Entanglement is Necessary for Emergent Classicality in All Physical Theories

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One of the most striking features of quantum theory is the existence of entangled states, responsible for Einstein's so called "spooky action at a distance." These states emerge from the mathematical formalism of quantum theory, but to date we do not have a clear idea of the physical principles that give rise to entanglement. Why does nature have entangled states? Would any theory superseding classical theory have entangled states, or is quantum theory special? One important feature of quantum theory is that it has a classical limit, recovering classical theory through the process of decoherence. We show that any theory with a classical limit must contain entangled states, thus establishing entanglement as an inevitable feature of any theory superseding classical theory.

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Introduction.—Entanglement and nonlocality are two of the features of quantum theory that clash most strongly with our classical preconceptions as to how the Universe works. In particular, they create a tension with the other major theory of the twentieth century: relativity [1]. This is most clearly illustrated by Bell's theorem [2,3], in which certain entangled states are shown to violate local realism by allowing for correlations that cannot be explained by classical causal structures [3]. In this Letter we ask whether entanglement is a surprising feature of nature, or whether it should be expected in any nonclassical theory. Could a scientist with no knowledge of quantum theory have predicted the existence of entangled states based solely on the premise that their classical understanding of the world was incomplete?

Any such scientist could reasonably postulate the existence of a classical regime—in that whatever theory describes reality must be able to behave like classical theory in some limit. Although this is a very natural assumption, given that we frequently observe systems behaving classically, we show that it imparts very strong constraints on the structure of any nonclassical theory. Indeed in Ref. [4], Landau and Lifshitz noted. Quantum mechanics occupies a very unusual place among physical theories: it contains classical mechanics as a limiting case, yet at the same time it requires this limiting case for its own formulation.

Thus to answer these questions, we explore all theories that have a classical limit [5,6]. This is formalized in quantum theory by *decoherence maps*, which take quantum systems to semiclassical states with respect to some basis. Physically, decoherence maps represent a quantum system interacting with some inaccessible environment resulting in the loss of quantum coherences. Inspired by this, we develop a generalization of decoherence maps for arbitrary operationally defined theories (see Refs. [7–9] for a related process-theoretic approach). We consider all theories that can decohere to classical theory and show that any such theory either contains entangled states or is classical theory. Thus, the existence of these classically counterintuitive entangled states present in quantum theory can be understood as arising from, and being necessary for, the existence of a classical world. This result hints towards the possibility that other counterintuitive features of quantum theory could be derived from its accommodation of a classical limit, and paves the way for deriving the features of postclassical and postquantum theories from the existence of this limit.

The outline of this Letter is as follows. In the following section we discuss the framework, describe the minimal characteristics expected of the generalized decoherence map, and introduce the class of theories that can exhibit a classical limit through decoherence. In the results section we formally state and outline a proof of our result and in the conclusion section we discuss the physical significance of our result. All technical proofs are given in the corresponding Supplementary Material.

Setup.—To begin to pose questions about how different physical features of theories relate we makes use of the generalized probabilistic theories (GPT) framework [10–13], which is broad enough to describe any operational description of nature. The framework is based on the idea that any physical theory must be able to predict the outcomes of experiments, and, moreover, that the theory should have an operational description in terms of those experiments. This framework is broad enough to describe arbitrary operationally defined theories including but not limited to quantum and classical theory. We provide a brief introduction to the framework and our notation below. For a full introduction to this framework see Refs. [10,11].

A primitive notion in this framework is the idea of a system $\mathcal{S}$, corresponding, for example, in quantum theory to an n-level quantum system. Such a system can be prepared in a variety of ways and so has an associated set of *states* $\Omega_{\mathcal{S}}$. One can perform measurements on the system to determine which state it has been prepared in, and the measurement outcomes are known as *effects* $e \in \mathcal{E}_{\mathcal{S}}$, which are maps $e:\Omega_{\mathcal{S}} \to [0,1]$, determining the probability that outcome e is observed given the system was in state s. Moreover, there will generally be *transformations* $T \in \mathcal{T}_{\mathcal{S}}$ that can be applied to the system; note that if $s \in \Omega_{\mathcal{S}}$ then $T \circ s \in \Omega_{\mathcal{S}}$ and, moreover, that if $e \in \mathcal{E}_{\mathcal{S}}$ then $e \circ T \in \mathcal{E}_{\mathcal{S}}$. Transformations are said to be *reversible* $T \in \mathcal{G}_{\mathcal{S}}$ if T^{-1} is also a valid transformation where $T \circ T^{-1} = \mathbb{1} = T^{-1} \circ T$.

