Lattice Homotopy Constraints on Phases of Quantum Magnets

Hoi Chun Po,[1,2] Haruki Watanabe,[3] Chao-Ming Jian,[4,5] and Michael P. Zaletel[6]
[1]*Department of Physics, University of California, Berkeley, California 94720, USA*
[2]*Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA*
[3]*Department of Applied Physics, University of Tokyo, Tokyo 113-8656, Japan*
[4]*Station Q, Microsoft Research, Santa Barbara, California 93106, USA*
[5]*Kavli Institute for Theoretical Physics, University of California, Santa Barbara, California 93106, USA*
[6]*Department of Physics, Princeton University, Princeton, New Jersey 08544, USA*
(Received 21 June 2017; published 22 September 2017)

The Lieb-Schultz-Mattis (LSM) theorem and its extensions forbid trivial phases from arising in certain quantum magnets. Constraining infrared behavior with the ultraviolet data encoded in the microscopic lattice of spins, these theorems tie the absence of spontaneous symmetry breaking to the emergence of exotic phases like quantum spin liquids. In this work, we take a new topological perspective on these theorems, by arguing they originate from an obstruction to "trivializing" the lattice under smooth, symmetric deformations, which we call the "lattice homotopy problem." We conjecture that all LSM-like theorems for quantum magnets (many previously unknown) can be understood from lattice homotopy, which automatically incorporates the full spatial symmetry group of the lattice, including all its point-group symmetries. One consequence is that any spin-symmetric magnet with a half-integer moment on a site with even-order rotational symmetry must be a spin liquid. To substantiate the claim, we prove the conjecture in two dimensions for some physically relevant settings.

DOI: 10.1103/PhysRevLett.119.127202

Quantum magnets arise naturally in Mott insulators, where strong Coulomb repulsion freezes the position of electrons and leaves behind their spin degrees of freedom. With strong frustration, quantum fluctuations can suppress spin ordering and lead to symmetric, quantum-entangled phases of matter that survive down to zero temperature. Quantum spin liquids, the spin analogues of fractional quantum Hall states, represent one of the most sought-after phases arising in this context [1]. They possess intrinsic topological order with emergent fractionalized excitations, which have been proposed as a useful resource for robust quantum computation [2,3].

Detecting whether a quantum magnet is a spin liquid, a many-body problem, is notoriously hard. Conventionally, the absence of symmetry breaking is regarded as an indicator for spin-liquid physics [3,4]. However, a symmetric quantum magnet could also be in a symmetry-protected topological (SPT) phase, like the spin-1 Haldane chain [5,6] and its generalizations [7], which does not support fractionalized excitations despite a nontrivial degree of entanglement. Conceptually, there is a sharp distinction between these phases: spin liquids are long-range entangled (LRE), and are necessarily either gapless or topologically ordered, while SPT phases are only short-range entangled (SRE). Experimentally, however, such distinction is subtle and one must rely on additional criteria to rule out *all* symmetric SRE (sym-SRE) phases before claiming discovery of a spin liquid.

Fortunately, it is possible to rule out all sym-SRE phases in certain quantum magnets on purely theoretical grounds. This line of reasoning was pioneered by Lieb, Schultz, and Mattis (LSM), who proved that any one-dimensional quantum magnet with both lattice-translation and spin-rotation symmetries cannot be sym-SRE if each unit cell contains a half-integral total spin [8]. Multiple generalizations of the LSM theorem have since been made, covering systems in higher dimensions and with less stringent physical assumptions [9–17]. We will collectively refer to these results as "LSM-like theorems." The common denominator of these generalizations is a constraint between the microscopic details of the system, specifically the lattice and the symmetry-transformation properties of the sites' Hilbert spaces, and the degree of ground-state degeneracy. Since any sym-SRE phase is expected to have a gapped, unique ground state on any thermodynamically large space without defects or boundaries, all sym-SRE phases are ruled out whenever the degeneracy is constrained to be nontrivial. In this sense, the LSM-like theorems are "no gos" for sym-SRE phases.

