, or frame fields, $e_\mu^A = \{e_0^A \equiv 0, e_i^A\}$ along with their "inverses" E_A^μ [40]. Here and throughout we use $\mu, \nu = 0, 1, 2,$ and $i, j = 1, 2$ to index ambient spacetime and space, respectively, while $A, B = 1, 2$ index flat internal space. The spatial metric g_{ij} is given as

$$g_{ij} = e_i^A e_j^B \delta_{AB}. \tag{1}$$

Parallel transport in spacetime is defined by demanding that the vielbeins are covariantly constant, i.e.,

$$\nabla_\mu e_\nu^A = \partial_\mu e_\nu^A - \Gamma_{\mu\nu}^\lambda e_\lambda^A + \omega_{\mu B}^A e_\nu^B = 0, \tag{2}$$

where ∇_μ is a covariant derivative with a spacetime index. These equations define both a spin connection $\omega_{\mu B}^A$ and a Christoffel connection $\Gamma_{\mu j}^i$. The Christoffel connection can be expressed in terms of derivatives of the metric, although we do not need the explicit expression here. Solving Eq. (2) for the spin connection, we find [41]

$$\omega_0 = \frac{1}{2}\epsilon_A{}^B(E_B^i \partial_0 e_i^A), \tag{3}$$

$$\omega_k = \frac{1}{2}\epsilon_A{}^B(E_B^i\partial_k e_i^A - E_B^i e_j^A\Gamma_{ki}^j), \qquad (4)$$

where we have defined the Abelian spin connection via $\omega_{\mu B}^A \equiv \epsilon_B{}^A\omega_\mu$, where $\epsilon_B{}^A$ is the Levi-Civita tensor.

Lastly, we review the transformation laws for these geometric fields. First, we note that the metric, the vielbeins, and the spin connection all transform as tensors under changes in the ambient coordinates (here we restrict to transformations that leave time invariant). Next, since the vielbeins are defined through the factorization Eq. (1), they suffer an $SO(2)$ gauge ambiguity

$$e_\mu^A \rightarrow e_\mu^{A'}S_{A'}{}^A, \qquad E_A^\mu \rightarrow E_{A'}^\mu S^{A'}{}_A, \qquad (5)$$

for $S = \exp(i\varphi\epsilon) \in SO(2)$; the spin connection transforms under rotations as an Abelian gauge field

$$\omega_\mu \rightarrow \omega_\mu + \partial_\mu\varphi. \qquad (6)$$

Anisotropic geometry.—Anisotropy naturally arises in condensed matter systems through symmetric rank two tensors, such as the effective mass tensor or dielectric tensor in crystals. Taking inspiration from this, we introduce anisotropy into the quantum Hall system through a symmetric tensor V, distinct from the spatial metric tensor g. To be consistent, we must be careful to account for the difference between spatial geometry—which we view as extrinsically imposed—and anisotropy—which we view as intrinsic. Our discussion here elaborates on and extends various observations made in Refs. [29,30] and is the first result of the Letter.

We choose a fairly general type of anisotropy parametrized by a quadrupolar background field $V^{AB}(\mathbf{x})$, which we take to have internal $SO(2)$ indices. We require that V^{AB} is symmetric and positive definite. We also define the inverse matrix v_{AB} satisfying

$$v_{AB}V^{BC} = \delta_A^C. \qquad (7)$$

Without loss of generality we can fix $\det V = 1$; changes to the determinant of V can be compensated by a uniform rescaling of coordinates, which would not introduce any anisotropy. In analogy with the spatial metric, we can factorize V and v as

$$V^{AB} = \Lambda_\alpha^A\Lambda_\beta^B\delta^{\alpha\beta}, \qquad v_{AB} = \lambda_A^\alpha\lambda_B^\beta\delta_{\alpha\beta}. \qquad (8)$$

Note that the indices $\alpha, \beta = 1, 2$ appearing in Eq. (8) are a new type of internal index. Rotations acting on this index are a new gauge redundancy, distinct from the internal $SO(2)$ rotational symmetry of the previous section. In order to distinguish between these two gauge groups, we refer to the new redundancy in the description of anisotropy as $\widehat{SO}(2)$.

