Quantifying Genuine Multipartite Correlations and their Pattern Complexity

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We propose an information-theoretic framework to quantify multipartite correlations in classical and quantum systems, answering questions such as what is the amount of seven-partite correlations in a given state of ten particles? We identify measures of genuine multipartite correlations, i.e., statistical dependencies that cannot be ascribed to bipartite correlations, satisfying a set of desirable properties. Inspired by ideas developed in complexity science, we then introduce the concept of weaving to classify states that display different correlation patterns, but cannot be distinguished by correlation measures. The weaving of a state is defined as the weighted sum of correlations of every order. Weaving measures are good descriptors of the complexity of correlation structures in multipartite systems.

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Introduction.—Statistical relations in measurement outcomes, i.e., correlations, are powerful tools to investigate multipartite systems, employed in (quantum) information theory, statistical mechanics, condensed matter theory, network theory, neuroscience, and complexity science [1–4]. Correlations describe global properties that cannot be inferred from the features of the system parts, e.g., phases of many-body systems [5]. They are also resources. Entanglement, a kind of quantum correlation, enables speed-up in quantum information processing [6].

Yet, the very notion of genuine multipartite correlations still generates discussion [7]. There is no consistent way to quantify dependencies that do not manifest bipartite correlations, encoding joint properties of $k > 2$ particles instead, while witnesses of multipartite entanglement of at least order k have been proposed [8–15]. A further problem is that computing correlations is not always sufficient to fully describe multipartite correlation patterns. Equally correlated networks of multivariate variables can display different structures and properties [16,17]. Also, quantum states can be correlated in inherently inequivalent ways [18–20].

Here we propose a framework to describe genuine multipartite correlations in classical and quantum systems. We identify distance-based measures that satisfy a set of desirable properties when parts of the systems are added or discarded, and local operations are performed. We show that adopting the relative entropy allows for simplifying computations and meeting even stronger constraints. We then introduce the notion of weaving to classify multipartite states by studying how correlations scale with their order. The weaving of a state is given by the weighted sum of genuine multipartite correlations of any order, inheriting the properties of correlation measures. We compute the weaving of correlated classical and quantum states. In such cases, states that have equal total correlations or highest order correlations, but display a different correlation pattern, take different weaving values.

Quantifying genuine multipartite correlations.—A finite dimensional N-partite quantum system $\mathcal{S}_N = \{\mathcal{S}_{[1]}, \mathcal{S}_{[2]}, \ldots, \mathcal{S}_{[N]}\}$ is described by a density matrix ρ_N, $\rho_{[i]}$ being the states of the subsystems $\mathcal{S}_{[i]}, i = 1,2,\ldots,N$. In this framework, classical probability distributions $p_{\alpha_1,\ldots,\alpha_N}$ of N-variate discrete variables are embedded in density matrices of the form $\sum_{\alpha_1,\ldots,\alpha_N} p_{\alpha_1,\ldots,\alpha_N} |\alpha_1,\ldots,\alpha_N\rangle \times \langle \alpha_1,\ldots,\alpha_N|, \sum_{\alpha_1,\ldots,\alpha_N} p_{\alpha_1,\ldots,\alpha_N} = 1$, where $\{\alpha_i\}$ are orthonormal basis elements in the Hilbert spaces of each subsystem $\mathcal{S}_{[i]}$. The correlations in $\mathcal{S}_N$ depend on the tensor product structure of its Hilbert space, induced by the partition $\{\mathcal{S}_{[i]}\}$. This is usually dictated by physical constraints, e.g., spatial separation of the subsystems. Indeed, even maximally entangled states are factorizable by changing the system structure [21,22]. The total correlations in the system represent the information encoded in ρ_N, which is unaccessible to an observer knowing the states of each subsystem, $\rho_{[i]}$. We extend the argument to define genuine multipartite correlations of order higher than k, $2 \leq k \leq N-1$, as the missing information to a more informed observer, who knows the states ρ_{k_j} of clusters forming a coarse grained partition $\{\mathcal{S}_{k_1}, \mathcal{S}_{k_2}, \ldots, \mathcal{S}_{k_m}\}$, $\sum_{j=1}^{m} k_j = N$, $k_j \leq k$, where each cluster $\mathcal{S}_{k_j}$ includes up to k subsystems, e.g., $\mathcal{S}_{k_1} = \{\mathcal{S}_{[1]}, \mathcal{S}_{[2]}, \ldots, \mathcal{S}_{[k_1]}\}$. Genuine N-partite correlations, the highest order, are the information that is still missing when clusters including subsets of up to $N-1$ subsystems are accessible, $k_j \leq N-1$.