One important direction for generalization is to make fuller use of the spatial symmetries of the system. This was partially addressed in Refs. [15–17], which showed that combinations of nonsymmorphic symmetries like glides and screws, being "fractions" of the lattice translations, can lead to stronger no gos. Ideally, to expose the strongest constraints one would attempt to utilize *all* spatial symmetries of the problem. However, the nonsymmorphic generalizations in Refs. [15–17] ignore all point-group symmetries (e.g., rotations), which fix at least one point in space. New techniques are required for the desired extension.

In this work, we address the problem of incorporating all spatial symmetries in deriving stronger LSM-like no gos, which are operative even when all earlier theorems are not. We will rely on two key insights. First, the presence of a no go should be insensitive to a smooth, symmetric deformation of the underlying lattice. We will refer to the study of such deformations as the "lattice homotopy problem." Second, there is a strong sense of locality in sym-SRE phases due to the limited range of entanglement, and therefore, compared to more exotic phases like spin liquids, they respond in a more conventional manner when fluxes are inserted into the system. Combining these observations, we conjecture that a quantum magnet which is nontrivial under lattice homotopy is obstructed from being sym-SRE.

In the following, we will elaborate on the conjecture, which encompasses all earlier LSM-like theorems for quantum magnets, and then offer a physical argument for its proof restricting to 2D systems with an internal symmetry group G being either finite Abelian or SO(3). As an example, we will show that sym-SRE phases are forbidden whenever a half-integer spin, carrying a projective representation of SO(3), sits at an even-order rotation center. Intuitively, this is because any symmetric deformation brings in an even number of spins, which cannot screen the half-integer moment at the center.

Statement of the conjecture.—Consider a quantum magnet with Hamiltonian $\hat{H}$ defined on a lattice Λ. For simplicity, we will first assume $\hat{H}$ is symmetric under the group $G = \mathrm{SO}(3)$ of spin rotations, and later discuss how the ideas apply to more general on-site symmetry groups. We are interested in whether $\hat{H}$ can be in a sym-SRE phase. As demonstrated by the LSM-like theorems, the microscopic data encoded in Λ may present an obstruction. The key ingredient in our argument will be the spatial distribution of half-integer vs integer spins. Therefore, as far as obstructions are concerned, we view Λ as a lattice of black and white circles, denoting half-integer and integer spins, respectively (Fig. 1). No obstruction is expected on a lattice composed only of integer spins, and we say such lattices are "trivial." In addition, the presence of obstructions should be insensitive to a smooth deformation of the lattice, provided that the deformation respects all spatial symmetries [Fig. 1(e)]. This motivates the following conjecture:

Conjecture—A sym-SRE phase is possible only when Λ is smoothly deformable to a trivial lattice.

Let us make precise what is meant by a "smooth deformation." We suppose the magnet is symmetric under a space group $\mathcal{S}$. By deformation, we refer first to a collective, $\mathcal{S}$-symmetric movement of sites. Second, when sites collide they "fuse;" since we only keep track of the integer vs half-integer nature of the sites, the fusion follows a $\mathbb{Z}_2$ rule [Figs. 1(a)–1(d)]. In this process, an even number of half-integer sites may annihilate. Generally, when a

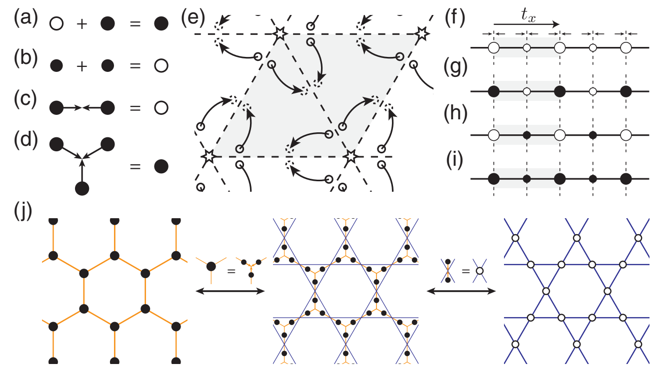

FIG. 1. Lattice homotopy. (a)–(d) Representations of the rotation group SO(3) fuse following a $\mathbb{Z}_2$ rule. Open and filled circles, respectively, denote the representations of integer and half-integer spins. (e) A smooth deformation of a lattice (circles) symmetric under mirror planes (dashed lines) and threefold rotations (about the stars). (f)–(i) There are two inequivalent sites (big and small circles) in each unit cell (shaded) of a mirror-symmetric 1D lattice. Under lattice homotopy, there are four distinct lattice classes. (j) Assuming the symmetries of (e), a honeycomb lattice of half-integer spins is equivalent to a kagome lattice of integer spins, as demonstrated by the depicted smooth deformation.

collection of sites are symmetrically collapsed at a point, the number of sites involved is determined by the degree of the point-group symmetry. We also allow the inverse of fusion, in which half-integer spins are created in pairs.

