Dynamics of Fragmented Condensates and Macroscopic Entanglement

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The relative phase of the order parameters in the collision of two condensates can influence the outcome of their collision in the case of weak coupling. With increasing interaction strength, however, the initially independent phases of the two order parameters in the colliding partners quickly become phase locked, as the strong coupling favors an overall phase rigidity of the entire condensate, and upon their separation the emerging superfluid fragments become entangled.

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Since the gauge symmetry is spontaneously broken in superfluids, it is reasonable to wonder under what conditions the relative phase of two superfluids is physically relevant. The Josephson effect [1,2], experiments with cold Bose or Fermi atoms [3–8], and the superfluid fragments emerging from nuclear fission [9–11] are just a few examples where that is the case. As we will discuss here, there are other situations when one would, however, expect that the relative phase of two condensates is physically irrelevant. However, the emerging overall picture of the role of the relative phase of two condensates appears to be more complex than envisaged so far. Recently Magierski, Sekizawa, and Wlazłowski (MSW) [12] reported on a rather surprising observation concerning the role the pairing field plays in the collisions of two heavy ions at energies near the Coulomb barrier. MSW observed a very strong dependence of the properties of the emerging fragments on the relative phase of the pairing condensates in the initial colliding nuclei. In a somewhat related study of $^{20}O + ^{20}O$ [13], the reported effect was rather weak, a result confirmed in the similar case of $^{44}Ca + ^{44}Ca$ [14], due to the small number of nucleons above the closed shell. The amplitude of the pairing field Δ in nuclei is of the order of 1 MeV, which is significantly smaller than the magnitude of the normal single particle field, which is of the order of 50 MeV. The character of the nuclear pairing correlations is recognized in literature of being of the Bardeen-Cooper-Schriefer (BCS) type [15], a theory which describes weak coupling pairing with Cooper pairs with sizes significantly larger than the average separation between fermions. The gain in binding energy due to pairing correlations, called condensation energy $E_{\rm cond} = -N(0)|\Delta|^2/2$, can hardly be greater than perhaps a few MeV. MSW report, however, that in the collision of ^{240}Pu on ^{240}Pu near the Coulomb barrier pairing effects can lead to changes in the total kinetic energy of the emerging fragments of up to 20 MeV and that the apparent height of the fusion barrier could be changed by 10 MeV or even more. These dramatic changes, with an energy significantly higher than the magnitude of the total pairing condensation energy, were correlated by MSW with the relative phase of the pairing fields in the two colliding partners prior to collisions.

The gauge symmetry breaking bears similarity with the rotational symmetry breaking in the case of deformed nuclei, when their relative orientations plays a noticeable role in heavy-ion fusion reactions and various decays. The MSW results, obtained by solving the time-dependent density functional theory (TDDFT) equations, can be reproduced semiquantitatively using a simple Ginzburg-Landau approach [16,17], or the formally equivalent static Gross-Pitaevskii (GP) description [18,19]. When the two nuclei touch, the phase of the condensate can change across the contact region, as in a domain wall, in a manner superficially similar to the tunneling current in a Josephson junction [1,2], albeit in the absence of a barrier.