The set of states describing clusters of up to k subsystems is

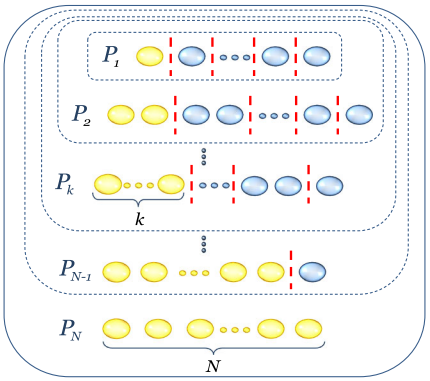

FIG. 1. Multipartite correlation hierarchy. Given a system of N particles (blue spheres), the sets P_k, $k = 1, 2, \ldots, N$, consist of states displaying up to k-partite correlations. The yellow k spheres identify the largest subset of a coarse grained partition (the dashed red lines separate each cluster). The amount of genuine k-partite correlations in a state is the difference between the distance to the sets P_{k-1} and P_k.

$$P_k := \left\{ \sigma_N = \otimes_{j=1}^{m} \sigma_{k_j}, \sum_{j=1}^{m} k_j = N, k = \max\{k_j\} \right\}. \quad (1)$$

For example, given $N = 3$, the set P_1 consists of the product states $\otimes_{j=1}^{3} \sigma_{[j]}$, P_2 includes P_1 and the products of bipartite and single-site states, i.e., $\sigma_2 \otimes \sigma_1$ and their permutations, while P_3 contains P_2 and the nonfactorizable density matrices σ_3. The complete chain reads $P_1 \subset P_2 \subset \ldots \subset P_{N-1} \subset P_N$, where P_N is the Hilbert space of the global system (Fig. 1). Note that a pure state in P_k is a k-producible state [23]. Genuine multipartite correlations of order higher than k are then quantified by the distance of the global state to the set P_k,

$$D^{k\to N}(\rho_N) := \min_{\sigma \in P_k} D(\rho_N, \sigma), \quad (2)$$

where the function D is non-negative, $D(\rho, \sigma) \geq 0$, $D(\rho, \sigma) = 0 \Leftrightarrow \rho = \sigma$, and contractive under completely positive and trace preserving (CPTP) maps Φ, $D(\rho, \sigma) \geq D(\Phi(\rho), \Phi(\sigma))$, $\forall \rho, \sigma$. Then, any distance identifies a measure of k-partite correlations,

$$D^k(\rho_N) := D^{k-1\to N}(\rho_N) - D^{k\to N}(\rho_N). \quad (3)$$

As expected, the total correlations are given by the distance to the set of statistically independent N-partite states, which equals the sum of the correlations of any order, $D^{1\to N}(\rho_N) = \min_{\sigma \in P_1} D(\rho_N, \sigma) = \sum_k D^k(\rho_N)$. For example, the genuine bipartite correlations in a tripartite state are computed as the difference between total correlations and genuine tripartite correlations, $D^2(\rho_3) = D^{1\to 3}(\rho_3) - D^{2\to 3}(\rho_3)$, which are nonzero if and only if the state is not factorizable with respect to any bipartite cut.

The minimization in Eq. (2) is cumbersome for a generic distance D, but significantly simplified by employing the relative entropy $S(\rho||\sigma) = -S(\rho) - \mathrm{Tr}(\rho \log \sigma)$, $\forall \rho, \sigma$: $\mathrm{supp}\rho \subseteq \mathrm{supp}\sigma$, ∞ otherwise, $S(\rho) = -\mathrm{Tr}(\rho \log \rho)$. In such a case, the closest product state to the global state is the product of its marginals, $\min_{\otimes_{i=1}^{m} \sigma_{k_i}} S(\rho_N || \otimes_{i=1}^{m} \sigma_{k_i}) = S(\rho_N || \otimes_{i=1}^{m} \rho_{k_i}) = \sum_{i=1}^{m} S(\rho_{k_i}) - S(\rho_N)$, $\mathrm{Tr}_{N-k}\rho_N = \rho_k$ [13,24]. Correlations of order higher than k are then given by the distance to $\tilde{P}_k \subset P_k$, $\tilde{P}_k := \{\otimes_{j=1}^{m} \rho_{k_j}, \sum_{j=1}^{m} k_j = N, k = \max\{k_j\}\}$. Therefore, the genuine k-partite correlations are measured by