Based on operational ideas we find that these sets of states, effects, and transformations have much more structure. Specifically, the state space has the structure of a finite-dimensional convex set. *Convexity* corresponds to the idea that if one can prepare a system in state s_1 or s_2 then it should be possible to prepare a probabilistic mixture of these two states, for example, conditioned on the outcome of a biased coin flip. If a state can we written as a convex combination of other states $\rho = \sum_i p_i s_i$ we say that s_i *refines* ρ (denoted $s_i \succ \rho \ \forall \ i$ [11]). States that cannot be refined by any other states are called *pure*, otherwise they are called *mixed*. Not all of the well-defined measures of purity in quantum and classical theory will translate to general theories, but there is a sufficient condition for if one state is purer than another that applies to all convex theories. In general, if state σ can be written as a probabilistic mixture involving ρ, but not vice versa (e.g., $\rho \succ \sigma$, $\sigma \not\succ \rho$), then ρ is strictly purer than σ. Operationally, σ can be prepared by an experiment that prepares a probabilistic mixture of states including ρ, but the converse is not true for ρ.

Finite dimensionality comes from the requirement that it should be possible to characterize the state of a system by performing only a finite number of distinct experiments. Moreover, this state space is typically assumed to be compact and closed. Transformations and effects should respect this convex structure, for example, probabilistically preparing state s_1 or s_2 followed by applying some transformation T should be operationally equivalent to probabilistically preparing state $T \circ s_1$ or $T \circ s_2$. This implies that transformations and effects should be *linear* maps.

It is typically useful not to only consider the physical states of the system, but also to consider sub- and supernormalized states. This extends the state space from a convex set $\Omega_{\mathcal{S}}$ in a d-dimensional vector space to a convex cone $\mathcal{K}_{\mathcal{S}}$ living in a $d+1$ dimensional vector space. The state space $\Omega_{\mathcal{S}}$ is recovered by enforcing normalization via the deterministic effect $u_{\mathcal{S}}$. That is that states $s \in \mathcal{K}_{\mathcal{S}}$ are normalized, and thus belongs to in $\Omega_{\mathcal{S}}$, if $u_{\mathcal{S}}(s) = 1$, subnormalized if $u_{\mathcal{S}}(s) < 1$ and supernormalized otherwise. Effects now extend to linear maps $e:\mathcal{K}_{\mathcal{S}} \to \mathbb{R}^+$ and transformations extend to linear maps on the cone. Reversible transformations, therefore, must be automorphisms of the cone that preserve the normalized state space. Beyond these minimal assumptions, we place no further constraints on the state space $\Omega_{\mathcal{S}}$, which can take the form of arbitrary convex sets (see the Supplemental Material [14], for example). In the statement of our results we use the notion of the *faces* of a convex set. These are defined in the Supplemental Material [14] but can be understood intuitively as the convex subsets that form the boundary of the convex set. For example, for the three dimensional cube the faces are the squares, edges, and vertices on the boundary of the cube.

The above is best illustrated with examples, the key examples here being quantum theory and classical probability theory. Given an n-level quantum system the convex cone is given by the set of positive semidefinite Hermitian matrices and the deterministic effect by the trace, such that normalized states are density matrices. Given an n-level classical system the convex cone is given by real vectors with non-negative entries and the deterministic effect by the covector with all entries 1 such that normalized states correspond to probability distributions over an n element set. Effects are then linear functionals on these cones in quantum theory corresponding to POVM elements and in classical theory to covectors with elements ≤ 1. Transformations are then linear maps between these cones, in quantum theory corresponding to *CP* maps and in classical theory to substochastic matrices. Reversible transformations are then cone automorphisms that preserve the normalized states; in quantum theory these will be unitary transformations and in classical theory permutations of the underlying set.

