Emergence of Multiscaling in a Random-Force Stirred Fluid

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We consider the transition to strong turbulence in an infinite fluid stirred by a Gaussian random force. The transition is defined as a first appearance of anomalous scaling of normalized moments of velocity derivatives (dissipation rates) emerging from the low-Reynolds-number Gaussian background. It is shown that, due to multiscaling, strongly intermittent rare events can be quantitatively described in terms of an infinite number of different "Reynolds numbers" reflecting a multitude of anomalous scaling exponents. The theoretically predicted transition disappears at $R_\lambda \leq 3$. The developed theory is in quantitative agreement with the outcome of large-scale numerical simulations.

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Introduction.—If an infinite fluid is stirred by a Gaussian random force supported in a narrow interval of the wave numbers $k \approx 2\pi/L$, then a very weak forcing leads to the generation of a random, close-to-Gaussian, velocity field. In this flow, the mean velocity $\bar{\mathbf{u}} = 0$, and one can introduce the large-scale Reynolds number $\mathrm{Re} = u_{\mathrm{rms}}L/\nu$, where $u_{\mathrm{rms}} = \sqrt{\overline{|\mathbf{u}|^2}}$ is the root-mean-square velocity, the overbar denotes a suitable average, and ν is the kinematic viscosity. Increasing the forcing amplitude or decreasing viscosity results in a strongly non-Gaussian random flow with moments of velocity derivatives obeying the so-called anomalous scaling. This means that the moments $\overline{(\partial_x u)^{2n}}/\overline{(\partial_x u)^2}^n \propto \mathrm{Re}^{\rho_{2n}}$ where the exponents ρ_n are, on the first glance, unrelated "strange" numbers. In this Letter, we investigate the transition between these two different random or chaotic flow regimes. First, we discuss some general aspects of the traditional problem of hydrodynamic stability and transition to turbulence.

Fluid flows can be described by the Navier-Stokes equations subject to boundary and initial conditions (the density is taken to be $\rho = 1$ without the loss of generality):

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f}, \quad (1)$$

where p is pressure. For an incompressible flow, the additional solenoidal condition $\nabla \cdot \mathbf{u} = 0$ needs to be satisfied. The characteristic velocity and length scales u and L, respectively, used for making the Navier-Stokes equations dimensionless, are somewhat arbitrary. In the problem of a flow past cylinder, it is natural to choose $f = 0$, $u = U$, and $L = D$, where U and D are the free-stream velocity and cylinder diameter, respectively. In a pipe or channel flow, $u = U = (1/H)\int_0^H u(y)dy \propto u_{\mathrm{centerline}}$ is the mean velocity averaged over the cross section, and $L = H$ is a half-width of the channel. In a fully turbulent flow in an infinite fluid, one typically takes $u = u_{\mathrm{rms}}$ and L equal to the integral scale of turbulence. Some other definitions will be discussed below.

Depending on the setup, a flow can be generated by pressure or temperature gradients, gravity, rotation, electromagnetic fields, etc., represented as forcing functions on the right side of (1). If viscosity $\nu \geq \nu_{tr}$ and the corresponding Reynolds number $\mathrm{Re} = (uL/\nu) \leq \mathrm{Re}_{\mathrm{tr}} = (uL/\nu_{\mathrm{tr}})$, the solution to (1) driven by the regular (not random) forcing $\mathbf{f}$ is laminar and regular. As an example, we may recall the parabolic velocity profile $u(y)$ in pipe and channel flows with a prescribed pressure difference between the inlet and outlet. In this case, the no-slip boundary conditions are responsible for the generation of the rate of strain $S_{ij} = (\partial_i u_j + \partial_j u_i)/2$. Another important example is the so-called Kolmogorov flow in an infinite fluid driven by the forcing function $\mathbf{f} = U(0, 0, \cos kx)$. In Benard convection, the relevant regular patterns are rolls appearing as a result of the instability of the solution to the conductivity equation. Thus, the remarkably successful science of the transition to turbulence deals mainly with various aspects of nonequilibrium order-disorder or laminar-to-turbulent transition.

