P H Y S I C A L R E V I E W L E T T E R S

All Entangled States can Demonstrate Nonclassical Teleportation

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Quantum teleportation, the process by which Alice can transfer an unknown quantum state to Bob by using preshared entanglement and classical communication, is one of the cornerstones of quantum information. The standard benchmark for certifying quantum teleportation consists in surpassing the maximum average fidelity between the teleported and the target states that can be achieved classically. According to this figure of merit, not all entangled states are useful for teleportation. Here we propose a new benchmark that uses the full information available in a teleportation experiment and prove that all entangled states can implement a quantum channel which cannot be reproduced classically. We introduce the idea of nonclassical teleportation witness to certify if a teleportation experiment is genuinely quantum and discuss how to quantify this phenomenon. Our work provides new techniques for studying teleportation that can be immediately applied to certify the quality of quantum technologies.

Quantum teleportation [1] is a cornerstone of quantum information science, and serves as a primitive in several quantum information tasks [2–4]. Since the first demonstrations [5–7], quantum teleportation has been implemented in a variety of physical systems and has become a test bed for quantum information platforms [8]. In the ideal setting, quantum teleportation refers to the situation where Alice shares a maximally entangled state with Bob, which she uses, in combination with classical communication, to faithfully transmit a quantum state to Bob, even if that state is unknown to her.

In order to test that Alice and Bob are performing quantum teleportation, a third party, which we refer to as the verifier, provides quantum systems to Alice in states $|\omega_x\rangle$, $x = 1, \ldots, |x|$, which are unknown to her, and asks her to transmit these states to Bob. By applying a Bell-state measurement on the input system and her share of the maximally entangled state, Alice projects Bob's system onto the states $\rho_{a|\omega_x}^B = U_a|\omega_x\rangle\langle\omega_x|U_a^\dagger$, where U_a is a known unitary operation that depends on the outcome a of the Bell state measurement. By classically communicating the outcome a to Bob, he can correct the unwanted unitary U_a, and then send his system to the verifier, who is able to check whether it is the same as the one provided to Alice. Note that for the purpose of verification, it is completely equivalent if Alice communicates the outcome a to the verifier instead of Bob, who can then check if the state Bob sent—which is now uncorrected—is equal to the state given to Alice, modulo the correction, see Fig. 1.

In any realistic teleportation scheme, the states and measurements used will not be perfect. In this case the states that Bob receives after Alice applies a measurement with POVM elements M_a^{VA} on systems V and A are given by

$$\rho_{a|\omega_x}^B = \frac{\text{tr}_{VA}[(M_a^{VA} \otimes \mathbb{1}^B)(|\omega_x\rangle\langle\omega_x|^V \otimes \rho^{AB})]}{p(a|\omega_x)}, \quad (1)$$

where ρ^{AB} is the state shared by Alice and Bob, and $p(a|\omega_x) = \text{tr}[(M_a^{VA} \otimes \mathbb{1}^B)(|\omega_x\rangle\langle\omega_x|^V \otimes \rho^{AB})]$ is the probability of the particular outcome a given that the verifier gives to Alice the state $|\omega_x\rangle$. The standard figure of merit used to quantify how well such a teleportation scheme performs is the average fidelity between the input and output states of the process [9],

$$\bar{F}_{\text{tel}} = \frac{1}{|x|} \sum_{a,x} p(a|\omega_x)\langle\omega_x|U_a\rho_{a|\omega_x}^B U_a^\dagger|\omega_x\rangle. \quad (2)$$

Clearly, in the case of a perfect teleportation scheme, $\bar{F}_{\text{tel}} = 1$, while in real experiments one always obtains a smaller value. In the other direction, in a classical teleportation scheme—one where Alice and Bob do not share any entanglement—the maximum fidelity of teleportation that they can obtain, called the classical average fidelity, is denoted by $\bar{F}_{\text{cl}}$. Thus, an imperfect teleportation scheme is certified to be nonclassical if $\bar{F}_{\text{tel}} > \bar{F}_{\text{cl}}$ [8]. It turns out that some entangled states can never lead to $\bar{F}_{\text{tel}} > \bar{F}_{\text{cl}}$, a famous example being bound entangled states [10]. Thus, according to this benchmark, these entangled states are useless for teleportation (although they can help in improving $\bar{F}_{\text{tel}}$ of a combined state [11]).