There is a final key aspect of a theory that we are yet to discuss, and that is how to form *composite* systems. Given two systems $\mathcal{S}_1$ and $\mathcal{S}_2$ with their associated state spaces or cones, effects and transformations, there should be a way to form a composite system, denoted $\mathcal{S}_1 \otimes \mathcal{S}_2$. Note that here we use the symbol $\otimes$ to denote the construction of a bipartite system, which need not be related to the vector space tensor product [15]. There are various operational constraints on this product $\otimes$ [10], for example, that if one can prepare system $\mathcal{S}_1$ in state s_1 and $\mathcal{S}_2$ in state s_2 then there should be a state, denoted $s_1 \otimes s_2$ which represents independently preparing the two systems in these two states. Similar statements and constraints can be made for the effects and transformations of the theory. These operational constraints, however, do not uniquely specify a way to form composite systems, as generally composite systems allow for more than just doing things on each system independently. Given local state spaces $\mathcal{S}_1$ and $\mathcal{S}_2$ there are therefore many different possible composite systems that could be formed (see, for example, Ref. [10]). An important feature of these composites is whether or not the bipartite state spaces exhibit entanglement, which we now define.

Definition 1 (Entanglement).—A state ψ belonging to the bipartite state space $\mathcal{S}_1 \otimes \mathcal{S}_2$ is entangled if and only if it cannot be written in the following form:

$$\psi = \sum_i p_i s_i \otimes s_i', \quad p_i \geq 0, \sum_i p_i = 1,$$

where $s_i \in \Omega_{\mathcal{S}_1}$, $s_i' \in \Omega_{\mathcal{S}_2}$; i.e., a state is entangled if it cannot be seen as the convex combination of product states.

In this Letter we will show that entangled states are a feature of any nonclassical theory which can decohere to classical theory. We prove this by showing that any theory without entanglement that decoheres to classical theory must be classical theory itself. As such, we need a way to define classical theory and theories without entanglement. Constraining a theory to have no entanglement is equivalent to fixing a particular choice of tensor product for the theory, that is, the *min-tensor* product [10]. Therefore, rather than defining the general requirements of a tensor product we will just consider this particular case.

Definition 2 (Min-tensor product $\boxtimes$).—The min-tensor product for combining systems $\mathcal{A}$ and $\mathcal{B}$ is defined by

$$\mathcal{K}_{\mathcal{A}\boxtimes\mathcal{B}} \coloneqq \mathsf{Conv}[\{a \otimes b | a \in \mathcal{K}_{\mathcal{A}}, b \in \mathcal{K}_{\mathcal{B}}\}]$$

$$u_{\mathcal{A}\boxtimes\mathcal{B}} = u_{\mathcal{A}} \otimes u_{\mathcal{B}},$$

where here $\otimes$ is the vector space tensor product.

Note that as classical theory exhibits no entanglement, systems compose under the minimal tensor product, $\otimes = \boxtimes$. We can now define theories without entanglement and classical theory.

Definition 3 (Generalized probabilistic theory without entanglement).—A GPT without entanglement is defined by a collection of systems $\{\mathcal{S}\}$, their associated effects $\mathcal{E}_{\mathcal{S}}$, and transformations between them $\mathcal{T}_{\mathcal{S}\to\mathcal{S}'}$, and, the composite of systems $\mathcal{S}$ and $\mathcal{S}'$ is given by the min-tensor product, $\mathcal{S}\boxtimes\mathcal{S}'$.

Definition 4 (Classical probabilistic theory).—An N-level classical system, denoted Δ_N has a state space which is an N vertex simplex. These compose under the min-tensor product and satisfy

$$\Delta_N \boxtimes \Delta_M = \Delta_{NM}.$$

Reversible transformations correspond to permutations of the vertices of the simplex. Effects are any linear functional $e: \mathcal{K}_{\mathcal{S}} \to \mathbb{R}^+$, $e: \Omega_{\mathcal{S}} \to [0,1]$.

An interesting feature of classical and quantum theory is that they obey the *no-restriction* hypothesis [10,16,17], which states all mathematically well-defined effects are allowed in the theory and can be experimentally realized. We do not make this assumption when considering theories that can decohere to classical theory. Finally, we must define and characterize the generalized decoherence-to-classical map, which we discuss in the following section.

