Microscopic Origin of Ideal Conductivity in Integrable Quantum Models

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Nonergodic dynamical systems display anomalous transport properties. Prominent examples are integrable quantum systems, whose exceptional properties are diverging dc conductivities. In this Letter, we explain the microscopic origin of ideal conductivity by resorting to the thermodynamic particle content of a system. Using group-theoretic arguments we rigorously resolve the long-standing controversy regarding the nature of spin and charge Drude weights in the absence of chemical potentials. In addition, by employing a hydrodynamic description, we devise an efficient computational method to calculate exact Drude weights from the stationary currents generated in an inhomogeneous quench from bipartitioned initial states. We exemplify the method on the anisotropic Heisenberg model at finite temperatures for the entire range of anisotropies, accessing regimes that are out of reach with other approaches. Quite remarkably, spin Drude weight and asymptotic spin current rates reveal a completely discontinuous (fractal) dependence on the anisotropy parameter.

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Introduction.—Obtaining a complete and systematic understanding of how macroscopic laws of thermodynamics emerge from concrete microscopical models has always been one of the greatest challenges of theoretical physics. Nonergodic dynamical systems, displaying a whole range of exceptional physical properties, have a special place in this context. One of their prominent features is unconventional transport behavior, which attracted a great amount of interest after the authors of Refs. [1,2] conjectured that integrable quantum systems behave as ideal conductors. Although this has been shown to hold almost universally [2], spin and charge transport in systems with unbroken particle-hole symmetries instead show normal (or even anomalous) diffusion [3–7]. Despite long efforts, the question whether the spin Drude weight in the isotropic Heisenberg spin chain at finite temperature and at half filling is precisely zero is still vividly debated [8–10], with a number of conflicting statements spread in the literature: while the prevailing opinion is that the spin Drude weight vanishes [9–15], other studies reach the opposite conclusion [16–21]. As the question is inherently related to asymptotic time scales in thermodynamically large systems, numerical approaches—ranging from exact diagonalization to DMRG [9,14,15,17,18,21,22]—are insufficient to offer a conclusive and unambiguous answer.

In this Letter, we rigorously settle the issue by closely examining the underlying particle content that emerges in thermodynamically large systems, and combine it with symmetry-based arguments to lay down the complete microscopic background of ideal (dissipationless) conductivity. Moreover, we present an efficient exact computational scheme for computing Drude weights with respect to general equilibrium states by employing a nonequilibrium protocol based on the hydrodynamic description developed in Refs. [23,24]. Applying our method to the anisotropic Heisenberg model, we find that while the thermal Drude weight shows continuous (smooth) dependence on the anisotropy parameter, the spin Drude weight is a discontinuous function that exhibits a striking fractal-like profile.

Drude weights.—Transport behavior in the linear response regime is given by conductivity $\sigma^{(q)}(\omega)$ associated with charge density q. The real part reads

$$\mathrm{Re}\sigma^{(q)}(\omega) = 2\pi\mathcal{D}^{(q)}\delta(\omega) + \sigma^{(q)}_{\mathrm{reg}}(\omega), \tag{1}$$

where $\sigma^{(q)}_{\mathrm{reg}}$ denotes the regular frequency-dependent part, whereas the magnitude of the singular part—the so-called Drude weight $\mathcal{D}^{(q)}$—signals a dissipationless (ballistic) contribution. The standard route to express $\mathcal{D}^{(q)}$ is via the Kubo formula, using the time-averaged current autocorrelation function [25,26]

$$\mathcal{D}^{(q)} = \lim_{\tau\to\infty}\lim_{L\to\infty}\frac{\beta}{2\tau L}\int_{t=0}^{\tau} dt\langle\hat{\mathcal{J}}^{(q)}(t)\hat{\mathcal{J}}^{(q)}(0)\rangle_{\beta,h}, \tag{2}$$

where $\langle\bullet\rangle_{\beta,h} = \mathrm{Tr}(\bullet\hat{\varrho}_{\beta,h})$, $\hat{\varrho}_{\beta,h} \propto \exp(-\beta\hat{H} + h\hat{N})$ denotes the grand canonical average at inverse temperature β and chemical potential h [27] in a system of length L, while $\hat{\mathcal{J}}(q) = \sum_i \hat{j}^{(q)}_i$, where current densities $\hat{j}^{(q)}$ are determined from local continuity equations, $\partial_t\hat{q}_i = \hat{j}^{(q)}_i - \hat{j}^{(q)}_{i+1}$. While the linear response formula (2) is suitable for efficient numerical simulations with DMRG techniques [9,28–30], it poses a formidable task for analytical approaches. The spin Drude weight is commonly expressed via the Kohn formula [31] (see also Refs. [1,2,32]) as the thermally averaged energy level curvatures under the application of a small twist ϕ (representing magnetic flux piercing the ring), $\mathcal{D}^{(s)} = (1/2L)\sum_n w_n \partial_\phi^2 E_n(\phi)|_{\phi=0}$, with $w_n \propto \exp(-\beta E_n)$ denoting the Boltzmann weights. Although the Kohn formula proves convenient for analytic considerations, it necessitates properly resolving second-order system-size corrections [12,20,33,34]. Alternatively, Drude weights may be conveniently defined as the time-asymptotic rates of the total current growth in the zero-bias limit $\delta\mu_q \to 0$ (with $\mu_e = \beta$ and $\mu_s = h$, cf. Fig. 2),

$$\mathcal{D}^{(q)} = \lim_{\delta\mu_q \to 0} \lim_{t\to\infty} \lim_{L\to\infty} \frac{\beta}{2t} \frac{\langle \hat{\mathcal{J}}^{(q)}(t;\delta\mu_q)\rangle_{\beta,h}}{\delta\mu_q}. \quad (3)$$

This formulation was previously employed in Ref. [30] to study thermal transport in the *XXZ* spin chain, and recently in a DMRG study [9] of spin and thermal Drude weights in the Hubbard and Heisenberg models. A related definition, with the bias appearing as a Hamiltonian perturbation, was defined in Ref. [35], and is shown to be equivalent (under some mild assumptions) to the Kubo formula (2).