However, notice that one has more information in a teleportation experiment than simply the value of $\bar{F}_{\text{tel}}$. In particular, the verifier has access to $\{|\omega_x\rangle\}_x$, $\{\rho_{a|\omega_x}^B\}_{a,x}$, and $\{p(a|\omega_x)\}_{a,x}$. Whenever these data cannot be explained by a classical teleportation scheme then nonclassical teleportation has clearly taken place. In principle, there could even
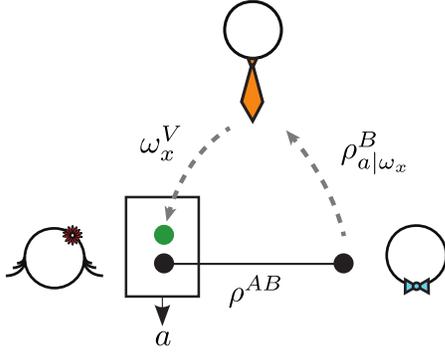
FIG. 1. Teleportation scenario: Alice and Bob share a bipartite state ρ^{AB}. A verifier, who wants to check whether this state is entangled, sends systems in one of the states ω_x^V to Alice, and asks her to transmit it to Bob. Alice applies a global measurement on the state given to her by the verifier and her share of ρ^{AB}, which produces the states $\rho_{a|\omega_x}^B$ for Bob. The verifier has to determine if ρ^{AB} is entangled based on the knowledge of $\{\omega_x^V\}_x$ and $\{\rho_{a|\omega_x}^B\}_{a,x}$.

exist a situation where $\bar{F}_{\text{tel}} \leq \bar{F}_{\text{cl}}$, but for which no classical teleportation scheme can explain the full data observed in the experiment.

The goal of this Letter is twofold. First we propose a method to quantify the nonclassicality of a teleportation scheme that uses the full data available. This method can be implemented by semidefinite programming (SDP) and provides nonclassical teleportation witnesses, which generalize the average fidelity of teleportation. Second, we prove that every entangled state can be used to implement a teleportation scheme that is nonclassical. This is true even with incomplete Bell state measurements, or when utilizing inefficient detectors.

Quantifying nonclassicality of teleportation.—For convenience, in what follows we will work with the set of unnormalized teleported states

$$\sigma_{a|\omega_x}^B = \text{tr}_{VA}[(M_a^{VA} \otimes \mathbb{1}^B)(\omega_x^V \otimes \rho^{AB})]$$
$$= \text{tr}_V[M_a^{VB}(\omega_x^V \otimes \mathbb{1}^B)], \quad (3)$$

where the state given to Alice by the verifier is now simply denoted ω_x^V, which need not be a pure state, and

$$M_a^{VB} = \text{tr}_A[(M_a^{VA} \otimes \mathbb{1}^B)(\mathbb{1}^V \otimes \rho^{AB})]. \quad (4)$$

The normalization factor $p(a|\omega_x) = \text{tr}[\sigma_{a|\omega_x}]$ is the probability that Alice receives outcomes a given that the input state was ω_x^V. Equation (3) describes teleportation as a collection of channels from V to B, labeled by a, that transform the input states ω_x^V into the (unnormalized) output states $\sigma_{a|\omega_x}^B$, according to the channel operators M_a^{VB}. Note that, due to the normalization condition $\sum_a M_a^{VA} = \mathbb{1}^{VA}$, the channel operators satisfy $\sum_a M_a^{VB} = \mathbb{1}^V \otimes \rho^B$, where ρ^B

is Bob's reduced state, which can be seen as a no-signaling condition.

Consider now the case where ρ^{AB} is a separable state, $\rho^{AB} = \sum_\lambda p_\lambda \rho_\lambda^A \otimes \rho_\lambda^B$, which we will see captures a completely general classical teleportation scheme. In this case the channel operators (4) become

$$M_a^{VB} = \sum_\lambda p_\lambda \text{tr}_A[(M_a^{VA} \otimes \mathbb{1}^B)(\mathbb{1}^V \otimes \rho_\lambda^A \otimes \rho_\lambda^B)]$$
$$= \sum_\lambda p_\lambda M_{a|\lambda}^V \otimes \rho_\lambda^B, \quad (5)$$

where $M_{a|\lambda}^V = \text{tr}_A[M_a^{VA}(\mathbb{1}^V \otimes \rho_\lambda^A)]$, and Eq. (3) becomes

$$\sigma_{a|\omega_x}^B = \sum_\lambda p_\lambda \text{tr}[M_{a|\lambda}^V \omega_x^V]\rho_\lambda^B. \quad (6)$$