Decoherence.—It is physically well motivated to postulate that, in any reasonable theory of nature, be it quantum or post-quantum, systems must be able to behave classically. Indeed the GPT framework is fundamentally built on the assumption that we have a *classical interface* with the world. We can choose, potentially using classical randomness, which experiment to perform, and we can characterize states, effects, and transformations in terms of classical probability distributions that we obtain from experiments. However, ultimately this classical interface should be explainable from the theory itself rather than just being an external structure. This is indeed the case in quantum theory, where we can view the classical interface as an effective description of *decohered* quantum systems. It therefore seems like any well-founded GPT should have an analogous decoherence mechanism so as to explain how it gives rise to the classical interface. We now consider the key features of quantum to classical decoherence which we then take to define decoherence for generalized theories.

For each quantum system $\mathcal{Q}$ there is a decoherence map $D_{\mathcal{Q}}$ and classical system with state space $\Delta_{N(\mathcal{Q})}$ where the decoherence map is given by $D_{\mathcal{Q}}[\rho] \coloneqq \sum_{i=1}^N \langle i|\rho|i\rangle |i\rangle\langle i|$. This map has the following key properties:

Definition 5 (Decoherence maps).—Purely decoherence maps, in quantum and general theories, obey the following properties. (1) Physicality: the decoherence map is a physical map, typically considered to be arising from an interaction with some environmental system that is then discarded, and hence must satisfy all of the constraints on transformations in a GPT. In particular, it must be *linear* and map states to states. (2) Idempotence: in quantum theory the decoherence map destroys the coherences between the basis states, and so applying it a second time does nothing to the state. In general, the decoherence map should restrict the state space to a classical subspace which is invariant under repeat applications. Therefore, applying it twice is the same as applying it once and $D_{\mathcal{S}}[D_{\mathcal{S}}[\sigma]] = D_{\mathcal{S}}[\sigma]\ \forall \sigma \in \Omega_{\mathcal{S}}$. (3) Purity decreasing: the decoherence map arises from losing information to an environment and as such it cannot increase our knowledge of the input state. Therefore, $D[\rho]$ cannot be strictly purer than ρ for any input state ρ. For example, in quantum theory a decoherence map will not map mixed states to pure states. In general, a state ρ is strictly purer than state σ if $\rho \succ \sigma$ and $\sigma \not\succ \rho$. Therefore, if $D_{\mathcal{S}}[\rho] \succ \rho$, then $\rho \succ D_{\mathcal{S}}[\rho]$, else $D_{\mathcal{S}}[\rho]$ is strictly purer that ρ.

In general, this map could decohere to any subtheory. However, we are interested, in particular, with decoherence maps that take systems $\mathcal{S}$ to classical systems.

Definition 6 (Decoherence to classical theory).—A theory decoheres to classical theory if it has a decoherence map (Definition 5) for each system which obey the following. (1) State space: the most obvious constraint is that image of the decoherence map is a classical state space:

$$D_{\mathcal{S}}(\Omega_{\mathcal{S}}) = \Delta_{N(\mathcal{S})}.$$

However, we do not just want to reproduce the states of classical theory, but the full theory, including its dynamical and probabilistic structure. (2) Effect space: classical effects should also arise from the original theory. That is, that for every effect in classical theory there is some effect in the full theory that behaves as the classical effect when we restrict to $\Delta_{N(\mathcal{S})}$. This can be formalized as for all $e_{\text{classical}}$ there exists $e \in \mathcal{E}_{\mathcal{S}}$ such that

$$e_{\text{classical}} = e \circ D_{\mathcal{S}}.$$

(3) (Reversible) transformations: similarly for (reversible) transformations we expect for any classical (reversible) transformation t there is a corresponding postclassical (reversible) transformation with the same action on the image of $D_{\mathcal{S}}$. This can be formalized as for all classical reversible transformations $T_{\text{classical}}$ there exists some reversible $\mathcal{T} \in \mathcal{T}_{\mathcal{S}}$ such that

$$T_{\text{classical}} = T \circ D_{\mathcal{S}}.$$

(4) Composites: finally, we expect decoherence to act suitably with composition; i.e., if system $\mathcal{S}_1$ decoheres to Δ_N via $D_{\mathcal{S}_1}$ and system $\mathcal{S}_2$ to Δ_M via $D_{\mathcal{S}_2}$, then the composite system $\mathcal{S}_1 \otimes \mathcal{S}_2$ can decohere to $\Delta_N \otimes \Delta_M = \Delta_{NM}$ via $D_{\mathcal{S}_1} \otimes D_{\mathcal{S}_2}$.

