Many-Particle Entanglement Criterion for Superradiantlike States

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We derive a many-particle entanglement criterion for mixed states using a relation between single-mode and many-particle nonclassicalities. The criterion relies on the measurement of collective spin observables. It works very well not only in the vicinity of the Dicke states, but also for the superpositions of Dicke states: superradiant ground states of finite or infinite number of particles and time evolution of single-photon superradiance from an extended sample where random phases appear. We also obtain a criterion for ensemble-field entanglement, which is successful for such kinds of states. We also observe an interesting phenomenon: even though the collective excitation of this many-particle system has a sub-Poissonian character, which results in entanglement, the wave function displays bunching.

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Quantum technologies, such as quantum information, computation, and metrology, necessitate the presence of entanglement. Many-particle entanglement plays a key role in these technologies [1,2]. For instance, entanglement can speed up computations and can increase the response time of devices. Entanglement can also reduce the noise of spin observables, e.g., in squeezed spin states (SSSs) [3], or it can create sub-shot-noise fluctuations, e.g., in Dicke states [4,5].

Single-photon superradiant or subradiant states, superpositions of Dicke states, have drawn great interest in the last decade due to their potential applications in ultrafast data readout [6] and long-lived data storage [7]. When the extent (L) of an ensemble is much larger than the wavelength (λ) of the light, $L \gg \lambda$, random phases between the emitters create collective states known as timed-Dicke states [8]. The nature of timed-Dicke states is much different than the Dicke states and the superradiant states of a small ensemble ($L \ll \lambda$). Recently, several experiments were conducted to explore the behavior of the radiation from such samples [9–11].

In an ensemble of spins, e.g., cold atoms or nitrogen-vacancy centers [12], it is usually not possible to address individual spins. So, searching for entanglement criteria based on the measurements of the collective spin is necessary. Such criteria and inseparability quantifications are demonstrated to work well for SSSs [13–16] and in the vicinity of the Dicke states [17,18]. An entanglement criterion for superradiant (timed-Dicke) states is also necessary.

In this Letter, we derive a many-particle entanglement (MPE) criterion ($\xi_{\rm new} < 0$) which works very well for superradiant states both in the Dicke limit ($L \ll \lambda$) and when the collective state is a superposition of single-atom excitations with random phases (positions) in the $L \gg \lambda$ limit. Criteria working in the vicinity of Dicke states [17] and for SSSs [13] fail in the superradiant states. Simulations show that $\xi_{\rm new}$ also works as a quantifier for superradiant states. We also present a criterion for ensemble-field entanglement ($\mu_{\rm new}$), which is quite successful for such states.

We determine the form of the MPE criterion $\xi_{\rm new}$ from a single-mode nonclassicality (SMNc) condition as follows. First, we realize that criteria for three *kinds* of nonclassicalities, (i) MPE, (ii) SMNc, and (iii) two-mode entanglement (TME) are intimately connected to each other. They can be derived from each other [19–21].

Second, we notice that two SMNc criteria, (a.ii) quadrature squeezing ($\Delta\hat{x}$) [22,23] and (b.ii) number squeezing (sub-Poissonian, $\Delta\hat{n} < \langle\hat{n}\rangle^{1/2}$) [24] form two distinct classes. They address reduced uncertainty in electric or magnetic fields and number of photons, respectively. MPE and TME criteria of each class are related to (a.ii) and (b.ii) as follows. (a.i) Spin-squeezing criterion ($\xi_{\rm spin}$) for MPE [13] and (a.iii) Duan-Giedke-Cirac-Zoller (DGCZ) criterion for TME [25] can be transformed into the (a.ii) quadrature-squeezing (SMNc) criterion, via Holstein-Primakoff (HP) [26] and beam-splitter (BS) [27–29] transformations, respectively [19–21]. Similarly, the (b.iii) Hillery-Zubairy (HZ) criterion [30], which is successful in identifying TME of superpositions of Fock states [31], transforms into the (b.ii) number-squeezing condition [24] via BS transformation [19–21]. So, criteria, enumerated as (a.i)–(a.iii) and (b.i)–(b.iii) form two distinct *groups* (classes). For a better visualization, see the Supplemental Material [32].