A sequence of such deformations defines an equivalence relation between lattices, and we refer to the enumeration of the resulting equivalence classes $[\Lambda]$ as the "lattice homotopy" problem. $\{[\Lambda]\}$ naturally forms an Abelian group under stacking, with the empty (trivial) lattice the identity element. The conjecture is that a sym-SRE obstruction is present whenever a lattice belongs to a nontrivial class. We note that all the previously known LSM-like theorems feature nontrivial lattices [8–17].

Thanks to its geometrical nature, lattice homotopy can often be computed in an intuitive manner. For instance, consider a 1D translation and mirror symmetric spin chain. Spins at generic positions can be smoothly brought to the mirror planes, where they will annihilate pairwise. Since this cannot change the color on the mirror plane, the only lattice invariant is the color at the two inequivalent mirror planes in a unit cell, giving a $[\Lambda] \in \mathbb{Z}_2 \times \mathbb{Z}_2$ classification [Figs. 1(f)–1(i)]. In fact, a no go for the three nontrivial elements was already proven in Ref. [17]. Together with the original LSM theorem invoking only translations, this proves the conjecture for the two 1D space groups.

As another example, a square lattice of spin-1/2 moments is nontrivial, but that of spin-1 is trivial. This is consistent with the known LSM-like theorems for the former [9–13], and the existence of sym-SRE phases for the latter [18]. A more intriguing example is a honeycomb lattice of half-integer spins, which is symmetric under both threefold rotations and mirrors. As shown in Fig. 1(j), the lattice is smoothly deformable to a kagome lattice of integer spins, and therefore belongs to the trivial class. Interestingly, this picture is consistent with a recent construction of sym-SRE wave functions [18,19].

It is conceptually revealing to generalize the internal symmetry group G in the discussion above beyond SO(3) spin rotations. We assume that the total symmetry group is a direct product of the internal and space group symmetries, $G \times \mathcal{S}$, where $\mathcal{S}$ acts by permuting the sites. (The case with "spin-orbit coupling" is an interesting future direction.) The role of "half-integer" vs "integer" spin is now played by the Abelian group of distinct projective representations of G, $\mathcal{H}^2[G, U(1)]$. The $\mathbb{Z}_2$ fusion of spins generalizes to group multiplication in $\mathcal{H}^2[G, U(1)]$, and the above conjecture naturally carries over to this more general setting.

The resulting group of lattice homotopy classes depends on G. For instance, suppose G is such that the projective representations have $\mathbb{Z}_3$ fusion, and consider again the 1D lattice with reflection symmetry. If two copies of a projective representation $[\omega]$ approach a mirror plane, they do *not* annihilate, since $[\omega]^2 = [\omega]^{-1}$ in $\mathbb{Z}_3$. Consequently, the projective representation on a mirror plane is not conserved, and the lattice homotopy classification collapses down to $\mathbb{Z}_3$.

Computing the lattice classification can be automated by a reduction to the properties of high-symmetry points (Wyckoff positions). We relegate details to Sec. I of the Supplemental Material [20]. In Table I, we tabulate the lattice classification results for all 2D space groups. Here, we present the case relevant to spins, $\mathcal{H}^2[G, U(1)] = \mathbb{Z}_2$—the general form, which extends readily to any finite Abelian $\mathcal{H}^2[G, U(1)]$, is tabulated in the Supplemental Material [20], Table I.

Proof of conjecture in 2D.—We now sketch a physical argument supporting the conjecture for quantum magnets symmetric under any of the 17 2D space groups, assuming $G = \mathrm{SO}(3)$ or is finite Abelian. The logic proceeds by first deriving three concrete conditions on Λ, each implying a no go for sym-SRE phases: (i) Bieberbach no go. A "fundamental domain" D is a region which tiles the plane under the action of translation and glide symmetries. If the total projective representation in D is nontrivial, $[\omega]_D = \prod_{r\in D}[\omega]_{\boldsymbol{r}} \neq 1$, then a sym-SRE phase is forbidden [17].