$$S^k(\rho_N) = S^{k-1\to N}(\rho_N) - S^{k\to N}(\rho_N). \quad (4)$$

For systems invariant under subsystem permutations, the subadditivity of the von Neumann entropy, $S(\rho_i) + S(\rho_j) \geq S(\rho_{ij})$, makes the closest product state $\tilde{\rho}_N^k$ the most "compact" one, being the tensor product state of $\lfloor N/k \rfloor$ clusters of k subsystems and a cluster of $N - \lfloor N/k \rfloor k = N \mod k$ subsystems, $S^{k\to N}(\rho_N) = \lfloor N/k \rfloor S(\rho_k) + (1 - \delta_{N \mod k,0}) S(\rho_{N \mod k}) - S(\rho_N)$.

We now verify the consistency of the framework. We identify reasonable properties characterizing measures of k-partite correlations, applicable for any order k, by generalizing the ones proposed for N-partite correlations [7]. We show that the quantities in Eqs. (2) and (3), for any distance D, satisfy the criteria $0D - 4D$ of invariance and monotonicity under local operations and changes in the system partition. We also prove that, by adopting the relative entropy, stronger yet desirable constraints $0S$–$5S$ are met.

($0D$–$0S$) The measures of k-partite correlations are faithful. They are non-negative, $D^k(\rho_N) \geq 0$, and vanish if and only if the state does not have k-partite correlations.

($1D$) Adding a disjoint n-partite system, $\mathcal{S}'_{N+n} := \mathcal{S}_N \cup \mathcal{S}_n$, cannot create correlations of order higher than n. If the state of $\mathcal{S}_N$ does not have correlations of order higher than n, $\rho_N = \otimes_{j=1}^{m} \sigma_{k_j}$, $\sum_{j=1}^{m} k_j = N$, $n \geq \max\{k_j\}$, then the state of $\mathcal{S}'_{N+n}$ is $\rho_N \otimes \rho_n$, which does not have correlations of order higher than n, the largest factor of the product being still a state of n or fewer subsystems. Thus, $D^{n\to N+n}(\rho_{N+n}) = D^{n\to N}(\rho_N) = 0$.

($1S$) Adding a disjoint n-partite system cannot increase correlations of order higher than n. One has $S^{n\to N}(\rho_N) = S(\rho_N || \tilde{\rho}_N^n) = S(\rho_N \otimes \rho_n || \tilde{\rho}_N^n \otimes \rho_n) \geq S(\rho_N \otimes \rho_n || \tilde{\rho}_{N+n}^n) = S^{n\to N+n}(\rho_{N+n})$. For example, given $N = 3$, adding a bipartite system, $n = 2$, cannot increase the tripartite correlations.

($2D$–$2S$) Local CPTP maps $\Pi_i \Phi_{[i]}, \Phi_{[i]} = I_1 \otimes \ldots \Phi_i \otimes \ldots \otimes I_N$, cannot create correlations of any order k, and cannot increase the amount of correlations higher than any order k. Local operations do not change the tensor product structure of a state, $\rho_N \in P_k \Rightarrow \Pi_i \Phi_{[i]}(\rho_N) \in P_k$, $\rho_N \notin P_k \Rightarrow \Pi_i \Phi_{[i]}(\rho_N) \notin P_k$, so they cannot create correlations of any order, $D^k(\rho_N) = 0 \Rightarrow D^k(\Pi_i \Phi_{[i]}(\rho_N)) = 0$, $\forall k$. Contractivity under CPTP maps guarantees $D^{k\to N}(\rho_N) \geq D^{k\to N}(\otimes_i \Phi_{[i]}(\rho_N))$, $\forall k$. This also implies