One can realize that the MPE criterion ($\xi_{\rm new}$) is missing in group (b), i.e., (b.i). We use the HP transformation and *guess* the form of $\xi_{\rm new}$ from (b.ii), the number-squeezing condition, as follows. We consider an ensemble of N two-level particles. We show that in the $N \to \infty$ limit, inseparability of a symmteric many-particle state implies the nonclassicality of collective (single-mode) excitations ($\hat{c}_e$) of the ensemble. This observation enables us to utilize the HP transformation, e.g., $S_+ = \hat{c}_e^\dagger \hat{c}_g = \hat{c}_e^\dagger \sqrt{N - \hat{c}_e^\dagger \hat{c}_e} \to \sqrt{N}\hat{c}_e^\dagger$ [23,26], to establish a connection between many-particle observables $\hat{S}_{\pm,z}$ and the single-mode operator $\hat{c}_e$. Here, $\hat{c}_e^\dagger$ ($\hat{c}_g$) creates (annihilates) a particle in the excited (ground) state [23]. Taking into account the relation between many-particle and single-mode states, a MPE criterion in terms of $\hat{S}_{\pm,z}$ becomes a SMNc criterion in terms of the $\hat{c}_e$ operator.

In group (a), $\Delta\hat{S}_x \to \Delta(\hat{c}_e^\dagger + \hat{c}_e) = \Delta\hat{x}_e$ leads to quadrature squeezing in the $N \to \infty$ limit. Noting the analogy, $\Delta(\hat{S}_+\hat{S}_-) \to \Delta(\hat{c}_e^\dagger\hat{c}_e) = \Delta\hat{n}_e$, we raise the following question. If we examine the uncertainty of $\hat{\mathcal{R}} = \hat{S}_+\hat{S}_-$, i.e., $\langle(\Delta\hat{\mathcal{R}})^2\rangle = \langle\hat{\mathcal{R}}^2\rangle - \langle\hat{\mathcal{R}}\rangle^2$, will we be able to obtain an inseparability criterion for many-particle systems?

We obtain a criterion ($\xi_{\rm new}$) which works better than our expectations. The strength of violation of this criterion ($\xi_{\rm new} < 0$, or larger squeezing in $\langle(\Delta\mathcal{R})^2\rangle$) accompanies the superradiant phase transition both for a finite and infinite number of particles; see Fig. 2. $\xi_{\rm new}$ also correctly predicts the temporal behavior of the entanglement of (timed) single-photon superradiance [8,33,34], see Fig. 4, for $N = 2000$ atoms placed randomly in a sphere larger than a wavelength [33,34]. The positions of the atoms induce random phases between different single-atom excited states; see Eq. (6) and above. We further test $\xi_{\rm new}$ for 2000 superposition states, where coefficients in front of the Dicke states are assigned randomly. Additionally, we show that our criterion for ensemble-field entanglement, $\mu_{\rm new} < 0$, also works very well for superradiantlike states, Figs. 3 and 4.

Superradiant to normal phase transition can be realized in Bose-Einstein condensates (BECs) by optically tuning the spin-orbit coupling strength [35,36]. Such a setup is a good candidate for continuous monitoring of $\xi_{\rm new}$ [5], with respect to the varying coupling parameter, for the superpositions of symmetric Dicke (superradiant) states, Fig. 2. Our simulations for superpostions of single-atom excited states with random phases, Fig. 4, show that $\xi_{\rm new}$ can reveal the entanglement in recent experiments on single-photon superradiance [9] and subradiance [11] in extended samples of atoms. Different subradiant states can possess different amounts of MPE [7]. Hence, regarding the long-lived data storage, $\xi_{\rm new}$ can be helpful in obtaining information about different superpositions of dark states. $\xi_{\rm new}$ can also be adapted to monitor if spin singlet (Cooper) pairings of electrons are broken into separable states in superconducting structures [37,38]. There are ongoing efforts for entangling more than 2 quantum emitters (QEs) near metal nanostructures via single-plasmon excitations [39,40]. Analogous to squeezing (quadrature) transfer from light to an ensemble [41], ongoing experiments can be visualized as number squeezing $\langle(\Delta\hat{n})^2\rangle$ transfer to the ensemble of QEs. Since $\langle(\Delta\hat{n})^2\rangle$ squeezing is in group (b), $\xi_{\rm new}$ is expected to work well for these many-body states.