In this Letter, we consider a completely different class of flows. In general, the unforced Navier-Stokes equations, being a very important and interesting object, do not fully describe the physical reality which includes Brownian motion, light scattering, random wall roughness, uncertain inlet conditions, stirring by "random swimmers" in biofluids, etc. For example, a fluid in thermodynamic equilibrium satisfies the fluctuation-dissipation theorem stating that there exists an exact relation between viscosity ν in (1) and a random noise $\mathbf{f}$ which is a Gaussian force defined by the correlation function [1]:

$$\overline{f_i(\mathbf{k},\omega)f_j(\mathbf{k}',\omega')} = (2\pi)^{d+1}D_0 d(k)P_{ij}(\mathbf{k})\delta(\omega+\omega')\delta(\mathbf{k}+\mathbf{k}'), \quad (2)$$

where D_0 is the amplitude of the forcing and $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2$ is the projection operator. The function $d(k)$ defines the distribution of forcing amplitude across the wave numbers. For example, in an equilibrium fluid, thermal fluctuations responsible for Brownian motion are generated by the forcing (2) with $D_0 d(k) = (k_B T\nu/\rho)k^2 \equiv D_0 k^2$ and k_B is the Boltzmann constant. It is clear that, in general, the function $d(k)$ in (2) depends on the physics of a flow.

The random-force-driven Navier-Stokes equation can be written in the Fourier space:

$$u_l(\mathbf{k},\omega) = G^0 f_l(\mathbf{k},\omega) - \frac{i}{2}G^0\mathcal{P}_{lmn}\int u_m(\mathbf{q},\Omega)u_n(\mathbf{k}-\mathbf{q},\omega-\Omega)d\mathbf{q}d\Omega, \quad (3)$$

where $G^0 = (-i\omega + \nu k^2)^{-1}$, $\mathcal{P}_{lmn}(\mathbf{k}) = k_n P_{lm}(\mathbf{k}) + k_m P_{ln}(\mathbf{k})$, and, introducing the zero-order solution $\mathbf{u}_0 = G^0\mathbf{f} \propto \sqrt{D_0}$, so that $\mathbf{u} = G^0\mathbf{f} + \mathbf{v}$, one derives the equation for the perturbation $\mathbf{v}$:

$$\begin{aligned} v_l(\hat{k}) = &-\frac{i}{2}G^0(\hat{k})\mathcal{P}_{lmn}(\mathbf{k})\int v_m(\hat{q})v_n(\hat{k}-\hat{q})d\hat{q} \\ &-\frac{i}{2}G^0(\hat{k})\mathcal{P}_{lmn}(\mathbf{k})\int [v_m(\hat{q})G^0(\hat{k}-\hat{q})f_n(\hat{k}-\hat{q}) + G^0(\hat{q})f_m(\hat{q})v_n(\hat{k}-\hat{q})]d\hat{q} \\ &-\frac{i}{2}G^0(\hat{k})\mathcal{P}_{lmn}(\mathbf{k})\int G^0(\hat{q})f_m(\hat{q})G^0(\hat{k}-\hat{q})f_n(\hat{k}-\hat{q})d\hat{q}, \end{aligned} \quad (4)$$

where the 4-vector $\hat{k} = (\mathbf{k},\omega)$. If Eqs. (3) and (4) are driven by a regular force or boundary and/or initial conditions, then at low Reynolds number (Re) it typically describes a regular (laminar) flow field $\mathbf{u}_0$ with $\mathbf{v} = 0$. With an increase of the Reynolds number $\mathrm{Re} \geq \mathrm{Re}_{\mathrm{inst}}$, this zero-order solution can become unstable, meaning that initially introduced small perturbations $\mathbf{v}$ grow in time. The further increase of Re leads first to weak interactions between the modes describing the "gas" of these perturbations, and, eventually, when $(\mathrm{Re} - \mathrm{Re}_{\mathrm{inst}}/\mathrm{Re}_{\mathrm{inst}}) \gg 1$, the mode coupling described by Eq. (4) becomes very strong. This regime we call "fully developed" or strong turbulence. The problem of hydrodynamic stability is notoriously difficult, and we know very little about the structure of the solution for perturbations in the nonuniversal range $\mathrm{Re} \approx \mathrm{Re}_{\mathrm{inst}}$.