In the presence of pairing correlations the ground state of a nucleus is a Bose-Einstein condensate (BEC) of Cooper pairs, which in theory is accurately described in the grand canonical ensemble, where only the average particle number is specified. The phase of the order parameter $\hat{\phi}$ is conjugate to the particle number $\hat{N}$, and thus in a system with well-defined particle number the phase is undefined [20,21]. However, as Anderson points out [22], in a bucket of liquid helium below the λ point "ϕ has become a classical variable, … any future experiment will be interpretable as though ϕ was fixed." This is also the prevalent approach in describing nuclei with well-defined pairing correlations, when the effect of particle projection is small. One can thus reasonably ask a common question in condensed matter physics, "Can a nucleus have a well-defined phase of the condensate with respect to another nucleus?" Since the total wave function of the two nuclei prior to their interaction is merely a product of two independent wave functions, one would expect that the interaction between two nuclei cannot depend on the phases of each initial wave function. A (relative separation) coordinate dependence of the phase of the pairing field indicates the presence of a current. The phases of the pairing fields can be changed by arbitrary and independent gauge transformations in each partner prior to the moment the two nuclei touch and, thus, one can generate a phase gradient in the "neck." An objection raised by Bertsch in discussions was that initial nuclei have well-defined proton and neutron numbers, unlike the anomalous densities that are the central objects in a DFT approach, and the phase of the wave function of each nucleus prior to the collision should be physically irrelevant. Clearly, a similar argument would not be accepted in the case of deformed nuclei, as a number of observables are impacted (α-decay penetrability, heavy-ion fusion cross sections, etc.). This kind of argumentation began at the inception of quantum mechanics, and many have wondered about similar problems; see Anderson [22] and the follow-up spirited discussion. As Anderson writes, "if the experimenter now cools down two entirely different, non-communicating buckets of liquid helium from $T > T_\lambda \to T \approx 0$, … upon opening an orifice between the two, would see initially with equal probability any *fixed* value of the phase difference, and thereafter no experiment he tried could recover the components of the wave-function which started out with different relative phases. He would not see *zero* interference current, … ." This situation corresponds theoretically to a fragmented condensate [23], and the inability of the experimenter to recover the initial state is due to the fact that the two buckets became macroscopically entangled after being in contact for some time. Macroscopic entanglement of up to hundreds to millions of particles have been put in evidence experimentally [24–28]. It is crucial to recognize that there are two qualitative steps in Anderson's gedanken experiment: the creation of the initial state and the subsequent emergence of the final state. This is also the situation in the MSW simulations, and the natural question arises of why these authors did not observe the outcome conjectured in Anderson's gedanken experiment, as the outcome of their collisions showed a strong dependence on the initial relative phase of the condensates, unlike what Anderson conjectured. We are not aware, however, of any experiments in which the dependence on the strength of the coupling on the outcome of a collision and of the entanglement have been studied.

There is, however, another qualitatively different situation, relevant to experiments performed in cold gases [3–8] or to superfluid fragments emerging from nuclear fission [9–11]. This happens when one cools down a bucket of helium from above the λ transition, and subsequently separates it into two parts kept always close to $T \approx 0$ and reunites them after they had different histories, and the two parts remain macroscopically entangled at all times [22]. In this situation, the relative phase of the two buckets is always rather well defined, but the particle numbers in the two buckets are not. (We will not discuss here the role the phase diffusion can play).

There will definitely be increasingly more studies of colliding superfluid nuclei and other systems in the future performed within the only practical microscopic framework available so far, the TDDFT. A correct interpretation of such numerical simulation results and a correct method to evaluate observables are stringent elements of our theoretical tools, tools which are still not yet ascertained. Nuclei contain many particles, are essentially macroscopic objects, and as Anderson has also noted [22], "… the central problem of measurement theory is not the quantum mechanics of atoms, which is simple and easy, but the fact that macroscopic everyday objects are very difficult indeed for the quantum theory to deal with properly." Many properties of nuclei (liquid drop mass formula, surface tension, compressibility, symmetry energy, hydrodynamics, collective motion, rotation, symmetry breaking, transport coefficients, etc.) can be and are often treated quite accurately using concepts characteristic for macroscopic systems.

In order to shed light on MSW's very startling observation, that the relative phase of the pairing fields in two colliding nuclei can have a dramatic role in the collision process, we will turn at first to a simpler system, in which the role of the relative phase of two condensates can be easily studied. In the presence of pairing correlations, nuclei can be treated as a BEC of interacting Cooper pairs, as in the case of electrons in superconductors [15], and the total wave function can be represented as an antisymmetrized product of Cooper pair wave functions. In the case of a weakly interacting Bose system at zero temperature, a GP equation is extremely accurate [29]. In the GP approximation a boson field operator $\hat{\psi}(\mathbf{r})$ is replaced with its nonvanishing average $\psi(\mathbf{r}) = \langle 0|\hat{\psi}(\mathbf{r})|0\rangle$ [a classic example of U(1) broken gauge symmetry] and the accuracy of the approximation is of order $\sim 1/\sqrt{N}$, where N is the total number of bosons. A BCS fermionic condensate is a system of weakly interacting Cooper pairs or bosons, and qualitatively a GP equation is appropriate and has been used numerous times in the past. The weakness of the interaction is typically characterized by the ratio of the pairing gap to the Fermi energy $\Delta/\varepsilon_F \ll 1$. In the weak coupling limit, all Cooper pairs have a zero momentum, as in a BEC.