In our simulations, we additionally observe the following two interesting phenomena. The wave function operator $\hat{\psi}(\mathbf{r})$ of a condensate becomes bunched (super-Poissonian) above a critical atom-field coupling ($g > g_c$); see Fig. 2(a) in the Supplemental Material [32]. This is opposite of the sub-Poissonian behavior of the collective ($\hat{c}_e$) excitations of the N-particle system [Fig. 2(b)] and the scattered field [see Fig. 2(b) in Supplemental Material]. Such a behavior also occurs in the ground state of an interacting BEC (without a field). Bunching and MPE emerge mutually when interaction (collisions) per particle exceeds the excitation (recoil) energy, $U_{\rm int}/N > \hbar\omega_{\rm exc}$; see Supplemental Material VI [32]. Incidentally, experiments on BECs [42–47] show that a condensate responds to an excitation *collectively* when the energy of the excitation $\hbar\omega_{\rm exc}$ is sufficiently less than the interaction energy per atoms $U_{\rm int}/N = g_s\int d^3\mathbf{r}|\psi(\mathbf{r})|^4/N$.

In the following, we first demonstrate the relation between single-mode and many-particle nonclassicalities. Next, we derive $\xi_{\rm new}$ and $\mu_{\rm new}$. We test them for Dicke states (Fig. 1), superradiant states in the Dicke limit (Figs. 2 and 3), and for the single-photon emission with random phases (Fig. 4).

Relation among nonclassicalities.—Criteria for the three kinds of nonclassicalities (i)–(iii) can be connected via BS and HP transformations as follows [19–21]. SMNc criterion can be transformed into TME criterion using a BS and vice versa [27–29]. Although nonclassicality can be transformed into TME partially [48–50], this relation can be utilized for converting TME witnesses [29,51] into SMNc criteria [19–21]. Such a relation is also encountered between TME and MPE in Ref. [52]. It is shown that the spin-squeezing (MPE) criterion [13] cannot be satisfied unless the two-modes describing this N-particle two-level system are entangled.

It is possible to obtain such a link also between (i) MPE and (ii) SMNc [20,26]. The Dicke state $|N, n_e\rangle$, in an ensemble of N two-level particles, is a symmetric superposition of all possible combinations $\binom{N}{n_e}$ of N-particle states, where n_e of them are excited [24]. Thus, a Dicke state is a strongly entangled state. There is a particular subset of symmetric Dicke states called atomic coherent states (ACSs) [24,53], $|\xi_{\rm ACS}\rangle_N = \sum_{n_e=0}^N \alpha_N(n_e)|N, n_e\rangle$. Despite $|N, n_e\rangle$, $|\xi_{\rm ACS}\rangle_N$ is separable. When $N \to \infty$, the Dicke state $|N, n_e\rangle$ mimics the n_eth Fock state of the single-mode $\hat{c}_e$ [54,55]. What is more, coefficients of ACS become the coefficients of a coherent state $|\alpha\rangle = \sum_{n=0}^\infty \alpha^n/\sqrt{n!}|n\rangle$ in this limit [54]. Since ACS is a separable state, both $|\xi_{\rm ACS}\rangle_N$ and $|\alpha\rangle$ possess no nonclassicality.

On the other hand, superposition of ACSs and coherent states, i.e., $|\psi_N\rangle = \sum_{i=1}^r \kappa_i|\xi_{\rm ACS}^{(i)}\rangle_N$ and $|\psi\rangle = \sum_{i=1}^r \kappa_i|\alpha^{(i)}\rangle$, are inseparable and nonclassical for $r \geq 2$ [56]. Therefore, in this mapping, inseparability of $|\psi_N\rangle$ implies nonclassicality of the single-mode state $|\psi\rangle$. It is more convenient to work in terms of operators (observables). Invoking the HP transformation [26], collective spin observables can be mapped onto the single-mode operators, e.g., $\hat{S}_+ \to \sqrt{N}\hat{c}_e^\dagger$. This mapping is shown to transform the (a.i) spin-squeezing criterion into the (a.ii) quadrature-squeezing condition [20]. Similar to Ref. [20], here we realize that, (b.i) $\hat{\mathcal{R}} = \hat{S}_+\hat{S}_-$ transforms to $\hat{c}_e^\dagger\hat{c}_e$, whose uncertainty leads to (b.ii) Mandels Q parameter.