This actually describes the most general classical teleportation scheme: a classical variable λ is sampled from p_λ and sent to Alice and Bob. Upon receiving λ Alice measures the verifiers' system V using the measurement operators $\{M_{a|\lambda}^V\}_a$ and obtains result a according to the distribution $p(a|\omega_x, \lambda) = \text{tr}[M_{a|\lambda}^V \omega_x^V]$. Bob, in turn, upon receiving λ prepares the state ρ_λ^B, which he then sends to the verifier as the teleported state.

Given the structure of this classical teleportation channel, we can test if a given set of teleportation data is nonclassical by solving the following optimization problem:

$$\text{given} \quad \{\sigma_{a|\omega_x}^B\}_{a,x},$$
$$\mathcal{T}_R(\sigma_{a|\omega_x}) = \min_{r,\{M_a^{VB}\}} r$$
$$\text{s.t.}$$
$$\frac{1}{(1+r)}\sigma_{a|\omega_x}^B + \frac{r}{(1+r)}\frac{1}{o_A}\frac{\mathbb{1}^B}{d} = \text{tr}_V[M_a^{VB}(\omega_x^V \otimes \mathbb{1}^B)]$$
$$\forall\, a,x,$$
$$M_a^{VB} \in \mathcal{S} \quad \forall\, a,$$
$$\sum_a M_a^{VB} = \mathbb{1}^V \otimes \frac{\rho^B + r\frac{\mathbb{1}^B}{d}}{1+r}, \quad (7)$$

where o_A is the number of outcomes a, $\mathcal{S}$ denotes the set of separable operators (i.e., of the form $\sum_\lambda \tau_\lambda \otimes \chi_\lambda$, with $\tau_\lambda \geq 0$ and $\chi_\lambda \geq 0$ for all λ). The optimal solution r^* of this problem gives the minimum amount of "white noise" that has to be added to the teleportation data such that the mixture admits a classical scheme. We call $\mathcal{T}_R(\sigma_{a|\omega_x}) = r^*$ the random teleportation robustness of the data $\{\sigma_{a|\omega_x}^B\}_{a,x}$ [12–14].

Note that although the set of separable operators has a complicated structure [15], we can nevertheless relax $\mathcal{S}$ in Eq. (7) to be the set of operators with positive partial

transposition (PPT) [16], which has a simple characterization in terms of a single semidefinite constraint. In this case the above test becomes an instance of a strictly feasible semidefinite program [17], which can be easily solved with available software [18]. Moreover, in the case of qubit teleportation, since the PPT criterion is necessary and sufficient for testing separability [19], Eq. (7) (without relaxation) is already an SDP, and therefore straightforward to solve. In higher dimensions other semidefinite relaxations of the set of separable operators have also been proposed and can be readily implemented [20].

Every entangled state leads to nonclassical teleportation.—As we show in Ref. [21], in the case that (i) one of Alice's measurement operators corresponds to a projection onto a maximally entangled state (e.g., $M_1^{VA} = |\Phi^+\rangle\langle\Phi^+|$ with $|\Phi^+\rangle = \sum_{i=1}^d |ii\rangle^{VA}/\sqrt{d}$) and (ii) the inputs ω_x are tomographically complete, $\mathcal{T}_R(\sigma_{a|\omega_x}) = E_R(\rho^{AB})$, where $E_R(\rho^{AB})$ [24] is the random robustness of the state ρ^{AB}, defined as

$$E_R(\rho^{AB}) = \min_{r,\sigma_S} \ r$$
$$\text{s.t.} \quad \frac{1}{1+r}\rho^{AB} + \frac{r}{1+r}\frac{\mathbb{1}}{d^2} = \Sigma_S,$$
$$\Sigma_S \in \mathcal{S}. \quad (8)$$

Since $E_R(\rho^{AB})$ is non-null if and only if ρ^{AB} is entangled [24], this result shows that every entangled state can lead to a nonclassical teleportation data. Second, since the only requirement is that one of the Alice's measurement operators is a projection onto a maximally entangled state, the demonstration of nonclassical teleportation can be done with partial Bell state measurements. This is experimentally good, since some setups naturally use these type of measurements due to the impossibility of performing a complete Bell state measurement with linear optics [25] or the use of inefficient detectors. Finally, it gives a one-to-one correspondence between how fragile the entanglement of a state is and how well this state can be used as a nonclassical teleportation channel.