It is natural to define an anisotropy metric

$$\hat{g}_{ij} \equiv e_i^A e_j^B v_{AB} = \delta^{\alpha\beta}e_i^A\lambda_A^\alpha e_j^B\lambda_B^\beta = \delta^{\alpha\beta}\hat{e}_i^\alpha\hat{e}_j^\beta, \qquad (9)$$

where we have introduced $e_i^A\lambda_A^\alpha = \hat{e}_i^\alpha$. We similarly define the inverse

$$\hat{G}^{ij} \equiv \hat{E}_A^i\hat{E}_B^j\delta^{AB}. \qquad (10)$$

With two metrics around, we must be careful to distinguish between tensor fields and their inverses. We use the convention that for the spatial metric only $g^{ij}g_{jk} = \delta_k^i$. Spatial indices are raised and lowered by this metric, while internal indices A and α are both raised and lowered by δ symbols. It would be a grave error to use $\hat{g}$ or $\hat{G}$ to manipulate indices.

The anisotropy data $\hat{g}$ and $\hat{e}$ can be used to construct connections and curvatures, just like their geometric relatives from the previous section. Any description of an anisotropic system in terms of $\hat{e}$ with fully contracted indices will automatically be spatially covariant. In particular, we may define a hat-covariant derivative $\hat{\nabla}$ satisfying

$$\hat{\nabla}_\mu\hat{g}_{ij} = 0, \qquad \hat{\nabla}_\mu\hat{e}_j^\alpha = 0. \qquad (11)$$

This defines for us implicitly an affine connection $\hat{\Gamma}$, as well as an $\widehat{SO}(2)$ spin connection $\hat{\omega}$ given by replacing all factors of the metric and vielbeins in Eqs. (3)–(4) with their hatted cousins. Clearly, $\hat{\omega}$ transforms as an Abelian $\widehat{SO}(2)$ gauge field under rotations in the internal $\{\alpha, \beta\}$ space, in analogy with Eq. (6).

Given these two geometries, we define a matrix-valued one-form $C_{\mu j}^i$,

$$C_{\mu j}^i = \Gamma_{\mu j}^i - \hat{\Gamma}_{\mu j}^i. \qquad (12)$$

We also define $C_\mu = \epsilon_i^j C^i{}_{j\mu}$ for future use. There are no more independent objects.

The cohomology class (or, informally, the Chern number) $\hat{\chi} = [1/(2\pi)]\int d\hat{\omega}$ is not independent of the Euler characteristic χ. We find

$$\hat{\chi} = \frac{1}{4\pi}\int \sqrt{\hat{g}}\hat{R} = \chi + N_{\text{discl}}, \qquad (13)$$

for some integer N_{discl}. Indeed, taking $V^{AB} = \delta^{AB}$ we have $\hat{\omega}_\mu = \omega_\mu$, and so $\hat{\chi} = \chi$. On the other hand, when the metric (can be and) is set to identity $g_{ij} = \delta_{ij}$ we find

$$N_{\text{discl}} = \frac{1}{2\pi}\int d\hat{\omega}\big|_{g_{ij}=\delta_{ij}}, \qquad (14)$$

where N_{discl} counts the number of singularities of the anisotropic connection. When V^{AB} comes from a nematic order parameter, this integer is related to the number of nematic disclination defects.

Anisotropic Chern-Simons theory.—We now consider a generic Abelian, anisotropic one-component FQH system in a curved space, coupled to a weak external electromagnetic field. The low energy theory for such a phase is a $U(1)_k$ Chern-Simons action coupled to our anisotropy connections, with $k = 2p + 1$. Note that an anisotropy tensor V^{AB} can be generated dynamically from the interplay between the dielectric tensor, band-mass curvature, in-plane magnetic field, quadrupolar interactions, etc. The only assumption we make is that such a V^{AB} exists. Although we are primarily interested in cases where the spatial metric is flat or nearly flat, we must first formulate

the theory in a general background, so as not to miss any allowed couplings, or introduce prohibited ones.

Given our previous discussions, the most general low energy effective action is [42]

$$S = \frac{2p+1}{4\pi} \int a da - \frac{1}{2\pi} \int \mathcal{A} da,$$ (15)

where

$$\mathcal{A}_\mu = A_\mu + s\omega_\mu + \varsigma\hat{\omega}_\mu + \xi C_\mu.$$ (16)

The coefficients s and ς must be quantized because ω and $\hat{\omega}$ are connections (as opposed to one-forms); however, ξ can be an arbitrary function of the anisotropy. For small anisotropy we expect ξ to be approximately independent of the anisotropy, and we focus on this situation throughout the remainder of the text. We note that a nonzero ξ explicitly breaks the apparent symmetry between the ambient and anisotropy metrics, and cannot be excluded on the basis of effective field theory.