Typical BEC systems have all particles in one cloud and the one-body density matrix acquires the form

$$\rho(\mathbf{r}_1, \mathbf{r}_2) = \langle 0|\hat{\psi}^\dagger(\mathbf{r}_1)\hat{\psi}(\mathbf{r}_2)|0\rangle \Rightarrow n_0\psi^*(\mathbf{r}_1)\psi(\mathbf{r}_2),$$

when $|\mathbf{r}_1 - \mathbf{r}_2| \to \infty$, and there is only one eigenvector with a macroscopic eigenvalue $n_0 = O(N)$, a situation known as the off-diagonal long-range order [16,30–32]. It is possible to have a fragmented BEC system [23], when two or more eigenvalues of the one-body density matrix $\rho(\mathbf{r}_1, \mathbf{r}_2)$ are macroscopically large. This is the case of two BEC clouds with particle numbers N_1 and N_2 in two spatially well separated potential traps $V_k(\mathbf{r})$, $\int d^3\mathbf{r}|\psi_k(\mathbf{r},t)|^2 = N_k$, $k=1,2$,

$$i\hbar\dot{\psi}_k(\mathbf{r}, t) = -\frac{\hbar^2}{2m}\Delta\psi_k(\mathbf{r}, t) + V_k(\mathbf{r})\psi_k(\mathbf{r}, t)$$
$$+g|\psi_k(\mathbf{r}, t)|^2\psi_k(\mathbf{r}, t) = \mu_k\psi_k(\mathbf{r}, t) = \mu_k\phi_k(\mathbf{r})e^{-i\mu_k t/\hbar}. \quad (1)$$

Let us consider now this fragmented BEC, when their initially spatially well-separated trapping potentials are moving towards each other, and their combined wave function at times before the two clouds come into contact is naturally given by

$$\Psi(\mathbf{r},t) = \psi_1(\mathbf{r},t) + e^{i\alpha}\psi_2(\mathbf{r},t), \tag{2}$$

$$\psi_k(\mathbf{r},t) = \phi_k(\mathbf{r}-\mathbf{r}_k-\mathbf{u}_k t)e^{im\mathbf{u}_k\cdot\mathbf{r}/\hbar - i\mu_k t/\hbar - im\mathbf{u}_k^2 t/2\hbar}, \tag{3}$$

with $V_k(\mathbf{r}) \to U_k(\mathbf{r},t) = V_k(\mathbf{r}-\mathbf{r}_k-\mathbf{u}_k t)$ and $e^{i\alpha}$ arbitrary. Using $\Psi(\mathbf{r},t)$ one can construct a coherent state $\exp[\tau\int d^3\mathbf{r}\Psi(\mathbf{r},t)\hat{\psi}^\dagger(\mathbf{r})]|0\rangle$, and the fragmented BEC state is obtained only after a specific particle projection is performed; see the discussion below and in connection with Eq. (4) and the Supplemental Material (SM) [33]. We will assume that the velocities $\mathbf{u}_k$ are significantly smaller in magnitude than the speed of sound [29] $c = \sqrt{g|\Psi(\mathbf{r},t)|^2/m}$ evaluated in the central part of the cloud, and therefore superfluidity is not endangered. At all times this combined wave function satisfies the time-dependent GP equation with $U(\mathbf{r},t) = U_1(\mathbf{r},t) + U_2(\mathbf{r},t)$:

$$i\hbar\dot{\Psi}(\mathbf{r},t) = \left[-\frac{\hbar^2}{2m}\Delta + g|\Psi(\mathbf{r},t)|^2 + U(\mathbf{r},t)\right]\Psi(\mathbf{r},t).$$

Before contact each component of the total wave function $\psi_k(\mathbf{r},t)$, see Eq. (1), satisfies its own time-dependent GP equation (1) with $V_k(\mathbf{r}) \to U_k(\mathbf{r},t)$. The arbitrary phase $\exp(i\alpha)$ can arguably influence the dynamics if $g \neq 0$. This is the phase in one of the two cases of liquid helium buckets discussed by Anderson [22]. Unlike the overall phase of the many-body wave function, this phase cannot be removed now, similarly to the relative orientation of two colliding deformed nuclei. In the case of two separated condensates, the overall order parameter is the sum of the two separated order parameters, similarly to magnetization, for example. (The action of the magnetic field on the spin coordinate of a fermion is formally identical to the action of the pairing field on the two components of the fermionic quasiparticle [34].) Magnetization is created by electric currents and magnetic moments, and when one brings two magnets into proximity, the two magnetic fields add up, even though the many-body electron wave functions for the two separated magnets are multiplied to each other. As in the case of a magnetic field, where the relative orientation of two magnetic fields is important, and in the case of the complex pairing field, the relative phase of the two fields is important, as is in the case of Josephson junctions, too. This relative phase is also arbitrary, but this relative phase can be controlled in some instances. In the vicinity of an isolated cloud one can apply for a finite interval of time a constant potential over the isolated cloud, a procedure performed in the case of cold atoms in experiments, equivalent to performing a local gauge transformation, and thus one can change the relative phase of two clouds [4–8].