In Ref. [21], we also prove a quantitative relation between the robustness of teleportation and the average fidelity of teleportation. Namely, for any set of teleported states coming from ρ^{AB} we have that

$$\mathcal{T}_R(\sigma_{a|\omega_x}) \geq \frac{\bar{F}_{\text{tel}}(\sigma_{a|\omega_x}) - \bar{F}_{\text{cl}}}{\bar{F}_{\text{cl}} - 1/d}. \quad (9)$$

This bound makes it clear that $\mathcal{T}_R(\sigma_{a|\omega_x})$ is stronger than $\bar{F}_{\text{tel}}$ as a quantifier of nonclassical teleportation for any set $\{\sigma_{a|\omega_x}\}$, since $\bar{F}_{\text{tel}}(\sigma_{a|\omega_x}) > \bar{F}_{\text{cl}}$ implies that $\mathcal{T}_R(\sigma_{a|\omega_x}) > 0$ [26]. Moreover, this bound can be tight: In the case of perfect teleportation using a maximally entangled state, a tomographically complete set of inputs, and a Bell state

measurement, the left-hand side becomes $\mathcal{T}_R(\sigma_{a|\omega_x}) = E_R(\rho^{AB})$ as discussed before. Moreover, since the state is maximally entangled we have that $E_R(\rho^{AB}) = d$ [24]. The right-hand side also equals d, since $\bar{F}_{\text{tel}} = 1$ and $\bar{F}_{\text{cl}} = 2/(d+1)$ [10].

Nonclassical teleportation witnesses.—An advantage of having a SDP formulation for teleportation is that it also provides linear constraints satisfied by any teleportation data that admit a classical scheme, which generalize the average fidelity of teleportation. These constraints work as nonclassical teleportation witnesses, which, similarly to the idea of entanglement witnesses [15], can be used to test the nonclassicality of any experimental teleportation data. In Ref. [21] we show that the random teleportation robustness $\mathcal{T}_R(\sigma_{a|\omega_x})$, given by Eq. (7), has the following dual formulation [17]:

$$\text{given} \ \{\sigma_{a|\omega_x}^B\}_{a,x},$$
$$\max_{\{F_{a|\omega_x}^B\},G^{VB}} \text{tr}\sum_{a,x} F_{a|\omega_x}^B \sigma_{a|\omega_x}^B - \text{tr}[G^B\rho^B]$$
$$\text{s.t.} \quad 1 + \frac{1}{o_A d}\text{tr}\sum_{a,x} F_{a|\omega_x}^B - \frac{1}{d}\text{tr}G^{VB} \geq 0,$$
$$-\sum_x \omega_x^V \otimes F_{a|\omega_x}^B + G^{VB} \in \mathcal{W} \quad \forall \ a, \quad (10)$$

The first constraint is a normalization condition, while the second says that $W_a = -\sum_x \omega_x^V \otimes F_{a|\omega_x}^B + G^{VB}$ is an entanglement witness for all a.

See Ref. [21] for explicit examples of nonclassical teleportation witnesses.

Examples.—Let us discuss the relevance of the present results through two concrete examples. We consider teleportation of the states $\{(|0\rangle\pm|1\rangle)/\sqrt{2}, (|0\rangle\pm i|1\rangle)/\sqrt{2}, |0\rangle, |1\rangle\}$ (which are tomographically complete) using the shared states

$$\rho_1^{AB} = p|\Phi^+\rangle\langle\Phi^+| + (1-p)\frac{\mathbb{1}^{AB}}{4} \quad (11)$$

and

$$\rho_2^{AB} = p|\Phi^+\rangle\langle\Phi^+| + (1-p)|01\rangle\langle01| \quad (12)$$