monotonicity of the highest order of nonzero correlations, $D^{\tilde{k}}(\rho_N) \geq D^{\tilde{k}}(\Pi_i \Phi_{[i]}(\rho_N))$, $\tilde{k} = \max\{k\} : D^{k\to N}(\rho_N) = 0$. Note that an operation on a cluster of n subsystems Φ_n, $n > 1$, can create correlations of order up to $k + n - 1$ from already existing k-partite correlations. A state with nonzero k-partite correlations reads $\otimes_{j=1}^{m} \sigma_{k_j}$, $\sum_{j=1}^{m} k_j = N$, $\max\{k_j\} \geq k$. A map Φ_n jointly applied to one subsystem of the largest cluster $\mathcal{S}_{\max\{k_j\}}$ and other $n-1$ subsystems generates correlations of order up to $\max\{k_j\}+n-1$. For example, the k-qubit state $|a_k\rangle = a|0\rangle^{\otimes k} + \sqrt{1-a^2}|1\rangle^{\otimes k}$, $a \in (0,1)$, has genuine k-partite correlations $S^k(|a_k\rangle) = 2[a^2 \log a^2 + (1-a^2)\log(1-a^2)]$. Correlating the state with an ancillary target qubit by a CNOT gate creates the state $|a_{k+1}\rangle = a|0\rangle^{\otimes(k+1)} + \sqrt{1-a^2}|1\rangle^{\otimes(k+1)}$, which has $k+1$-partite correlations, $S^k(|a_{k+1}\rangle) = S^k(|a_k\rangle)$.

(3D) Partial trace of n subsystems cannot increase correlations of order higher than $k < N - n$. Let $\tilde{\rho}_N^k$ be the closest N-partite state with up to k-partite correlations. Then, by contractivity of the distance function, $D^{k\to N}(\rho_N) = D(\rho_N, \tilde{\rho}_N^k) \geq D(\rho_{N-n}, \mathrm{Tr}_n \tilde{\rho}_N^k) \geq D^{k\to N-n}(\rho_{N-n})$.

(3S) Partial trace of $N-k$ subsystems can create up to k-partite correlations from existing N-partite correlations. Let us consider the classical N-bit state $\rho_N^c = (|0\rangle\langle 0|^{\otimes N} + |1\rangle\langle 1|^{\otimes N})/2$, which has N-partite correlations. The marginal state $\mathrm{Tr}_{N-k}\rho_N^c = (|0\rangle\langle 0|^{\otimes k} + |1\rangle\langle 1|^{\otimes k})/2$, $\forall k$, has genuine k-partite correlations. Contractivity ensures $S^N(\rho_N) \geq S^k(\rho_k)$. The property is then proven, as k-partite correlations are not necessarily present in ρ_N^c. One has $S^k(\rho_N^c) = \lfloor N/(k-1) \rfloor - \lfloor N/k \rfloor + \delta_{N \bmod k,0} - \delta_{N \bmod (k-1),0} = \lceil N/(k-1) \rceil - \lceil N/k \rceil$. For example, given $N = 5$, the global state does not have genuine 4-partite correlations, $S^2(\rho_5^c) = 2$, $S^3(\rho_5^c) = S^5(\rho_5^c) = 1$, $S^4(\rho_5^c) = 0$. Indeed, the state $\rho_3^c \otimes \rho_2^c$ has more information about the global state ρ_5^c than $\rho_2^c \otimes \rho_2^c \otimes \rho_1^c$; thus, there are genuine 3-partite correlations, but the state $\rho_4^c \otimes \rho_1^c$ is not more informative than the 3-vs-2 product (the relative entropy distance to the global state is equal). Genuine 4-partite correlations are distilled by tracing away one subsystem, $S^4(\rho_4^c) = 1$.