By analyzing both the GP equation, see SM [33], and the collision of superfluid nuclei, we arrived at a totally unexpected and surprising result, that the strength of the interaction g plays a qualitative role in the dynamics. By increasing the strength of the interaction from zero (corresponding to the case of noninteracting bosons or absence of pairing correlations in nuclei) to a relatively large value, the character of the collision changes dramatically, but in a continuous manner.

We observe the establishment of a common phase of the combined condensate for large values of the coupling constant, which clearly can be attributed to the phase rigidity in superfluids [16,17,20,22,35]. While the two partners are in contact, the phase of the condensate becomes spatially constant over the entire system, and the phase gets locked. We illustrate the phase locking mechanism for both Fermi and Bose superfluid systems: with the collision of two superfluid nuclei described within the extension of TDDFT formalism to superfluid fermionic systems [36] by changing the strength of the pairing correlations, see Fig. 1, and with the case of the collision of two BECs with relevant results in the SM [33].

One can limit the analysis to a one-dimensional model as only matter, momentum, and energy transfer between two

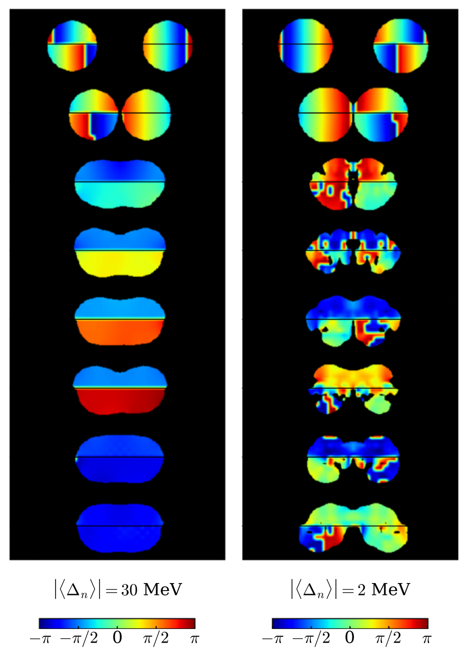

FIG. 1. The evolution of the phase of the pairing field (time runs top to bottom) in the head-on collision of $^{120}Sn + ^{120}Sn$ [9–11], simulated with the phenomenological energy density functional SLy4 and pairing as described in Ref. [37]. The right-hand and the left-hand columns correspond to a realistic or artificially increased pairing field strength, respectively. The upper and lower half of each frame corresponds to an initial phase difference between the two initial pairing condensates of 0 and π, respectively. Even though the pairing field magnitudes are constant before the colliding nuclei come into contact, their phases change in time and space (first two top frames); see Eq. (3). The phase locking of the pairing field is clearly manifest after fusion in the left-hand column, but absent in the right-hand column.

colliding partners along the line joining the two partners (which can rotate in space though) are controlling most of the dynamics, similarly to the case of the Josephson junction in the case of superconductors, when only dynamics across the junction is typically analyzed. In the absence of the interaction ($g \equiv 0$), the GP equation is linear, and each wave function $\psi_k(\mathbf{r}, t)$ satisfies independently the Schrödinger equation

$$i\hbar\dot{\psi}_k(\mathbf{r}, t) = -\frac{\hbar^2}{2m}\Delta\psi_k(\mathbf{r}, t) + U(\mathbf{r}, t)\psi_k(\mathbf{r}, t),$$