To give an example, consider quantum theory for qubits, which decohere to classical theory for bits by applying the standard dephasing map in (for example) the computational basis $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$. Measurements in the computational basis provide the classical measurements, utilising the quantum effects $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$. Two qubits can decohere independently to classical bits, which can then interact and all permutations of composite bits can be achieved by the classical CNOT and bit flip operations, which are provided by the quantum CNOT unitary and the Pauli-X unitary. Also note that the dephasing map obeys all conditions in Definition 5.

Results.—We are now in a position to prove our main result. If a theory can decohere to classical theory and does not have entanglement, then the original systems must be composites including a classical system, $\Omega_{\mathcal{S}} = \Delta_N \otimes \Omega_f$, and the decoherence map simply discards any nonclassical subsystems. More succinctly: theories with nontrivial decoherence must have entangled states.

The proof of this is provided in the Supplementary Material along with all necessary mathematical definitions and background to understand the proof. However, we will also provide an outline of the proof here. We first show—by considering the consequences of decoherence for single systems—that the state space $\Omega_{\mathcal{S}}$ has the following geometric properties:

Result 1: If $D_{\mathcal{S}}[\Omega_{\mathcal{S}}] = \Delta_N$, where $D_{\mathcal{S}}$ obeys Definitions 5 and 6, then the state space $\Omega_{\mathcal{S}}$ has the following properties. (1) $\Omega_{\mathcal{S}}$ is the minimal face of a set of faces $f_1, f_2, \ldots, f_N$ that are isomorphic $f_i \cong f_j \ \forall\ i, j$, disjoint (share no states) $f_i \cap f_j = \emptyset \ \forall\ i, j$ and are exposed, (2) each face f_i decoheres uniquely to a pure classical state s_i, $D_{\mathcal{S}}[f_i] = s_i$.

We then consider the consequences of decoherence on composite systems. Essentially, as the resulting classical systems must be able to interact under classical dynamics we deduce the following additional constraints.

Result 2: If $D_{\mathcal{S}}[\Omega_{\mathcal{S}}] = \Delta_N$, where $D_{\mathcal{S}}$ obeys Definitions 5 and 6 and $D_{\mathcal{S}}[\Omega_{\mathcal{S}}] \boxtimes D_{\mathcal{S}}[\Omega_{\mathcal{S}}] = \Delta_{N^2}$ which enjoys the full set of classical dynamics in Definition 4, then (1) The classical faces are linearly independent, $\mathsf{Span}[f_i] \cap \mathsf{Span}[\bigcup_{k\neq i} f_k] = \{0\} \ \forall\ i$. (2) $\Omega_{\mathcal{S}}$ is the convex hull of these classical faces $\Omega_{\mathcal{S}} = \bigvee_i f_i = \mathsf{Conv}[\{f_i\}]$. Given these constraints on the state space it is then simple to show that it is a composite of a classical state space with a state space isomorphic to these faces, and the decoherence map simply discards the nonclassical subsystem.

Result 3: For any theory that decoheres to classical theory as per Definitions 5 and 6, and whose composition rule $\otimes$ is given by the minimal tensor product $\boxtimes$, all state spaces are of the form

$$\Omega_{\mathcal{S}} = f \boxtimes \Delta_N$$

and the decoherence map is of the form

$$D_{\mathcal{S}} = (s \circ u)_f \otimes \mathbb{1}_{\Delta_N},$$

where u is the discarding effect and s is some fixed internal state of f. E.g., decoherence of nonclassical systems comprises of discarding them.

Therefore, if we restrict ourselves to considering nonclassical theories, the only decoherence-to-classical map possible is the trivial map, where we discard our nonclassical systems. For example, in quantum theory this would correspond to all quantum systems regardless of their state or dimension decohering to the zero-dimensional classical state, and the resulting classical theory being trivial.