Here we are interested in a simplified problem of a flow generated by a Gaussian random force (2) with a well-understood zero-order solution $\mathbf{u_0} = G^0\mathbf{f}$ which is *not* a result of an instability of a regular laminar flow but is prescribed by a choice of a random force (2). The advantages of this formulation are clear from (4) describing the dynamics of perturbation $\mathbf{v}$ driven by an induced forcing given by $O(f^2) \propto D_0$, the last term in (4). It is easy to see [1,2] that a dimensionless expansion parameter, related to a Reynolds number (see below), is $\Gamma_0^2 = (D_0 L^4/\nu^3)\Delta$, where $\Delta = \int d(k)d\mathbf{k}$, and, since we keep $L = O(1)$, $\nu = O(1)$, and $\Delta = O(1)$, the variable forcing amplitude D_0 can be treated as a dimensionless expansion parameter. Thus, as $D_0 \to 0$, all contributions to the right side of (4) can be neglected, and, if $\mathbf{f}$ stands for the Gaussian random function, then the lowest-order solution $\mathbf{u}_0$ is a Gaussian field. However, there always exist low-probability rare events with $|\mathbf{v}| \geq |\mathbf{u}_0|$ responsible for the strongly non-Gaussian tails of the probability density function (PDF). Thus, in this flow Gaussian velocity fluctuations coexist with the low-probability powerful events where a substantial fraction of kinetic energy is dissipated. At even higher Reynolds numbers (see below), the nonlinearity in (4) dominates the entire field. This complicated dynamics has been observed in experiments on a channel flow with rough ("noisy") walls [3].

This regime is characterized by the generation of velocity fluctuations $\mathbf{v}(\mathbf{k},t)$ in the wave number range $k > 2\pi/L$, where the "bare" forcing $\mathbf{f}(\mathbf{k}) = 0$, which is the hallmark of turbulence. The above example shows that, at least in some range of the Reynolds number, low- and high-order moments may describe very different physical phenomena. The transition between these two chaotic or random states of a fluid is a topic of interest to us in this Letter.

In what follows, instead of Re, we will use the more common Taylor Reynolds number defined as $R_\lambda \equiv \sqrt{5/3\mathcal{E}\nu}u_{\mathrm{rms}}^2$ with $\mathcal{E}$ being the mean energy dissipation rate and $u_{\mathrm{rms}}^2 = 2\mathcal{K}$ with $\mathcal{K}$ the kinetic energy of velocity fluctuations. At high Reynolds numbers, the two are related as $R_\lambda^2 \propto \mathrm{Re}$.

Two cases are of special interest. In the low-Reynolds-number regime (below transition), when $R_\lambda < R_\lambda^{\mathrm{tr}}$, the integral ($L$), dissipation ($\eta$), and Taylor ($\lambda$) length scales are of the same order. Therefore, $(\partial_x u)_{\mathrm{rms}} = [u(x+\eta) - u(x)]_{\mathrm{rms}}/\eta \approx [u(x+L) - u(x)]_{\mathrm{rms}}/L$ and, since we are interested in instability of a Gaussian flow, the moments

$$M_n^< = \frac{\overline{(\partial_x u)^{2n}}}{\overline{(\partial_x u)^2}^n} = (2n-1)!!$$

independent of the Reynolds number. In this case, since the $2n$th-order moment can be expressed in powers of the variance, this means that $(\partial_x u)_{\rm rms}$ is a single parameter (derivative scale) representing statistical properties of the flow in this regime. This is not always the case. The rms velocity derivative in high-Reynolds-number turbulent flows, $(\partial_x u)_{\rm rms} = \sqrt{\overline{(\partial_x u)^2}}$, is only one of an infinite number of independent parameters needed to describe the field and in the vicinity of transition $\mathrm{Re} \geq \mathrm{Re}^{\rm tr}$:

$$M_n^> = \frac{\overline{(\partial_x u)^{2n}}}{\overline{(\partial_x u)^2}^n} = (2n-1)!!C_n \mathrm{Re}^{\rho_n},$$

where the proportionality coefficients $C_n = O(1)$ [4,5].