TABLE I. The lattice homotopy classification for the 17 wallpaper groups, assuming $\mathbb{Z}_2$ projective representations, as in the case for spin-rotation invariant quantum magnets.

Lattice homotopy	Wallpaper group No. [24]
$\mathbb{Z}_2$	1, 4, 5, 13, 14, 15
$(\mathbb{Z}_2)^2$	3, 8, 12, 16, 17
$(\mathbb{Z}_2)^3$	7, 9, 10, 11
$(\mathbb{Z}_2)^4$	2, 6

(ii) Mirror no go. Let ℓ be a mirror-line parallel to a translation $T_\parallel$. We define the projective representation per unit length of ℓ, $[\omega]_\ell = \prod_{\boldsymbol{r}\in\ell'}[\omega]_{\boldsymbol{r}}$, by letting the product runs over a unit-length interval ℓ' of ℓ as defined by $T_\parallel$. If $[\omega]_\ell$ does not have a "square-root,", i.e., if no $\zeta \in \mathbb{Z}_n$ satisfies $[\omega]_\ell = \zeta^2$, then a sym-SRE phase is forbidden [17]. (iii) Rotation no go. Let $\boldsymbol{r}$ be a site with rotational point-group symmetry C_m and projective representation $[\omega]_{\boldsymbol{r}}$. If $[\omega]_{\boldsymbol{r}}$ does not have an "mth root,", i.e., if no $\zeta \in \mathbb{Z}_n$ satisfies $[\omega]_{\boldsymbol{r}} = \zeta^m$, then a sym-SRE phase is forbidden. We then show that these no gos forbid a sym-SRE phase in a 2D lattice Λ if and only if $[\Lambda] \neq 1$. Both the Bieberbach and mirror no gos were derived in an earlier work [17], so here we focus on illustrating the key ideas behind the derivation of the "rotation no go"—the key missing piece for establishing the conjecture in two dimensions—with further details given in Sec. II of the Supplemental Material [20].

Derivation of the rotation no go.—For simplicity, we will illustrate the ideas using systems symmetric under $G = \mathrm{SO}(3)$ and C_2 rotation. Roughly speaking, we will modify the Hamiltonian by inserting a pair of C_2-related spin fluxes, and show that when a half-integer moment lies on a C_2-invariant point, the system has a symmetry-protected degeneracy. We will then argue that, despite the presence of fluxes, such degeneracy remains impossible in sym-SRE phases, and thereby arriving at a no go.

We begin with the following observation: While a sym-SRE phase has a gapped, unique ground state on $\mathbb{R}^d$, it may possess symmetry-protected ground-state degeneracy in the presence of defects or boundaries (a notable example being the edge states of the AKLT chain). In contrast to LRE phases, however, the degeneracies in a sym-SRE phase should be "localized" to the defect regions (for example, each edge of the AKLT chain carries an independent twofold degeneracy). Physically, this arises because a sym-SRE phase can only respond to local data, defined with respect to the correlation length ξ, so it should not be possible to "share" a degeneracy between two distant defect regions (note we are only considering bosonic models; certain fermionic SPTs violate this assumption [25]).

To formalize this intuition we introduce the notion of *degeneracy localization* (Sec. III of the Supplemental Material [20]). A "defect region" is a region in which the Hamiltonian is not local-unitarily equivalent to the Hamiltonian of the bulk [17] (examples could include an impurity spin, dislocation, or external flux), and we let $\{R^{(i)}: i = 1, \ldots, N_D\}$ be a collection of defect regions of finite extent, which are separated from each other on distances $r \gg \xi$. We say the system exhibits degeneracy localization if each $R^{(i)}$ can be modeled as an emergent d_i-dimensional, degenerate, degree of freedom, so that the total ground-state subspace $\mathcal{H}'_{\mathrm{GS}}$ is $(\prod_{i=1}^{N_D} d_i$ dimensional. This implies that if $\hat{U}$ is a local operator taking the ground-state subspace $\mathcal{H}'_{\mathrm{GS}}$ into itself (e.g., a symmetry),