(4D–4S) Distilling n subsystems by fine graining cannot create correlations of order higher than $k+n$, for any k. Fine graining a subsystem into a cluster of n subsystems, $\mathcal{S}_{[i]} \to \mathcal{S}_{i'} = \{\mathcal{S}_{[i_j]}\}, j = 1, \ldots, n+1$, changes the system partition into $\mathcal{S}'_{N+n} = \{\mathcal{S}_{[1]}, \mathcal{S}_{[2]}, \ldots, \mathcal{S}_{[i-1]}, \{\mathcal{S}_{i_j}\}, \mathcal{S}_{[i+1]}, \ldots, \mathcal{S}_{[N]}\}$. If the state of the system $\mathcal{S}_N$ has correlations of order up to k, $\rho_N = \rho_{k\leq N} \otimes (\otimes_{l_j \leq k} \rho_{l_j})$, $\sum_j l_j = N - k$, the fine-graining map creates at most correlations of order $k+n$, $\rho_{N+n} = \rho_{k+n\leq N+n} \otimes (\otimes_{l\leq k} \rho_l)$. Hence, $D^{k+n\to N+n}(\rho_{N+n}) = D^{k\to N}(\rho_N) = 0$.

(5S) Total correlations are superadditive. It is given a coarse grained partition $\{\mathcal{S}_{k_1}, \mathcal{S}_{k_2}, \ldots, \mathcal{S}_{k_l}\}$, $\sum_{j=1}^{l} k_j = N$. The total correlations in each cluster $\mathcal{S}_{k_j} = \{\mathcal{S}_{[\sum_{m=1}^{j-1} k_m + 1]}, \mathcal{S}_{[\sum_{m=1}^{j-1} k_m + 2]}, \ldots, \mathcal{S}_{[\sum_{m=1}^{j-1} k_m + k_j]}\}$ are quantified by the multi-information between the single subsystems forming the cluster, $S^{1\to k_j}(\rho_{k_j}) = \sum_{n=1}^{k_j} S(\rho_{[\sum_{m=1}^{j-1} k_m + n]}) - S(\rho_{k_j})$, a non-negative extension of the bipartite mutual information [25–27]. Exploiting subadditivity, one has $\sum_{j=1}^{l} S^{1\to k_j}(\rho_{k_j}) = \sum_{i=1}^{N} S(\rho_{[i]}) - \sum_{j=1}^{l} S(\rho_{k_j}) \leq \sum_{i=1}^{N} S(\rho_{[i]}) - S(\rho_N) = S^{1\to N}(\rho_N)$, where the inequality is saturated for product states $\rho_N = \otimes_j \rho_{k_j}$. That is, the sum of the total correlations in each cluster is upper bounded by the total correlations in the global system. For product states, subadditivity also implies additivity for correlations higher than k, for every k, $\rho_N = \otimes_j \rho_{k_j} \Rightarrow \tilde{\rho}_N^k = \otimes_j \tilde{\rho}_{k_j}^k \Rightarrow S^{k\to N}(\rho_N) = \sum_j S^{k\to k_j}(\rho_{k_j})$.

While being intuitive and simple to phrase, the discussed properties are not met by heavily employed measures and indicators of multipartite correlations. Covariances of local observables $O_{[i]}$, $\langle \Pi_i O_{[i]} \rangle_{\rho_N} - \Pi_i \langle O_i \rangle_{\rho_N}$, do not satisfy such criteria. They can vanish, for any choice of $\{O_{[i]}\}$, in the presence of classical and quantum multipartite correlations [7,28,29]. An alternative correlation witness measures the ability of multipartite systems to extract work from local environments [7]; yet it is still unproven whether the quantity satisfies properties 0D–4D. Another measure of correlations above order k is the (relative entropy) distance of the global state to the state with maximal von Neumann entropy, among the ones with the same marginal states of k subsystems, $S(\rho_N || \bar{\sigma}_N^k), \bar{\sigma}_N^k := \max_{\sigma_N : \mathrm{Tr}_{N-k}\sigma_N = \rho_k} S(\sigma_N)$ [30–34]. Remarkably, independent lines of thinking converged to the very same definition. However, such a measure, as well as a related one given by the trace norm of the cumulant of the state [29,35], violates contractivity under local operations in both classical and quantum scenarios [17,36]. This happens because local operations do change a state while preserving its tensor product structure, thus changing the set of states with the same k marginals.