Below, this anomalous state of a fluid we call strong turbulence, as opposed to the close-to-Gaussian low-Reynolds-number flow field, considered above. In a transitional, low-Reynolds-number, flow we are interested in here, the forcing, Taylor, and dissipation scales are of the same order: $L \approx \eta \approx \lambda$. This implies that $M_n^>$ can also be written as $\approx (2n-1)!!R_\lambda^{\rho_n}$ if $O(1)$ constants are omitted. Furthermore, since $\lambda^2 = u_{\rm rms}^2 / \overline{(\partial_x u)^2}$, we find that, close to transition, the Reynolds number based on the Taylor length scale is

$$R_\lambda = \sqrt{\frac{5}{3\mathcal{E}\nu}} u_{\rm rms}^2 \approx \sqrt{\frac{5L^4}{3\mathcal{E}\nu} \overline{(\partial_x u)^2}}. \quad (5)$$

The physical meaning of this parameter can be seen readily: Multiply and divide (5) by ν and by the dissipation scale η^2. This gives

$$R_\lambda \propto \frac{L^2}{\eta^2} \times \eta^2 \sqrt{\frac{\bar{\mathcal{E}}}{\nu^3}} \approx \frac{L^2}{\eta^2},$$

where $\eta^4 \bar{\mathcal{E}}/\nu^3 = O(1)$. The effective Reynolds number $O(L^2/\eta^2)$, which is the measure of the spread of the inertial range in k space, is a coupling constant, familiar from dynamic renormalization group applications to randomly stirred fluids. To describe strong turbulence, one must introduce an infinite number of "Reynolds" numbers

$$R_{\lambda,n} = \sqrt{\frac{5L^4}{3\mathcal{E}\nu} \overline{(\partial_x u)^{2n}}^{1/n}} \propto R_\lambda^{\rho_{2n}/n} \propto \frac{L^2}{\eta^2} \frac{\overline{\mathcal{E}^n}^{1/n}}{\bar{\mathcal{E}}}, \quad (6)$$

where close to transition points where $\eta \approx L$ we set $R_\lambda \equiv R_{\lambda,1} \approx \mathrm{Re}$. The expressions for the exponents ρ_{2n} can be written in terms of ξ_n, the inertial-range scaling exponents of longitudinal structure functions, as

$$\rho_{2n} = 2n + \frac{\xi_{4n}}{\xi_{4n} - \xi_{4n+1} - 1}; \qquad \xi_n = \frac{0.383n}{1 + \frac{n}{20}}. \quad (7)$$

This expression, which was derived in the "mean-field approximation" in [5,6], agrees extremely well with all available experimental and numerical data (see Refs. [6–9]). Theoretical predictions of anomalous exponents in a random-force-stirred fluid are compared with the results of numerical simulations [7] in the top panel in Fig. 1. The same exponents have been observed in a channel flow [8] and Benard convection [9], indicating universality of small-scale features in turbulent flows.