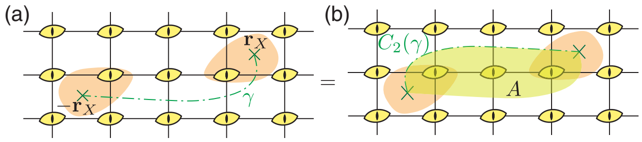

FIG. 2. Flux insertion. (a) A C_2 symmetric lattice with a pair of X fluxes (crosses) inserted at $\pm r_X$, which leads to "defect regions" (shaded) near the fluxes. Far away from $\pm r_X$, flux insertion amounts to choosing a defect line (dash-dot) and twisting the local Hamiltonian by X along the line. (b) As $X^{-1} = X$, the system retains a twisted C_2' symmetry, since the transformed defect line can be brought back to the original by applying a gauge transformation on the region A.

then its projection into $\mathcal{H}'_{\rm GS}$ can be "factorized" as $\hat{U}|_{\rm GS} = \otimes_{i=1}^{N_D} U^{(i)} + \mathcal{O}(e^{-r/\xi})$ for some d_i-dimensional matrix $U^{(i)}$ acting only on the degeneracy localized at the region $R^{(i)}$. In other words, degeneracy localization passes the locality structure from the full Hilbert space onto $\mathcal{H}'_{\rm GS}$. The discussed intuition about sym-SRE phases can then be summarized by the following physical assumption: A bosonic sym-SRE phase exhibits degeneracy localization.

We now use this assumption to prove the C_2 rotation no go with $G = \mathrm{SO}(3)$. Recall that a (projective) representation of SO(3) is classified by $[\omega]_{\boldsymbol{r}} \in \mathbb{Z}_2 = \{1, -1\}$, which encodes the phase factor for the commutator of two orthogonal π rotations at site $\boldsymbol{r}$, say $\hat{X}_{\boldsymbol{r}}\hat{Z}_{\boldsymbol{r}} = [\omega]_{\boldsymbol{r}}\hat{Z}_{\boldsymbol{r}}\hat{X}_{\boldsymbol{r}}$, where $\hat{X}$, $\hat{Z}$ are π rotations about $\hat{x}$, $\hat{z}$. Let the C_2-invariant point be the origin. Clearly, the no-go condition is unmet whenever $[\omega]_{\mathbf{0}} = 1$, and, hence, it suffices to prove a no go with $[\omega]_{\mathbf{0}} = -1$.

To this end, we modify the Hamiltonian by introducing a pair of X fluxes at the C_2-related points $\pm\boldsymbol{r}_X$ for some arbitrarily large $|\boldsymbol{r}_X|$ [Fig. 2(a)]. An "X flux" is analogous to a twist in boundary condition, and is microscopically defined as follows [26]. We choose a line segment γ connecting $\pm\boldsymbol{r}_X$, and for each local term $\hat{h} = \sum_j \hat{O}_L^j \hat{O}_R^j$ in the Hamiltonian intersecting γ, where $\hat{O}_L^j$ and $\hat{O}_R^j$ are, respectively, localized to the left and right of γ, we replace it by $\hat{h}' \equiv \sum_j \hat{O}_L^j (\hat{X}\hat{O}_R^j\hat{X}^\dagger)$ to obtain $\hat{H}'$. Note that while the flux insertion points $\pm\boldsymbol{r}_X$ are fixed and correspond to defects in the system, the choice of γ is arbitrary, and one can deform $\gamma \to \gamma'$ by applying the gauge transformation $\prod_{\boldsymbol{r}\in A}\hat{X}_{\boldsymbol{r}}$ in the region A enclosed by $\gamma' - \gamma$. Also, the orientation of γ is immaterial as X is an order-two symmetry.