Because of the presence of this intimate link between the three kinds of nonclassicalities, one can group the criteria into two [57]. (a) In the first group we can place the spin-squeezing criterion [13] for MPE, the quadrature-squeezing condition for single-mode states [22,23], and the DGCZ [25] criterion (and its product form [59]) with the Simon-Peres-Horodecki (SPH) [60,61] criterion for TME. (b) The second group contains the HZ criterion [30] (which is a subset of conditions by Shchukin and Vogel [58]) for two-mode states, Mandel's Q parameter as the SMNc, and a MPE criterion ($\xi_{\rm new}$) we derive below.

In the following, we find the MPE criterion missing in group (b) by examining the uncertainty of the operator $\hat{\mathcal{R}} = \hat{S}_+\hat{S}_-$.

Many-particle entanglement.—Derivation of $\xi_{\rm new}$ follows arguments similar to the spin-squeezing condition by Sorensen *et al.* [13]. Nevertheless, longer expressions show up due to the calculation of higher order moments. A many-particle system is separable if the N-particle density matrix (DM) can be written in the form

$$\hat{\rho} = \sum_k P_k \rho_1^{(k)} \otimes \rho_2^{(k)} \otimes \ldots \otimes \rho_N^{(k)}, \tag{1}$$

where $\hat{\rho}_i^{(k)}$ is the DM of the ith particle and P_k is the classical probability for mixed states. Uncertainty of the $\hat{\mathcal{R}} = \hat{S}_+\hat{S}_-$ operator becomes larger than $\langle(\Delta\hat{\mathcal{R}})^2\rangle \geq \sum_k P_k(\langle\hat{\mathcal{R}}^2\rangle_k - \langle\hat{\mathcal{R}}\rangle_k^2)$ if we use the Cauchy-Schwartz inequality $\sum_k P_k\langle\hat{\mathcal{R}}\rangle^2 \geq (\sum_k P_k\langle\hat{\mathcal{R}}_k\rangle)^2$. We express the collective operators in terms of the single atom spins $\hat{s}_\pm^{(i)}$, e.g., $\hat{\mathcal{R}} = \hat{S}_+\hat{S}_- = \sum_{i_1=1}^N \sum_{i_2=1}^N \hat{s}_+^{(i_1)}\hat{s}_-^{(i_2)}$. We evaluate the difference $\langle\hat{\mathcal{R}}^2\rangle_k - \langle\hat{\mathcal{R}}\rangle_k^2$ using many Cauchy-Schwartz inequalities and relations among single particle operators; see the Supplemental Material II [32]. We show that the DM (1) satisfies the inequality $\sum_k P_k\langle\hat{\mathcal{R}}^2\rangle_k - \sum_k P_k\langle\hat{\mathcal{R}}\rangle_k^2 \geq \eta_N$. We conclude that $\langle(\Delta\hat{\mathcal{R}})^2\rangle \geq \eta_N$ for a separable state. So, we define the parameter

$$\xi_{\rm new} = \langle(\Delta\hat{\mathcal{R}})^2\rangle_\rho - \eta_N, \tag{2}$$

whose negativity ($\xi_{\rm new} < 0$) witnesses the inseparability of the many-particle system. η_N is a relatively long expression of collective spins presented in Eq. (43) of the Supplemental Material II [32].