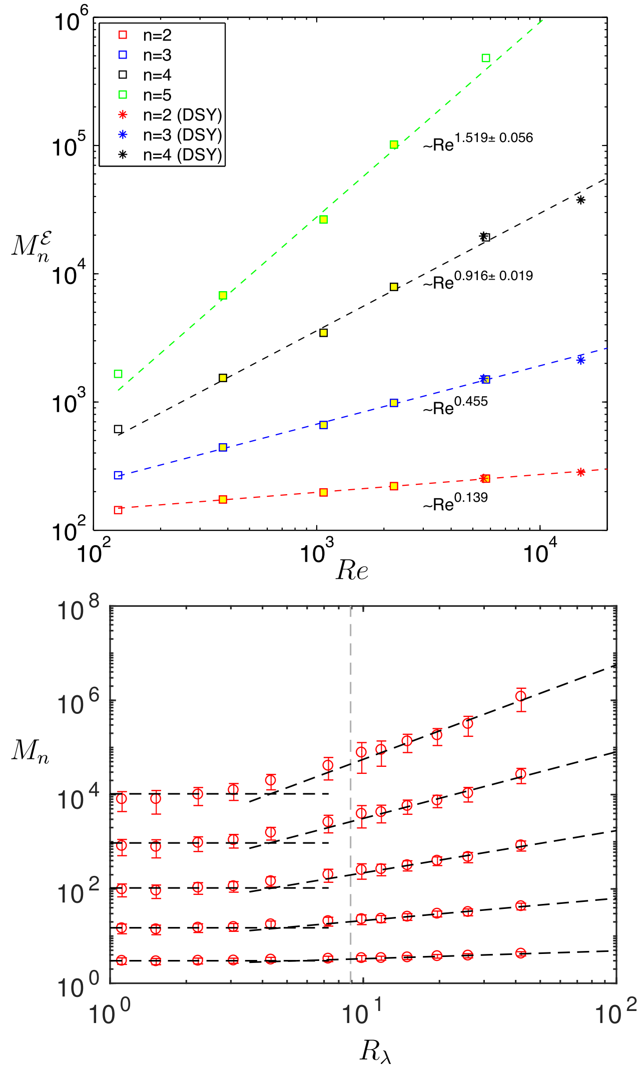

FIG. 1. Top panel: Normalized moments $M_n^{\mathcal{E}} = \overline{\mathcal{E}^n}/\bar{\mathcal{E}}^n$ as a function of the large-scale Reynolds number. Dashed lines: Theoretical predictions and numerical simulations of Refs. [5]. Squares are from Ref. [6] and asterisks from our large DNS database (see, e.g., Ref. [7]). Bottom panel: Transition region. Moments of velocity gradients in the low-R_λ transitional range (present work). From bottom to top, $n = 2$, 4, 6, 8, and 10. Horizontal dashed lines: Gaussian values at $(2n-1)!!$ for reference. The vertical dashed line is at 8.91 for reference.

A critical observation for the analysis that follows is that the n-dependent Reynolds numbers defined in [7] provide a description of regions of the flows with different magnitudes of the local Reynolds numbers: The higher the n, the stronger the fluctuations represented by it. As explained in more detail below, $R_{\lambda,n} \geq R_\lambda^{\rm tr}$ describes a locally turbulent flow.

Transition between Gaussian and anomalous flows.—In this Letter, the transition to turbulence is identified with the first appearance of non-Gaussian anomalous fluctuations of velocity derivatives. The concept is illustrated in the bottom panel in Fig. 1, where moments of velocity derivatives from well-resolved numerical simulations (described below) are plotted against Reynolds numbers $R_\lambda \equiv R_{\lambda,1} \geq 2$. We can see that transition points of different moments, expressed in terms of $R_\lambda \equiv R_{\lambda,1}$, are different, and below we denote them $R_{\lambda,1}^{\rm tr}(n)$. It is important that the transition point for low-order moments M_n has been found at $R_\lambda \equiv R_{\lambda,1} \approx 9$, first discovered in Ref. [6] and analytically derived in [2,10]. This result can be explained as follows.

In accord with the widely accepted methodology, consider the $R_{\lambda,1} \equiv R_\lambda$ dependence of the normalized nth derivative moment M_n in a flow driven by a relatively weak force f and large viscosity ν. Then, gradually decreasing the viscosity, one reaches the critical magnitude $\nu = \nu_{\rm tr}$ corresponding to $R_\lambda^{\rm tr}(n) = R_\lambda^-(n)$, which is the upper limit for Gaussianity of the nth moment. Then, consider the same flow but at a very large Reynolds number (small viscosity). In this strongly turbulent case, the large-scale low-order moments, M_4, for example, are dominated by a huge turbulent viscosity $\nu_T \propto \mathcal{E}^{1/3}L^{4/3}$, the largest effective viscosity, accounting for velocity fluctuations at the scales $r < L$ [1]. The effective Reynolds number, corresponding to the integral scale L, is $R_\lambda^+ \propto \sqrt{L^4/[\mathcal{E}\nu_T(L)]}(\partial_x u)^2_{\rm rms}$. This way, one reaches the smallest possible Reynolds number $R_\lambda \approx 9$ of strongly turbulent (anomalous) flow (see Fig. 1, bottom panel). If, in accord with experimental and numerical data, we assume that the transition is smooth and at a transition point the Reynolds number is a continuous function meaning that $R_\lambda^- = R_\lambda^+$, where $R_\lambda^\pm$ stand for the magnitudes just above and below transition, we can write