Though the two fluxes are C_2 related, the choice of the defect line γ naively spoils the C_2 symmetry. However, the change $\gamma \to C_2(\gamma)$ can be removed by a gauge transformation [Fig. 2(b)]. Consequentially, $\hat{H}'$ is symmetric under a twisted-C_2 operation: $\hat{C}_2{}' = (\prod_{\boldsymbol{r}\in A}\hat{X}_{\boldsymbol{r}})\hat{C}_2$, where $\partial A = \gamma - C_2(\gamma)$. In addition, $\hat{Z} \equiv \prod_{\boldsymbol{r}}\hat{Z}_{\boldsymbol{r}}$ remains a symmetry of $\hat{H}'$. Computing the commutation relation between the two symmetries, one finds

$$\hat{C}_2{}'\hat{Z}\hat{C}_2{}'^{-1}\hat{Z}^{-1} = \prod_{\boldsymbol{r}\in A}\hat{X}_{\boldsymbol{r}}\hat{Z}_{\boldsymbol{r}}\hat{X}_{\boldsymbol{r}}^{-1}\hat{Z}_{\boldsymbol{r}}^{-1} = \prod_{\boldsymbol{r}\in A}[\omega]_{\boldsymbol{r}} = [\omega]_{\mathbf{0}}, \quad (1)$$

where in the last equality we used the fact that A is C_2 symmetric, and by symmetry $[\omega]_{\boldsymbol{r}} = [\omega]_{-\boldsymbol{r}}$. Since both $\hat{Z}$ and $\hat{C}_2{}'$ are symmetries of $\hat{H}'$, they leave the ground space $\mathcal{H}'_{\rm GS}$ invariant. We can therefore project Eq. (1) into $\mathcal{H}'_{\rm GS}$, and obtain the corresponding relation $\hat{C}_2{}'|_{\rm GS}\hat{Z}|_{\rm GS} = [\omega]_{\mathbf{0}}\hat{Z}|_{\rm GS}\hat{C}_2{}'|_{\rm GS}$.

When $[\omega]_{\mathbf{0}} = -1$, there is (at least) a twofold degeneracy that we will now show is impossible in a sym-SRE phase, provided the degeneracy localization assumption holds. If the system was sym-SRE, degeneracy localization implied $\hat{Z}|_{\rm GS} = \hat{Z}|_{\rm GS}^{(+)} \otimes \hat{Z}|_{\rm GS}^{(-)}$, where $\pm$ denotes the fluxes at $\pm\boldsymbol{r}_X$. In addition, as C_2 exchanges the two fluxes, the local degeneracies satisfy $d_+ = d_-$, and without loss of generality we can choose a basis in which $C_2'|_{\rm GS}$ is simply $(\hat{C}_2{}'|_{\rm GS})|\alpha_+\alpha_-\rangle = |\alpha_-\alpha_+\rangle$, where $\alpha_\pm$ denotes the independent degenerate states "trapped" at $\pm\boldsymbol{r}_X$. In this basis, the commutation relation reads $\hat{C}_2{}'|_{\rm GS}\hat{Z}|_{\rm GS}\hat{C}_2{}'|_{\rm GS}^\dagger = \hat{Z}|_{\rm GS}^{(-)} \otimes \hat{Z}|_{\rm GS}^{(+)} = -\hat{Z}|_{\rm GS}^{(+)} \otimes \hat{Z}|_{\rm GS}^{(-)}$. A solution to this requires $\hat{Z}|_{\rm GS}^{(-)} = \nu\hat{Z}|_{\rm GS}^{(+)}$ for some $\nu \in \mathrm{U}(1)$ satisfying $-\nu = \nu$, leading to a contradiction. Hence the claim.

In closing, we remark that our no gos are circumvented if the system becomes LRE. An example is discussed in Sec. IV of the Supplemental Material [20].

Discussion and outlook.—In conclusion, we have conjectured that all LSM-like theorems for quantum magnets, where microscopic degrees of freedom forbid symmetric short-range entangled phases, can be understood intuitively as topological obstructions to smoothly deforming the underlying lattice into a trivial one. We proved the conjecture in 2D for quantum magnets that are either spin-rotation invariant, or possess on-site unitary finite-Abelian symmetries.

Our 2D arguments, in fact, cover all 80 layer groups, which are symmetries of 2D lattices embedded in three dimensions (see Sec. V of the Supplemental Material [20] for more details). They also extend to some genuinely 3D lattices—in particular, the three no gos remain true, where mirror lines and C_m rotation-invariant points in two dimensions become planes and lines in three dimensions. Such extensions can have immediate implications on spin-liquid candidates. As an example, we note that both the Bieberbach and mirror no gos are silent for the pyrochlore quantum spin ice $Yb_2Ti_2O_7$ [27], but the C_2-rotation no go remains active if we model the system as a spin-rotation invariant quantum magnet. Yet, we caution that spin-orbit coupling is strong in the actual material [27], and so this idealization is not immediately justified.