Ranking correlation patterns by weaving.—Having determined how to quantify genuine multipartite correlations, one observes that equally correlated states, in terms of total correlations, can display different values of correlations for some order k, and thus different properties. Assuming N even, a product of $N/2$ Bell states, e.g., $|\psi_{N/2}\rangle = [1/\sqrt{2}(|00\rangle + |11\rangle)]^{\otimes N/2}$, has the same total correlations of the N-partite GHZ state $|\mathrm{GHZ}_N\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$, as measured by the relative entropy, $S^{1\to N}(|\psi_{N/2}\rangle) = S^{1\to N}(|\mathrm{GHZ}_N\rangle) = N$, while the latter exhibits correlations of higher order. On the same hand, the highest order of correlations is not sufficient to describe multipartite states. Both the GHZ and the $N/2$-excitation Dicke state $[1/\sqrt{\binom{N}{N/2}}]\sum_i \mathcal{P}_i(|0\rangle^{\otimes N/2} \otimes |1\rangle^{\otimes N/2})$, where the sum is over the permutations $\{\mathcal{P}_i\}$ of the group $\mathrm{Sym}(N)$, have two bits of N-partite correlations. Yet, they

TABLE I. Genuine k-partite correlations, N-partite correlations, total correlations, and weaving (asymptotic scaling for $N \to \infty$) for the product of $N/2$ maximally correlated two-bit states; the maximally correlated N-bit state; the product of $N/2$ Bell states; the N-partite GHZ state (the expressions hold for N odd as well); N-partite Dicke states with one and $N/2$ excitations, being $h(x) = x \log x$, and the functions $f_{D,1}$, $f_{D,N/2}$ given in Ref. [39]; the maximally correlated N-partite classical state of dimension d; and the product of $N/2$ maximally entangled two-qudit states.

ρ_N, N even	$S^k, k < N$	S^N	$S^{1\to N}$	$W_S, \omega_k = k-1$
$[(\lvert 00\rangle\langle 00\rvert + \lvert 11\rangle\langle 11\rvert)/2]^{\otimes N/2}$	$N/2\delta_{k,2}$	0	$N/2$	$N/2$
$(\lvert 0\rangle\langle 0\rvert^{\otimes N} + \lvert 1\rangle\langle 1\rvert^{\otimes N})/2$	$\lceil N/(k-1)\rceil - \lceil N/k\rceil$	1	$N-1$	$\sim 1.13N \log N - N$
$[\frac{1}{\sqrt{2}}(\lvert 00\rangle + \lvert 11\rangle)]^{\otimes N/2}$	$N\delta_{k,2}$	0	N	N
$\frac{1}{\sqrt{2}}(\lvert 0\rangle^{\otimes N} + \lvert 1\rangle^{\otimes N})$	$\lceil N/(k-1)\rceil - \lceil N/k\rceil$	2	N	$\sim 1.13N \log N$
$\frac{1}{\sqrt{\binom{N}{1}}}\sum_i \mathcal{P}_i(\lvert 0\rangle^{\otimes(N-1)} \otimes \lvert 1\rangle)$	$f^k_{D,1}$	$2/N[h(N) - h(N-1)] \sim 0$	$h(N) - h(N-1) \sim \log N$	$\sim 2.61N$
$\frac{1}{\sqrt{\binom{N}{N/2}}}\sum_i \mathcal{P}_i(\lvert 0\rangle^{\otimes N/2} \otimes \lvert 1\rangle^{\otimes N/2})$	$f^k_{D,N/2}$	2	N	$\sim 0.01N^2$
$\sum_{i=1}^d \lvert i\rangle\langle i\rvert^{\otimes N}/d$	$(\lceil N/(k-1)\rceil - \lceil N/k\rceil)\log d$	$\log d$	$(N-1)\log d$	$\sim(1.13N \log N - N)\log d$
$(\sum_{i=1}^d \lvert ii\rangle/\sqrt{d})^{\otimes N/2}$	$N \log d\,\delta_{k,2}$	0	$N \log d$	$N \log d$

have different uses for information processing [37,38], and it is impossible to transform them into each other by local operations and classical communication [18].