Supplementing the action (15) with an appropriate gauge-fixing condition [8], we integrate out a to derive the generating functional [43]

$$W = \frac{\nu}{4\pi} \int A dA + \frac{\nu s}{2\pi} \int A d\omega + \frac{\nu\varsigma}{2\pi} \int A d\hat{\omega} + \frac{\nu\xi}{2\pi} \int A dC,$$ (17)

where $\nu = 1/(2p+1)$ and we have dropped purely gravitational terms. The electric charge density is given by

$$\rho = \frac{\nu}{2\pi} B + \frac{\nu s}{4\pi} R + \frac{\nu\varsigma}{4\pi} \hat{R} + \frac{\nu\xi}{4\pi} \epsilon^{ij}\partial_i C_j,$$ (18)

which implies the total particle number on a sphere

$$\nu^{-1} N = N_\phi + \mathcal{S} + \varsigma N_{\text{discl}},$$ (19)

where $\mathcal{S} = 2\bar{s} = 2(s + \varsigma)$ is the shift [1,21]. We see that anisotropy provides a natural way to split the mean orbital spin $\bar{s}$ into two parts: one that comes from the geometric spin and another one that couples to anisotropy. Thus we refer to ς as anisospin by analogy. To remind the reader that the anisospin bears a resemblance to the ordinary orbital spin, we denote it by ς, the Greek "final sigma."

Hall viscosity and response to anisotropy.—We next consider the response of the stress tensor to applied strains, defined from the generating functional as

$$\tau^\mu{}_A = \frac{\delta W}{\delta e^A_\mu} = \lambda^\mu{}_A{}^\lambda{}_B e^B_\lambda + \eta^\mu{}_A{}^\lambda{}_B \partial_0 e^B_\lambda.$$ (20)

We focus on the nondissipative Hall viscosity

$$(\eta^H)^\mu{}_A{}^\lambda{}_B = \frac{1}{2}(\eta^\mu{}_A{}^\lambda{}_B - \eta^\lambda{}_B{}^\mu{}_A).$$ (21)

In the rotationally invariant case, it has one independent component $\eta^H = \mathcal{S}\bar{\rho}/4$, proportional to the shift [19,22], and the average density $\bar{\rho}$. In the presence of anisotropy however, two new—and in general nonuniversal—contributions to the viscosity tensor emerge. To study these, we follow Haldane and introduce the contracted Hall tensor [44,45]

$$\eta^H_{AB} = \frac{1}{2}\epsilon^{CD} e^{A'}_\mu e^{B'}_\nu \epsilon_{A'A}\epsilon_{B'B}\eta^\mu{}_C{}^\nu{}_D.$$ (22)

From the generating functional Eq. (17) we find [46]

$$\eta^H_{AB} = \frac{\bar{\rho}}{2}[s\delta_{AB} + \varsigma v_{AB} + \xi(v_{AC}v_{CB} - \delta_{AB})].$$ (23)

In the isotropic limit this reduces to $\eta^H_{AB} = \eta^H \delta_{AB}$. We see that the Hall viscosity and the shift are only proportional in the special cases that either the anisotropy, or both ς and ξ, vanish. The contributions from ξ and ς can be distinguished through their scaling with V. In Supplemental Material [36], we use the Kubo formalism to derive the Hall tensor η^H for a microscopic model of noninteracting electrons with band-mass anisotropy.

The anisospin ς can also be calculated independent of s via the response to anisotropy. To see this, we define an "anisotropy current" [30]

$$\mathcal{N}^A_\alpha = \frac{1}{2\lambda} \frac{\delta W}{\delta \lambda^\alpha_A},$$ (24)

where $\lambda = \det(\lambda^\alpha_A)$. Following the logic of Eqs. (17)–(23), we find for the (contracted) odd part of the response of $\mathcal{N}$ to $\partial_0\lambda^\alpha_A$,

$$\vartheta^H_{AB} = \frac{\bar{\rho}}{2}(\varsigma + \xi)v_{AB}.$$ (25)

Note that $\vartheta^H_{\alpha\beta}$ contains only ς and ξ, but not s.