$$R_\lambda^{\rm tr}(2) = \sqrt{\frac{5}{3\mathcal{E}\nu_{\rm tr}}}u^2_{\rm rms} = \sqrt{\frac{5}{3\mathcal{E}\nu_T(L)}}u^2_{\rm rms},$$

where the effective viscosity of turbulence at the largest (integral) scale calculated in Refs. [2,10,11] is given by

$$\nu_T \equiv \nu(L) \approx 0.084\frac{\mathcal{K}^2}{\mathcal{E}}, \tag{8}$$

where $\mathcal{K} = u^2_{\rm rms}/2$ stands for kinetic energy of velocity fluctuations. Substituting this into the previous relation gives

$$R_\lambda^{\rm tr}(2) = \sqrt{\frac{5}{3\mathcal{E}\nu}}u^2_{\rm rms} = \sqrt{20/(3\times 0.084)} = 8.91 \approx 9, \tag{9}$$

very close to the outcome of numerical simulations. The coefficient $C_\mu = 0.084$, derived in [9–11], is to be compared with $C_\mu = 0.09$ widely used in engineering turbulent modeling for half a century [12]. It follows from the relations (5) and (6):

$$R_\lambda^{\rm tr}(n) \equiv R_{\lambda,1}^{\rm tr}(n) = (R_{\lambda,n}^{\rm tr})^{n/\rho_{2n}}. \tag{10}$$

Here we point out that, by its definition, $R_{\lambda,n}$ probes increasingly large fluctuations of velocity gradients as n increases. Thus, one expects a transition to anomalous scaling for fluctuations of different intensity when the corresponding Reynolds number $R_{\lambda,n}$ is greater than the transitional $R_\lambda \approx 8.91$. In other words, the n-dependent transitional Reynolds number $R_{\lambda,n}^{\rm tr} \approx 8.91$ is independent of n. This can also be understood in the context of Landau's theory of transition [1], where crossing the transitional Reynolds number results in instability for perturbations but not the zeroth-order solution. In addition to theoretical considerations, the n independence is supported by the agreement with numerical simulations below.

The Reynolds-number dependence of normalized moments of velocity derivative is shown in Fig. 1. The data in the bottom panel in Fig. 1 were generated from a new set of direct numerical simulations (DNS) at very low Reynolds numbers. As in Ref. [7], numerical solutions to Navier-Stokes equations are obtained from Fourier pseudospectral calculations with second-order Runge-Kutta integration in time. The turbulence is forced numerically at the large scales, using a combination of independent Ornstein-Uhlenbeck processes with Gaussian statistics and finite-time correlation. Only low wave number modes within a sphere of radius $k_F \approx 2$ in wave number space are forced. In order to obtain different Reynolds numbers, viscosity is changed accordingly while the forcing at large scales remains constant. In this approach, thus, large scales, and thus the energy flux, remain statistically similar. The resolution is at least $k_{\rm max}\eta \approx 3$ at the highest Reynolds number, which was found to produce converged results at the Reynolds numbers investigated here.

Velocity fields are saved at regular time intervals that are sufficiently far apart (of the order of an eddy-turnover time) to ensure statistical independence between them. For each field, velocity gradient moments are computed and averaged over space. The ensemble average is computed across these snapshots in time and is used to compute confidence intervals also shown in Fig. 1.