A closer inspection, however, reveals that these three no gos only prove the conjecture for some but not all of the 230 3D space groups (Sec. VI of the Supplemental Material [20] includes simple examples for which the current set of no gos are insufficient.) New techniques will be required, and we describe some partial results in Sec. VII of the Supplemental Material [20]. We also note that it would be most useful if only time-reversal $\mathcal{T}$ was required in the no gos, with the role of projective representation played by the Kramers degeneracy from $\mathcal{T}^2 = -1$. However, it is not clear how to extend our flux-insertion proof to this case. In addition, actual materials are composed of itinerant fermions carrying spin and the quantum-magnet description is often an approximation. It would be useful to know if our results extend to this more general case. With a Mott gap, it naively seems that there should be a sharp notion of "where" the spins of the electrons lie (at least up to the lattice equivalence relations), but certain examples suggest this may not be the case [28,29]. Finally, we note that our conjecture has interesting connection to the study of crystalline SPTs [30–36], which we comment on briefly in Sec. VIII of the Supplemental Material [20].

We thank A. Vishwanath for discussions and collaboration on related works. M. Z. is indebted to conversations with D. Else, M. Cheng, M. Freedman, C. Galindo-Martinez, and M. Hermele. C. M. J.'s research at the KITP is funded by the Gordon and Betty Moore Foundation's EPiQS Initiative through Grant No. GBMF4304. H. W. acknowledges support from JSPS KAKENHI Grant No. JP17K17678.