Anisospin for realistic systems.—Next, let us consider the case where anisotropy enters through the band-mass tensor m^{-1}_{ij}, and through a distortion of the interaction potential.

$$H = m^{-1}_{ij}\pi_i\pi_j + U(|x - x'|; \epsilon_{ij}),$$ (26)

where π_i is the momentum (independent of the anisotropy) and $U(|x - x'|; \epsilon_{ij})$ is the Coulomb potential in a medium with a homogeneous—but not necessarily isotropic—dielectric tensor ϵ_{ij} [47,48]. We assume that these tensors are diagonal and unimodular, with $m^{-1}_{ij} = \text{diag}[\alpha_m, (1/\alpha_m)]$ and $\epsilon_{ij} = \text{diag}[\alpha_\epsilon, (1/\alpha_\epsilon)]$.

To simplify the problem we can make a global coordinate rescaling to move all of the anisotropy into the interaction. We are then left in Eq. (23) with a single matrix $v_{ij} = \epsilon_{ik}m_{kj} = \text{diag}[(\alpha_\epsilon/\alpha_m), (\alpha_m/\alpha_\epsilon)]$. Next, note that each cyclotron orbit in the Nth Landau level carries orbital angular momentum

$$s_N = (2N - 1)/2$$ (27)

about its guiding center, an effect that originates from the now-isotropic kinetic term. Hence, for FQH states in the Nth Landau level we expect for the geometric spin $s = s_N$. This implies

$$\varsigma = \mathcal{S}_N/2 - s_N,$$ (28)

where $\mathcal{S}_N$ is the shift for the state in the Nth Landau level. We see that the shift $\mathcal{S}$ decomposes into cyclotron and interaction contributions.

Alternatively, we could have rescaled the interaction to move the anisotropy into the band-mass tensor. This would lead to a different matrix $v'_{ij} = m^{-1}_{ik}\epsilon^{-1}_{kj} = \text{diag}[(\alpha_m/\alpha_\epsilon),$

$(\alpha_\varepsilon/\alpha_m)$]. However, note that the anisotropy is a coordinate on the real projective line $\mathbb{R}P^1 \approx S^1$, since the overall scale of the Hamiltonian is unimportant. The two rescalings v and v' correspond to two different coordinate patches covering $\mathbb{R}P^1$. In the first case the coordinate is $\alpha = (\alpha_m/\alpha_\varepsilon)$, while in the second case it is $\alpha' = 1/\alpha$. Either coordinate choice is valid away from the poles of $\mathbb{R}P^1$, and so both parametrizations produce equivalent results for all observable quantities. The effect of moving all of the anisotropy into the mass tensor results in swapping the values of s and ς, which is consistent with the transformation law of the Hall tensor Eq. (23) under coordinate rescalings.

Anisotropic momentum polarization.—We have numerically calculated the Hall viscosity for a variety of anisotropic quantum Hall states produced from (26), using DMRG on an infinite cylinder. In this geometry, the Hall viscosity is related to the momentum polarization P_{pol}, which is the additional momentum in the azimuthal (x_2) direction when the cylinder is cut in the x_1 direction [49–51]. For anisotropic systems,

$$P_{\text{pol}} = -\frac{\eta_{22}^H}{2\pi}L^2 + O(1). \qquad (29)$$

The coefficient η_{22}^H is given by Eq. (23). For the anisotropic systems considered here, the extensive part of the momentum polarization depends on both the orbital spin s and the anisospin ς. The $O(1)$ constant is related to the central charge [50]; studying its response to the anisotropy is more computationally demanding than the L^2 term and is an interesting direction for future work.

When a quantum Hall problem projected into a single Landau level is written in a second-quantized basis, interaction anisotropy and mass anisotropy have an identical effect on the matrix elements of the Hamiltonian, and therefore lead to identical ground states in the orbital basis. Therefore, we can test both types of anisotropy in (26) in a single simulation. To compute momentum polarization from these states we first compute the real space entanglement spectrum (RSES) across the cut, and then average the momentum eigenvalues of all the levels in the RSES, weighted by their entanglement eigenvalues. The RSES depends on the shapes of the single-particle orbitals (which are modified by mass anisotropy but not by interaction anisotropy), so the two types of anisotropies give different results even though the orbital basis wave functions are identical. Additionally, for interaction anisotropy we can compute momentum polarization from Eq. (8b) of Ref. [50], which for isotropic single-particle orbitals gives equivalent results for less computational effort.