We introduce the concept of weaving to rank classical and quantum multipartite states by a single index, overcoming such ambiguities. The idea is to construct a consistent information-theoretic descriptor of correlation patterns by counting well-defined genuine multipartite correlations of every order. A weaving measure is built as the weighted sum of multipartite correlations,

$$W_D(\rho_N) = \sum_{k=2}^{N} \omega_k D^k(\rho_N) = \sum_{k=1}^{N-1} \Omega_k D^{k\to N}(\rho_N), \qquad (5)$$

where $\omega_k = \sum_{i=1}^{k-1} \Omega_i$, $\omega_k \in \mathbb{R}^+$. For any function D, a weaving measure is contractive under local operations and partial trace, $W_D(\rho_N) \geq W_D(\Pi_i \Phi_{[i]}(\rho_N))$, $W_D(\rho_N) \geq W_D(\rho_k)$, as it is a sum of contractive quantities (properties $2D$ and $3D$). The relative entropy of weaving $W_S(\rho_N)$ is also additive, $W_S(\otimes_j \rho_{k_j}) = \sum_j W_S(\rho_{k_j})$, being a sum of additive terms (property $5S$). Weaving is then easy to compute too, being obtained by global and marginal entropies.

The choice of the weights determines the meaning of a weaving measure. For $\omega_k = 1$, $\forall k$, it is a measure of total correlations. For $\omega_l = \delta_{kl}$, $\forall l$, it quantifies genuine k-partite correlations. As observed, computing correlations is not sufficient to discriminate different multipartite states. Thus, we study correlation scaling, that is, how the information about the global system scales by accessing partitions containing clusters of increasing size. This is captured, for example, by choosing weights proportional to the correlation order. We calculate the relative entropy measures of genuine multipartite correlations and weaving, selecting $\omega_k = k-1 \Rightarrow \Omega_i = 1$, $\forall i$, for highly correlated classical and quantum states of N particles (Table I). The quantity unambiguously ranks states with equal total correlations, or N-partite correlations. As expected, the weaving of states in tensor product form, e.g., the Bell state products, scales linearly $O(N)$ with the number of particles. Indeed, the correlations in the global state are the sum of the correlations in each product factor. The GHZ state shows superlinear scaling $O(N \log N)$ instead. However, the highest asymptotic value $O(N^2)$ for N qubits is found in the $N/2$ excitation Dicke state. Such a state has nonzero correlations at any order, $f^k_{D,N/2} \neq 0$, $\forall k$ [39], while the GHZ state has zero correlations whenever $\lceil N/(k-1)\rceil = \lceil N/k\rceil$. Weaving is proportional to the logarithm of the subsystem dimension d.

The concept of weaving solves issues that emerged in previous studies. A measure of "neural complexity" was proposed to study correlation scaling between binary variables [40]. The quantity, which we generalize to the quantum scenario, reads $C(\rho_N) = \sum_{k=1}^{N-1} k/N S^{1\to N}(\rho_N) - \langle S^{1\to k}(\rho_k)\rangle$, where the average term is calculated over the $\binom{N}{k}$ clusters of k subsystems $\mathcal{S}_k$. A geometric lower bound is given by the weighted distances to the entropy maximizers with the same k marginals, $C(\rho_N) \geq C^g(\rho_N) = \sum_k k/N S(\rho_N, \bar{\sigma}^k_N)$ [16,17,34,41–43]. The interest in complexity measures was spurred by the association with enhanced neuronal activity, evaluating the functionality of equally correlated neural networks. This generated a debate about whether complexity is the resource governing information transmission in the brain [44]. Such quantities have been also applied to study chaotic systems and cellular automata [17]. Yet, complexity measures fall short as benchmarks of multipartite correlations. The neural complexity is not additive under tensor products,

e.g., $C(\rho_2 \otimes \rho_1) = 4/3C(\rho_2)$, while the geometric complexity is not contractive under local operations, or under partial trace, requiring nonanalytical methods to be computed [17,43,45].

Conclusion.—We proposed a consistent information-theoretic definition of genuine multipartite correlations, and described how to quantify them. While we did not discuss the distinction between classical and quantum correlations, our result suggests a strategy to characterize genuine multipartite quantum correlations, an open question despite recent progresses [10,24,46,47]. Having defined k-partite correlations as in Eq. (2), classical and quantum contributions can be identified via the method employed for total correlations [24], then studying quantum correlations on their own.

We also introduced weaving, a descriptor of correlation patterns. Weaving is an alternative to complexity measures, i.e., a measure of how hard it is to determine the properties of a system from knowing its parts [48], which satisfies desirable constraints. An important question to address is its operational meaning. Specifically, the quantum contribution to the weaving of a state may be a further computational resource. This would confirm the intuition that interplaying complexity science and (quantum) information theory can advance both disciplines [49].

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