Discussion.—In this Letter we have shown that if a theory has a nontrivial decoherence mechanism, such that decoherence is not simply discarding the system, then the theory must have entangled states. It therefore seems that entanglement, rather than being a surprising feature of nature, is an entirely inevitable feature of any postclassical theory. A natural question to ask is what other features of quantum theory can be reproduced simply by demanding that the theory has a classical limit.

There are myriad other physical features that could be implied from the existence of a classical limit such as information causality [18], bit symmetry [19], and macroscopic locality [20] to name but a few. Of particular interest would be deriving genuine device-independent nonlocality. The existence of entangled states is in general a necessary but insufficient condition for observing violations of Bell inequalities. For example, nonseparable states are present in the local theory of Spekken's toy model [21]. On the other hand, it has been shown that all entangled states in quantum theory display some hidden nonlocality [22,23]. By determining the additional structure present in quantum theory that gives this correspondence between entanglement and nonlocality, it could be possible to derive the violation of Bell inequalities from purely physical postulates. Given the simplicity of the postulates used to derive the existence of entangled states, it is plausible that the postulates that give rise to Bell nonlocality are similarly mundane.

This notion of decoherence has allowed us to define a new class of GPTs—those with a classical limit. There is clear physical motivation to consider this class. For example, if a theory were not to have such a limit then one would have to posit the existence of two fundamentally distinct types of systems, the classical systems (which are how we interact with the world) along with postclassical systems (which we cannot directly probe). Such a fundamental distinction appears unnatural, and so it seems that decoherence is a necessary feature of any sensible operational theory. However, while being a physically well-motivated class, it nonetheless provides a great deal of mathematical structure and as such gives a more powerful framework for studying generalized theories.

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[1] A. Einstein, Annalen der Physik **49**, 769 (1916).
[2] J. S. Bell, Rev. Mod. Phys. **38**, 447 (1966).
[3] C. J. Wood and R. W. Spekkens, New J. Phys. **17**, 033002 (2015).
[4] L. D. Landau and E. M. Lifshitz, Course of Theoretical Physics (Elsevier, 2013), p. 3.
[5] W. H. Zurek, Nat. Phys. **5**, 181 (2009).
[6] E. Joos, H. D. Zeh, C. Kiefer, D. J. Giulini, J. Kupsch, and I.-O. Stamatescu, *Decoherence and the Appearance of a Classical World in Quantum Theory* (Springer Science & Business Media, New York, 2013).
[7] J. Selby and B. Coecke, Entropy **19**, 174 (2017).
[8] C. M. Lee and J. H. Selby, arXiv:1701.07449.
[9] B. Coecke, J. Selby, and S. Tull, arXiv:1701.07400.
[10] J. Barrett, Phys. Rev. A **75**, 032304 (2007).
[11] G. Chiribella, G. M. D'Ariano, and P. Perinotti, Phys. Rev. A **81**, 062348 (2010).
[12] L. Hardy, arXiv:1104.2066.
[13] L. Masanes, A. Acin, and N. Gisin, Phys. Rev. A **73**, 012112 (2006).
[14] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.119.080503 for proofs and mathematical background.
[15] Under the assumption of local tomography [17], the vector spaces containing the cones will indeed compose under the standard vector space tensor product. However, there is still freedom to choose how the bipartite cone within this composite vector space [10].
[16] P. Janotta and R. Lal, Phys. Rev. A **87**, 052131 (2013).
[17] L. Hardy, arXiv:quant-ph/0101012.
[18] M. Pawłowski, T. Paterek, D. Kaszlikowski, V. Scarani, A. Winter, and M. Żukowski, Nature (London) **461**, 1101 (2009).
[19] M. P. Müller and C. Ududec, Phys. Rev. Lett. **108**, 130401 (2012).
[20] M. Navascués and H. Wunderlich, *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences* (The Royal Society, London, 2009).
[21] R. W. Spekkens, Phys. Rev. A **75**, 032110 (2007).
[22] F. Buscemi, Phys. Rev. Lett. **108**, 200401 (2012).
[23] L. Masanes, Y.-C. Liang, and A. C. Doherty, Phys. Rev. Lett. **100**, 090403 (2008).