[1] L. Balents, Spin liquids in frustrated magnets, Nature (London) **464**, 199 (2010).
[2] M. Freedman, A. Kitaev, M. Larsen, and Z. Wang, Topological quantum computation, Bull. Am. Math. Soc. **40**, 31 (2003).
[3] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, Non-Abelian anyons and topological quantum computation, Rev. Mod. Phys. **80**, 1083 (2008).
[4] Y. Shimizu, K. Miyagawa, K. Kanoda, M. Maesato, and G. Saito, Spin Liquid State in an Organic Mott Insulator with a Triangular Lattice, Phys. Rev. Lett. **91**, 107001 (2003).
[5] F. Haldane, Continuum dynamics of the 1-D Heisenberg antiferromagnet: Identification with the O(3) nonlinear sigma model, Phys. Lett. **93A**, 464 (1983).
[6] I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Rigorous Results on Valence-Bond Ground States in Antiferromagnets, Phys. Rev. Lett. **59**, 799 (1987).
[7] X. Chen, Z.-X. Liu, and X.-G. Wen, Two-dimensional symmetry-protected topological orders and their protected gapless edge excitations, Phys. Rev. B **84**, 235141 (2011).
[8] E. Lieb, T. Schultz, and D. Mattis, Two soluble models of an antiferromagnetic chain, Ann. Phys. (N.Y.) **16**, 407 (1961).
[9] I. Affleck and E. H. Lieb, A proof of part of Haldane's conjecture on spin chains, Lett. Math. Phys. **12**, 57 (1986).
[10] I. Affleck, Spin gap and symmetry breaking in CuO_2 layers and other antiferromagnets, Phys. Rev. B **37**, 5186 (1988).
[11] M. Yamanaka, M. Oshikawa, and I. Affleck, Nonperturbative Approach to Luttinger's Theorem in One Dimension, Phys. Rev. Lett. **79**, 1110 (1997).
[12] M. Oshikawa, Commensurability, Excitation Gap, and Topology in Quantum Many-Particle Systems on a Periodic Lattice, Phys. Rev. Lett. **84**, 1535 (2000).
[13] M. B. Hastings, Lieb-Schultz-Mattis in higher dimensions, Phys. Rev. B **69**, 104431 (2004).
[14] X. Chen, Z.-C. Gu, and X.-G. Wen, Classification of gapped symmetric phases in one-dimensional spin systems, Phys. Rev. B **83**, 035107 (2011).
[15] S. A. Parameswaran, A. M. Turner, D. P. Arovas, and A. Vishwanath, Topological order and absence of band insulators at integer filling in non-symmorphic crystals, Nat. Phys. **9**, 299 (2013).
[16] R. Roy, Space group symmetries and low lying excitations of many-body systems at integer fillings, arXiv:1212.2944.
[17] H. Watanabe, H. C. Po, A. Vishwanath, and M. P. Zaletel, Filling constraints for spin-orbit coupled insulators in symmorphic and nonsymmorphic crystals, Proc. Natl. Acad. Sci. U.S.A. **112**, 14551 (2015).
[18] C.-M. Jian and M. Zaletel, Existence of featureless paramagnets on the square and the honeycomb lattices in $2 + 1$ dimensions, Phys. Rev. B **93**, 035114 (2016).
[19] P. Kim, H. Lee, S. Jiang, B. Ware, C.-M. Jian, M. Zaletel, J. H. Han, and Y. Ran, Featureless quantum insulator on the honeycomb lattice, Phys. Rev. B **94**, 064432 (2016).
[20] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.119.127202, which includes Refs. [21–23], for further discussions on the details for computing the lattice homotopy classification, proof of the conjecture in more general settings, elaboration on the degeneracy localization assumption and how it is circumvented in topologically ordered phases, extensions to three-dimensional systems, and connection to symmetry-protected topological phases.
[21] C. Miller, The second homology group of a group; relations among commutators, Proc. Am. Math. Soc. **3**, 588 (1952).
[22] J.-L. Brylinski and R. Brylinski, Universal quantum gates, *Mathematics of Quantum Computation* (Chapman and Hall/CRC, London, 2002).
[23] H. Watanabe, H. C. Po, M. P. Zaletel, and A. Vishwanath, Filling-Enforced Gaplessness in Band Structures of the 230 Space Groups, Phys. Rev. Lett. **117**, 096404 (2016).
[24] *International Tables for Crystallography*, 5th ed., edited by T. Hahn, Space-group symmetry Vol. A (Springer, New York, 2006).
[25] N. Tarantino and L. Fidkowski, Discrete spin structures and commuting projector models for two-dimensional fermionic symmetry-protected topological phases, Phys. Rev. B **94**, 115115 (2016).
[26] M. Barkeshli, P. Bonderson, M. Cheng, and Z. Wang, Symmetry, defects, and gauging of topological Phases, arXiv:1410.4540.
[27] K. A. Ross, L. Savary, B. D. Gaulin, and L. Balents, Quantum Excitations in Quantum Spin Ice, Phys. Rev. X **1**, 021002 (2011).
[28] H. C. Po, H. Watanabe, M. P. Zaletel, and A. Vishwanath, Filling-enforced quantum band insulators in spin-orbit coupled crystals, Sci. Adv. **2**, e1501782 (2016).

[29] H. C. Po, A. Vishwanath, and H. Watanabe, Symmetry-based indicators of band topology in the 230 space groups, Nat. Commun. **8**, 50 (2017).
[30] X. Chen, F. J. Burnell, A. Vishwanath, and L. Fidkowski, Anomalous Symmetry Fractionalization and Surface Topological Order, Phys. Rev. X **5**, 041013 (2015).
[31] C. Wang, C.-H. Lin, and M. Levin, Bulk-Boundary Correspondence for Three-Dimensional Symmetry-Protected Topological Phases, Phys. Rev. X **6**, 021015 (2016).
[32] M. Cheng, M. Zaletel, M. Barkeshli, A. Vishwanath, and P. Bonderson, Translational Symmetry and Microscopic Constraints on Symmetry-Enriched Topological Phases: A View from the Surface, Phys. Rev. X **6**, 041068 (2016).
[33] M. Hermele and X. Chen, Flux-Fusion Anomaly Test and Bosonic Topological Crystalline Insulators, Phys. Rev. X **6**, 041006 (2016).
[34] H. Song, S.-J. Huang, L. Fu, and M. Hermele, Topological Phases Protected by Point Group Symmetry, Phys. Rev. X **7**, 011020 (2017).
[35] R. Thorngren and D. V. Else, Gauging spatial symmetries and the classification of topological crystalline phases, arXiv:1612.00846.
[36] D. Else (to be published).