In Fig. 1, we test $\xi_{\rm new}$ on Dicke states for $N = 16$ particles (or $S = 8$). $|S, m = \mp S\rangle$, alternatively the $|N, n_e = 0\rangle$ and $|N, n_e = N\rangle$ [23] states are separable. They are $|g_1, g_2, \ldots, g_N\rangle$ and $|e_1, e_2, \ldots, e_N\rangle$ in the single particle basis, where g_i (e_i) means that the ith particle is in the ground (excited) state [24,53]. The number of terms, so the inseparability, increases up to $|S, m = 0\rangle$. Linear entropy [62–66], an entanglement monotone [67], follows the expected result, such that it increases up to the $|S, m = 0\rangle$ state. Our criterion $\xi_{\rm new}$—$\langle(\Delta\hat{\mathcal{R}})^2\rangle$ is more squeezed for more negative values of $\xi_{\rm new}$—also follows the similar trend. Duan recently introduced a new criterion [15,17,18], which not only serves for detecting the inseparability but it also reports that (if $\xi_{\rm Duan} > n$) at least n number of particles are entangled [68]. In Fig. 1, we scaled $\xi_{\rm Duan}$ with the number 17. Hence, for $m = 0$ it witnesses that at least 16 (all of the) particles are entangled. Duan's criterion is priceless for use in quantum teleportation [70] and in the research connecting gravitation and entanglement [71–73], since it quantifies the depth of entanglement.

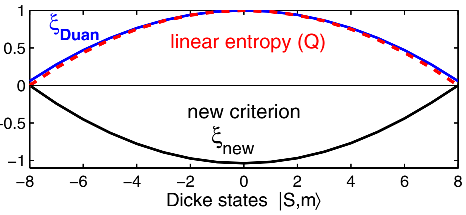

FIG. 1. Linear entropy Q [63,64], many-particle entanglement criterion by Duan [17], and the new many-particle inseparability criterion $\xi_{\rm new}$ successfully predict the entanglement in Dicke states $|S, m\rangle$. $Q, \xi_{\rm Duan} > 0$, and $\xi_{\rm new} < 0$ imply entanglement.

In Fig. 2, we calculate $\xi_{\rm new}$ for the ground state of the Dicke Hamiltonian

$$\hat{\mathcal{H}} = \hbar\omega_{eg}\hat{S}_z + \hbar\omega_a\hat{a}^\dagger\hat{a} + g/\sqrt{N}(\hat{S}_+ + \hat{S}_-)(\hat{a}^\dagger + \hat{a}) \tag{3}$$

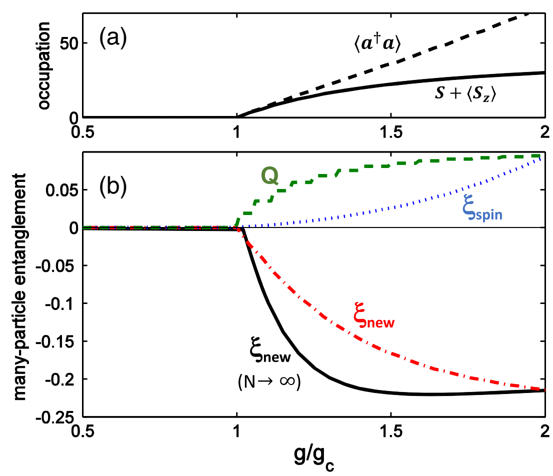

FIG. 2. (a) The number of photons and the excitation of the ensemble in the ground state of the Dicke Hamiltonian (3). Above the critical atom-photon coupling strength, $g > g_c$, superradiant phase transition occurs. (b) Many-particle entanglement. Linear entropy Q and the new criterion $\xi_{\rm new}$ (squeezing in $\langle(\Delta\hat{\mathcal{R}})^2\rangle$) accompanies the order parameters of the phase transition. $Q > 0$ and $\xi_{\rm spin}, \xi_{\rm new} < 0$ imply entanglement. $\xi_{\rm Duan}$ (not plotted) cannot witness the entanglement.

in the thermodynamic limit ($N \to \infty$) [26]. We also simulate for finite N in the symmetric subspace [74]. Here, g is the atom-photon coupling strength where for $g > g_c = \sqrt{\omega_{eg}\omega_a}/2$ the superradiant phase is observed [75].