The intersection points of curves describing Gaussian moments (horizontal dashed lines) and those corresponding to the fully turbulent anomalous scaling give the transitional $R_\lambda^{\rm tr}(n)$ for each moment. These are compared to the theoretical prediction (10) with $R_{\lambda,n}^{\rm tr} \approx 8.91$ in Fig. 2. This

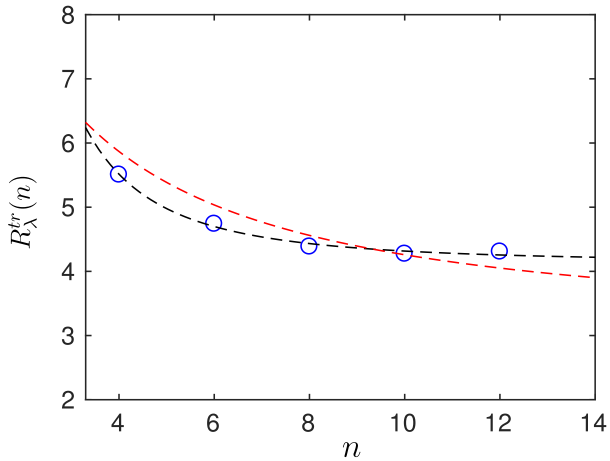

FIG. 2. Transitional Reynolds number $R_{\lambda,1}(n)$ of the nth moment of velocity derivative. Blue: Numerical simulations of the present work. Red: Theoretical prediction (10) with $R^{\rm tr}_{\lambda,n} = 8.91$.

result can be understood as follows: In accord with theoretical predictions, the transitional Reynolds number $R^{\rm tr}_{\lambda,n} \approx 9$ in each statistical realization, independent of n. This implies that each moment M_n becomes fully anomalous only when the corresponding n-dependent $R_{\lambda,n}$ becomes larger than 8.91. Thus, a transition can occur at a global R_λ lower that this transitional Reynolds number (Fig. 1, bottom). In particular, if $R_{\lambda,1} < R^{\rm tr}_\lambda \approx 9$, the transition is triggered by the low-probability violent velocity fluctuations $\overline{(\partial_x u)^n}^{1/n} > (\partial_x u)_{\rm rms}$ coming from the tails of the probability density.

It is also interesting to evaluate the limiting, smallest, transitional Reynolds number following (10) in the limit $n \to \infty$. The relations (5), (6), and (10) give $R_\lambda = R^{\rm tr}_{\lambda,1} \to 2.92$. Evaluated on a popular model $\xi_n = \frac{n}{9} + 2[1 - (\frac{2}{3})^{n/3}]$ [13], one readily derives $R^{\rm tr}_{\lambda,1} \to 3.81$. While the dependence on n differs slightly from the present results at low n, both models predict that, for $R_\lambda \lesssim 3$, no transition to strong turbulence defined by anomalous scaling of moments of velocity derivatives exists. More generally, since the function ξ_n is, under very general conditions, concave and nondecreasing in n [14], then from (7) we have $\rho_{2n} \propto n$ in the $n \to \infty$ limit [15]. Thus, (10) implies that $R^{\rm tr}_{\lambda,1}$ would tend to a constant as $n \to \infty$ for a very large class of physically meaningful functions ξ_n. The implication of this result is that, under very general conditions, there is a minimum Reynolds number under which no moment displays anomalous scaling.

Summary and conclusion.—In this Letter, the problem of a transition between two different *random* states has been studied both analytically and numerically. It has been shown that, while the Gaussian state can be described in terms of the Reynolds number based on the variance of velocity fluctuations, the description of the intermittent state of *strong* turbulence requires an infinite number of "Reynolds numbers" $R_{\lambda,n}$ reflecting the multitude of anomalous scaling exponents of different-order moments (n) of velocity derivatives. This novel concept enables one to account for both typical and violent extreme events responsible for the emergence of anomalous scaling in the "subcritical" state when the widely used Reynolds number $R_{\lambda,1} < R^{\rm tr}_\lambda$ is small. It has also been demonstrated that, in accord with the theory, the critical $R^{\rm tr}_{\lambda,n} \approx 9$ is independent of n. The proposed theory is in good quantitative agreement with the results of large-scale direct numerical simulations presented above. The role of turbulent bursts in low-Reynolds-number flows in various physicochemical processes and the problem of universality will be discussed in future communications.

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