We compute the Hall viscosity by fitting the computed momentum polarization vs circumference L for a number of different system sizes. In Fig. 1 we show results for the integer quantum Hall effect with $\nu = 1, 2$, the Laughlin state with $\nu = 1/3$, and the hierarchy state at $\nu = 2/5$. The solid curves are fits to Eq. (23), where s and ς are given by Eqs. (27)
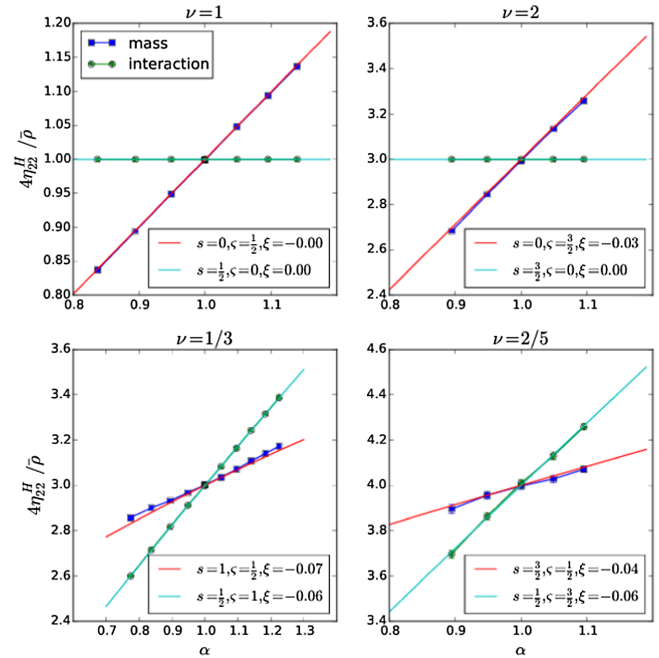

FIG. 1.   Hall viscosity η_{22}^H as a function of anisotropy α, for four different quantum Hall states. Data are obtained by introducing anisotropy into either the mass (blue squares) or interaction (green circles) part of the Hamiltonian. The lines correspond to Eq. (23), using the values of s and ς given in Eqs. (27) and (28), but allowing ξ to fluctuate to fit the data. The value at $\alpha = 1$ is the shift ($\mathcal{S}_N$). Data was obtained using system sizes $L = 10$–20 and bond dimensions up to 5400. The data are plotted such that in the isotropic case they are equal to the shift.

and (28), and ξ is allowed to vary. We find in all cases that the best fit occurs for $-0.1 \leq \xi \leq 0$. Finite-circumference effects introduce anisotropy-dependent oscillations in P_{pol}. The values of ξ we extract may therefore be overestimates, since they could reflect these systematic errors. Reducing these finite-size effects would require larger bond dimensions and would (with our present computational resources) simply replace finite-size error with finite-bond-dimension error (finite-bond-dimension error is very small in the data we present). A finite-size scaling analysis (presented in Supplemental Material [36]), suggests that ξ is small but nonzero. We have also assumed that ξ is independent of λ, though this is not required. This, along with finite-size effects, may explain the deviations from the fit we observe at large anisotropy in $\nu = 1/3$.

We thus see that the effect of anisotropy that couples only to either the kinetic energy, or to the interaction potential, is to split the contributions to the shift $\mathcal{S}$ into a single-particle "Landau orbit" contribution, and a many-body "guiding center" contribution. Such a splitting was first noted by Haldane [45,52]. The values of s and ς obtained are precisely those suggested in the previous section [53].

Conclusions.—We have introduced a framework for studying the low energy properties of anisotropic quantum Hall states. Using it, we constructed a family of low energy

theories for anisotropic Abelian FQH states, and studied their linear response. We have found a new quantized topological number ς, dubbed anisposin, related to the nondissipative linear response to anisotropy. We have shown that in the presence of homogeneous anisotropy the relation between the shift and the Hall viscosity is modified—while the former remains quantized for any value of the anisotropy (as long as it preserves the inversion symmetry), the Hall viscosity is quantized only in the isotropic case.

We have numerically investigated the Hall viscosity of a variety of quantum Hall states coupled to both band-mass and interaction anisotropy. We have shown that the aniso-spin for these systems realizes a splitting of the shift between Landau orbit and interaction contributions, first pointed out in Ref. [45].

We believe that our formalism will have many applications, including a detailed investigation of the dynamics of gapped collective excitations in FQH systems, nematic phase transitions, and "hidden" geometric degrees of freedom [52]. The correspondence between anisotropy and bimetric geometry allows one to construct anisotropic CFT trial states and study corresponding Berry phases, which we discuss in a forthcoming work. Finally, our geometric description may help to build a bridge between FQH physics and bimetric theories of massive gravity.

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