ξ_{new} not only successfully predicts the presence of the many-particle inseparability, but also its negativity (squeezing in $\langle(\Delta\hat{\mathcal{R}})^2\rangle$) accompanies the order parameters ($\langle\hat{a}^\dagger\hat{a}\rangle$ and $\langle\hat{S}_z\rangle$) of the transition. In Fig. 2(b), we observe that value of the linear entropy Q (an entanglement monotone [67]) also accompanies the transition [64–66]. The spin-squeezing criterion of Sorensen *et al.* [13] cannot witness the inseparability where $\xi_{\text{spin}} < 0$ implies the entanglement. The criterion of Duan [15,17,18], not plotted in Fig. 2(b), does not exceed 1 (no entanglement) for the superradiant ground state which is a superposition of many Dicke states.

In both Fig. 1 and Fig. 2(b), linear entropy Q and ξ_{new} exhibit parallel behavior. So, we became curious if this is true also for random states. For $N = 16$, in the 2^{16} dimensional space, we generated random states and examined if ξ_{new} and Q display parallel behavior. Even though ξ_{new} managed to detect the inseparability of all 2000 states, when $Q > 0$, the two did not exhibit parallel behavior in general.

Ensemble-field entanglement.—Commonly used two-mode criteria can be put in a stronger form using the Schrödinger-Robertson (SR) inequality and the partial tranpose of the operators [76]. For instance, the product form of the DGCZ criterion [59,77], belonging to group (a), can be put in a stronger form by using the variances $\tilde{H}_1' = \hat{x}_1 + \hat{x}_2$ and $\tilde{H}_2' = \hat{p}_1 - \hat{p}_2$ in the SR inequality [76]. Similarly, a stronger form of the HZ criterion, in group (b), can be obtained using the $\tilde{H}_1 = \hat{a}_1^\dagger\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_1$ and $\tilde{H}_2 = i(\hat{a}_1^\dagger\hat{a}_2 - \hat{a}_2^\dagger\hat{a}_1)$ in the SR inequality; see Eq. (11) in Ref. [76].

References [78–80] show that it is possible to obtain a criterion for the ensemble-field entanglement by making the substitutions $\hat{x}_1 \to \hat{S}_x$ and $\hat{p}_1 \to \hat{S}_y$ in $\tilde{H}_{1,2}'$. Similar to Refs. [78–80], we perform the substitutions $\hat{a}_1^\dagger \to \hat{S}_+$ and $\hat{a}_1 \to \hat{S}_-$ in $\tilde{H}_{1,2}$,

$$\tilde{H}_1 = \hat{S}_+\hat{a}_2 + \hat{S}_-\hat{a}_2^\dagger \quad \text{and} \quad \tilde{H}_2 = i(\hat{S}_+\hat{a}_2 - \hat{S}_-\hat{a}_2^\dagger), \tag{4}$$

and obtain the parameter

$$\mu_{\text{new}}^{\text{SR}} = (\langle(\Delta\tilde{H}_1)^2\rangle - 2\langle\hat{S}_z\rangle)(\langle(\Delta\tilde{H}_2)^2\rangle - 2\langle\hat{S}_z\rangle) - |\langle-\hat{S}_+\hat{S}_- + 2\hat{S}_z\hat{a}\hat{a}^\dagger\rangle|^2 - \langle\Delta\tilde{H}_1\Delta\tilde{H}_2\rangle_s^2, \tag{5}$$

where $\mu_{\text{new}}^{\text{SR}} < 0$ witnesses the presence of the ensemble-field entanglement.

In Fig. 3, we plot $\mu_{\text{new}}^{\text{SR}}$ for a finite or infinite number of particles. We observe that negativity of $\mu_{\text{new}}^{\text{SR}}$, squeezing in the product (5), accompanies the order parameters given in Fig. 2(a). For the purposes of comparison, we also calculate

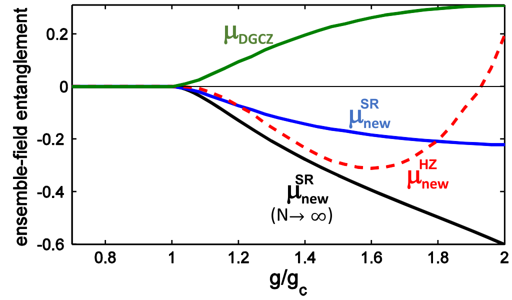

FIG. 3. Ensemble-field entanglement in the ground state of the Dicke Hamiltonian (3) for finite or infinite number of particles. $\mu < 0$ witnesses the entanglement. $\mu_{\text{new}}^{\text{HZ}}$ is obtained from the HZ [30] criterion. $\mu_{\text{new}}^{\text{SR}}$ is obtained from the stronger form of the HZ criterion [76].