and a full Bell state measurement. The results of the SDP (7) are provided in Fig. 2 as a function of the average fidelity of teleportation (2) when we vary $0 \leq p \leq 1$. First notice that for the same values of $\bar{F}_{\text{tel}}$ the two states give different values for $\mathcal{T}_R$. This means that, although the two states perform equally as quantified by the average fidelity, when quantified instead by the random teleportation robustness ρ_2^{AB} produces teleportation data which is more nonclassical than ρ_1^{AB} does. Second, there is a parameter region for which ρ_2^{AB} is useless for teleportation according to $\bar{F}_{\text{tel}}$ (i.e., $\bar{F}_{\text{tel}} \leq \bar{F}_{\text{cl}}$), but $\mathcal{T}_R$ still certifies that the
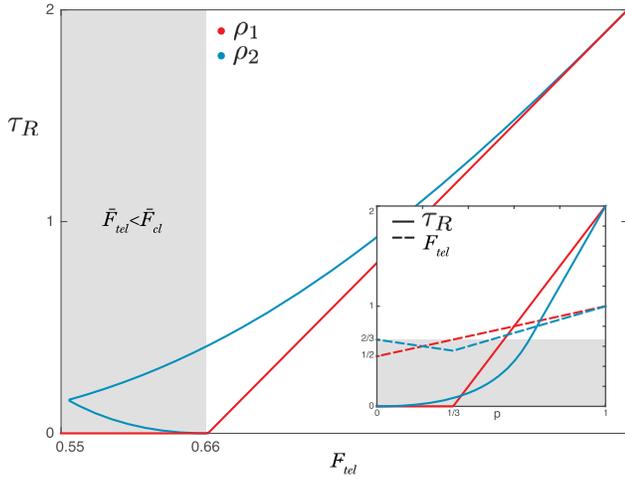
FIG. 2.   Average fidelity of teleportation [33] versus random teleportation robustness for ρ_1^{AB} (red)—Eq. (11)—and ρ_2^{AB} (blue)—Eq. (12). The gray regions show the region where the average fidelity of teleportation is below the classical average fidelity $\bar{F}_{cl} = 2/3$ [34]. The inset shows the same quantities as a function of the noise parameter p for the two states.

teleportation data it produces could not arise from any classical teleportation scheme.

Connection to other notions of nonlocality.—The present study also makes clear some connections between quantum teleportation and other ideas discussed in quantum foundations, such as EPR steering [27,28] and Bell inequalities with quantum inputs [29]. EPR steering is sometimes phrased in terms of a task where Bob wants to certify that he shares entanglement with Alice, but he does not trust her. He then asks her to perform some measurements on her share of the state and applies a test based on the post-measured states he obtains. Notice that this is exactly the teleportation scenario as presented in Fig. 1, but with the crucial distinction that the inputs to Alice's measuring devices are classical variables x, as opposed to the quantum variables ω_x in teleportation. Crucially, due to this difference, not every entangled state is useful for demonstrating steering [30]. Another similar situation is the recently introduced Bell scenario with quantum inputs [29] (see also Ref. [31] for variations), which was later interpreted as the task of measurement-device-independent entanglement detection [32]. The scenario is the same as in quantum teleportation, but now Bob also applies a measurement with a quantum input to his share of the state. We thus see that teleportation relates to Bell inequalities with quantum inputs in exactly the same way that EPR steering relates to Bell nonlocality.

Conclusions.—In this Letter we have studied quantum teleportation using the full data available in an experiment. We have shown that this allows us to test directly whether the data has any classical explanation via the method of semidefinite programming. Using the full data, every entangled state can be certified to implement nonclassical

teleportation, and we show that this can be tested in an experimentally friendly way using a teleportation witness. This overthrows the popular belief that not all entangled states are useful for teleportation (in particular bound entangled states), a conclusion which was based upon a single figure of merit, the average fidelity of teleportation, which our teleportation witnesses generalize.

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[12] Note that we take the data $\{\sigma^B_{a|\omega_x}\}_{a,x}$ to implicitly contain the states to be teleported $\{\omega^V_x\}_x$.

[13] One can also define other notions of teleportation robustness in a straightforward manner, by changing the white noise to

more general types of noise. We focus here on the random robustness only for ease of presentation.

[14] A MATLAB code implementing this SDP is available at https://github.com/paulskrzypczyk/nonclassicalteleportation.

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[26] Notice that the denominator of the right-hand side is non-negative. This is because for a general set of input states $\bar{F}_{cl} \geq 1/d$, as $1/d$ is achieved by the trivial strategy whereby Bob sends $\rho^B_{a|\omega_x} = \mathbb{1}^B/d$ to the verifier, and Alice outputs with any probability distribution $p(a|\omega_x)$.

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