and after the two potential wells have passed each other, each wave function $\psi_k(\mathbf{r}, t)$ will split in between the two potential wells. Obviously, the linear combination of the wave functions $\Psi(\mathbf{r}, t) = \psi_1(\mathbf{r}, t) + e^{i\alpha}\psi_2(\mathbf{r}, t)$, which satisfies the same Schrödinger equation, depends on the relative phase. While for weak coupling g the dynamics is α dependent, when the strength of the interaction g is gradually increased, the dependence of the final outcome on the relative phase α becomes weaker and weaker the stronger the interaction gets, and the two cases $\alpha = 0$ and $\alpha = \pi$ in their final state become almost identical; see Fig. 1 for nuclei in 3D and the SM for bosons [33]. When the coupling constant is sufficiently large, the two boson clouds penetrate each other and their final states are relatively little affected irrespective of the value of α, and both clouds emerge with the initial number of particles practically unchanged and with very small excitation energies as well [33]. The role of the particle-particle interaction is to lead upon contact to a very rapid phase locking between the two condensates after which the properties of the final state depend very weakly on the phase $\exp(i\alpha)$. The strength of the interaction g controls the speed at which the information is transmitted throughout the cloud. In the case of strong coupling, after the relatively short time needed to send "messages" between the two partners, the properties of the emerging final state are largely α independent and the two clouds become completely entangled upon separation. The total wave function corresponds in this case to a coherent state in the particle number difference, $N_- = N_1 - N_2$, and to a macroscopically entangled state of two large objects. This conclusion is in agreement with Anderson's conjecture [22] concerning the inability of an experimenter to recover the initial relative phase of the condensates α after establishing the contact between the two independently cooled liquid helium buckets from above T_λ. This also clarifies the content of Anderson's conjecture that only when the superfluid correlations are "strong" enough the role of the initial relative phase is erased. This is also consistent with the generalized phase rigidity due to the term in the Ginzburg-Landau equation $n_s\hbar^2|\nabla\phi|^2/2m$ (where n_s is the superfluid density) in the free energy of superfluids [16,17,20,22,35], which is an emerging term, whose presence and strength are dictated by the interactions, and which is absent in noninteracting systems.

This dependence on α of the properties of the emerging fragments in the case of "weak" superfluid correlations reflects particle number difference fluctuations between the two initial partners; see also SM [33]. The combined wave function of two superfluid nuclei (with even particle numbers), depending on two arbitrary gauge angles τ and α, can be written as [38] (here, for simplicity for one kind of nucleons only)

$$|\Psi(\tau, \alpha) = \prod_k [u_k + e^{i2\tau}e^{i2\alpha}v_k a_k^\dagger a_{\bar{k}}^\dagger] \times \prod_l [u_l + e^{i2\tau}e^{-i2\alpha}v_l a_l^\dagger a_{\bar{l}}^\dagger]|0\rangle, \qquad (4)$$

where k and $\bar{k}$ and l and $\bar{l}$ denote pairs of time-reversed states in the two nuclei and $u_{k,l}$ and $v_{k,l}$ are the corresponding amplitudes of the Bogoliubov-Valatin quasiparticles. Integrating $\Psi(\tau, \alpha)$ over τ with the weight $e^{-i\tau N_+}$ will select the wave function with the total particle number $N_+ = N_1 + N_2$. Integrating $\Psi(\tau, \alpha)$ over α with the weight $e^{-iN_-\alpha}$ will select the exact particle difference $N_- = N_1 - N_2$ between the two nuclei. In the case of weak coupling, an additional projection over the relative phase α is required to ensure that the particle number difference between the two initial partners has the expected value, namely, exactly zero ($\Delta N \equiv 0$) in the case of two identical nuclei; see also SM [33]. One can expect that total kinetic energy and fusion rates distributions would become wider in the case of superfluid colliding nuclei.

When comparing our simulations of ^{240}Pu fission [37] with realistic pairing interactions with simulations in which the pairing field was artificially increased to ≈3–4 MeV [9–11], we observed a similar transition to a phase locking pattern: realistic nuclear pairing strength is relatively weak, the phase locking does not typically occur on the way from saddle to scission, and the phase and the magnitude of the pairing fields fluctuate strongly in both space and time. In the case of strong pairing [9–11], even though the time from saddle to scission is about 10 times shorter, the evolution is almost identical to the dynamics of an ideal or perfect fluid and the fission fragments emerge strongly entangled. While one might naively expect a faster rate of energy transfer from collective to intrinsic degrees of freedom, the fluctuations of the pairing field are greatly suppressed (due to larger gaps and larger critical velocities) and the evolving fissioning nucleus stays cool.

In conclusion, we have established that the initial relative phase of two colliding condensates plays an increasingly smaller role in the case of strong interactions, when a phase locking over the entire system is established fast (unless the entire system is very extended and the signal propagation time is large as well), and after the separation the final macroscopic (large) fragments emerge entangled.

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