$\mu_{\text{new}}^{\text{HZ}}$: We perform the substitution $\hat{a}_1^\dagger \to \hat{S}_+$ in the HZ criterion [30], $|\langle\hat{a}_1^2\hat{a}_2^{\dagger 2}\rangle|^2 > \langle\hat{a}_1^{\dagger 2}\hat{a}_1^2\hat{a}_2^{\dagger 2}\hat{a}_2^2\rangle$, which is weaker than Ref. [76]. In Fig. 3, we see that $\mu_{\text{new}}^{\text{HZ}}$ cannot witness the ensemble-field entanglement for $g > 1.9g_c$. The ensemble-field version of the DGCZ criterion [78–80], μ_{DGCZ}, cannot reveal the presence of entanglement at all.

Single-photon superradiance is one of the few (almost) exactly solvable many-body systems [33,34] and it is gaining importance due to its technological applications [6,7]. The temporal behavior of a timed-Dicke state [8], prepared initially in the state $|\psi(0)\rangle = \sum_{j=1}^N e^{i\mathbf{k}_0\cdot\mathbf{r}_j}|g_1, g_2, \ldots e_j, \ldots, g_N\rangle$, can be given as [33,34]

$$|\psi(t)\rangle = \sum_{j=1}^N \beta_j(t)|g_1..e_j..g_N\rangle|0\rangle + \sum_{\mathbf{k}}\gamma_{\mathbf{k}}(t)|g_1..g_N\rangle|1_{\mathbf{k}}\rangle. \tag{6}$$

The solutions of $\beta_j(t)$ and $\gamma_{\mathbf{k}}(t)$ are studied in Refs. [33,34] explicitly. We test our criteria ξ_{new} and μ_{new} also for the single-photon superradiance of 2000 atoms randomly placed at positions $\mathbf{r}_j$; see Fig. 3 in the Supplemental Material [32]. The spatial extent of the ensemble is 10 times larger than a wavelength $\lambda_0 = 2\pi/k_0$.

In Fig. 4, we observe that the initial many-particle entanglement is lost after $t > 1/\Gamma_N$, where the collective

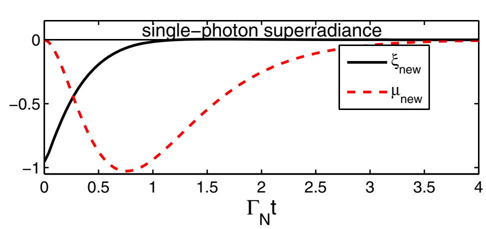

FIG. 4. Temporal behavior of many-particle entanglement (ξ_{new}) and ensemble-field entanglement (μ_{new}) for single-photon superradiance of $N = 2000$ atoms placed randomly in a sphere larger than wavelength. (μ_{new} is scaled for visual purposes.)

decay rate $\Gamma_N \sim N\gamma$ can be much larger than the single atom decay rate γ. This is something expected from Eq. (6), since the particles decay to the separable state, where $\beta_j(t) \sim e^{-\Gamma_N t}$ [33,34]. We also examine the entanglement of the ensemble with the central mode ($\mathbf{k}_0$). Initially $\mu_{\rm new} = 0$ since $\gamma_{\mathbf{k}}(0) = 0$. For $t > 0$, $\mu_{\rm new}$ witnesses the inseparability as $\beta_j(t)$ and $\gamma_{\mathbf{k}}(t)$ are mixed in $|\psi(t)\rangle$. At $t \simeq 4/\Gamma_N$, $\mu_{\rm new}$ approaches to zero again since the system ends up with the $\gamma_{\mathbf{k}}$ states eventually.

Finally, we anticipate that derivations, $\langle(\Delta\hat{\mathcal{R}})^2\rangle$, leading to $\xi_{\rm new}$ can be utilized for calculating the entanglement depth [17] of the system, using the methods in Refs. [17,69,81].

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