In ergodic dynamical systems, $\mathcal{D}^{(q)} = 0$ is a consequence of the decay of dynamical correlations in Eq. (2). Integrable systems on the other hand feature stable *interacting* particles, representing collective thermodynamic excitations which undergo completely elastic (nondiffractive) scattering [36]; see the Supplemental Material for further details [37]. Such dynamical constraints result in a macroscopic number of conserved quantities $\hat{Q}_k$, which prevent generic current-current correlations from completely decaying. This yields Mazur bounds [38,39], $\mathcal{D}^{(q)} \geq (1/2L)\sum_k \langle \hat{\mathcal{J}}^{(q)}\hat{Q}_k\rangle_{\beta,h}^2/\langle \hat{Q}_k^2\rangle_{\beta,h}$, which formally give exact results if *all* extensive conserved quantities, satisfying $\langle \hat{Q}_k^2\rangle_{\beta,h} \sim \mathcal{O}(L)$, are included. When $\hat{\mathcal{J}}^{(q)}$ belongs to a conserved current, $[\hat{\mathcal{J}}^{(q)}, \hat{H}] = 0$ (e.g., energy current $\hat{\mathcal{J}}^{(e)}$ in the Heisenberg model [2,40]), the Drude weight is trivially finite and reads $\mathcal{D}^{(q)} = \lim_{L\to\infty}(1/2L)\langle(\hat{\mathcal{J}}^{(q)})^2\rangle_{\beta,h}$. Conversely, when $\hat{\mathcal{J}}^{(q)}$ is not fully conserved, $\mathcal{D}^{(q)} > 0$ *if and only if* there exist at least one extensive conserved quantity $\hat{Q}$ with a nontrivial overlap $\langle \hat{\mathcal{J}}^{(q)}\hat{Q}\rangle_{\beta,h} > 0$.

Spin transport in the XXZ model.—We proceed by concentrating on the anisotropic Heisenberg model

$$\hat{H} = \sum_{i=1}^{L} \hat{S}_i^x \hat{S}_{i+1}^x + \hat{S}_i^y \hat{S}_{i+1}^y + \Delta\left(\hat{S}_i^z \hat{S}_{i+1}^z - \frac{1}{4}\right) \quad (4)$$

in the entire range of anisotropy parameter $\Delta \in \mathbb{R}$. For $|\Delta| > 1$ ($|\Delta| \leq 1$) the thermodynamic spectrum is gapped (gapless). We focus here on the elusive case of spin current $\hat{\mathcal{J}}^{(s)}$. The presence of chemical potential $h \neq 0$, which couples to $\hat{N} = \sum_i \hat{S}_i^z$ breaks particle-hole symmetry and renders $\mathcal{D}^{(s)} > 0$ for all $\Delta \in \mathbb{R}$ by virtue of a nontrivial Mazur bound [2]. At half filling $h = 0$, however, the situation becomes more subtle. Since $\hat{\mathcal{J}}^{(s)}$ is *odd* under the spin-reversal transformation $\hat{R} = \prod_i \hat{S}_i^x$, namely, $\hat{R}\hat{\mathcal{J}}^{(s)}\hat{R} = -\hat{\mathcal{J}}^{(s)}$, $\mathcal{D}^{(s)}$ can only be finite if there exists an extensive conserved quantity $\hat{Q}$ of odd parity and finite overlap $\langle \hat{\mathcal{J}}^{(s)}\hat{Q}\rangle_{\beta,h} \neq 0$ [2]. In spite of substantial numerical evidence, clearly pointing towards $\mathcal{D}^{(s)} > 0$ for $|\Delta| < 1$, the long search for appropriate conservation laws only ended recently with a nontrivial bound obtained in Ref. [41], followed by a further improved bound derived in Ref. [42] using a family of odd-parity charges stemming from noncompact representations of the quantized symmetry algebra $\mathcal{U}_\mathfrak{q}(\mathfrak{sl}_2)$. Specifically, for commensurate values of anisotropy $\Delta = (\mathfrak{q} + \mathfrak{q}^{-1})/2$, where $\mathfrak{q} = \exp(i\pi m/\ell)$ with $m < \ell$ ($\ell > 2$) being two co-prime integers, the high-temperature bound (i.e., in the vicinity of $\beta \to 0$) of Ref. [42] reads explicitly

$$\mathcal{D}^{(s)} \geq \frac{\beta}{16} \frac{\sin^2(\pi m/\ell)}{\sin^2(\pi/\ell)} \left(1 - \frac{\ell}{2\pi}\sin(2\pi/\ell)\right), \quad (5)$$

showing an unexpected "fractal" (nowhere-continuous) dependence on the anisotropy parameter Δ. At this stage, a few obvious questions come to mind: (i) is the bound (5) tight, or does it eventually smears out with the inclusion of extra (yet unknown) conservation laws? (ii) What is its value precisely at the isotropic point $\Delta = 1$, where the bound (5) becomes trivial? (iii) What is the physical origin of the charges found in Refs. [41,42]? We subsequently provide natural and definite answers to these questions by expressing the Drude weight in terms of ballistically propagating particle excitations on the model.

Particle content of the XXZ model.—Thermodynamic ensembles in integrable models are completely characterized by their particle content [43–45]. Local statistical properties are encoded in *macrostates*, corresponding to a complete set of mode density distributions $\rho_a(u)$, where index a labels distinct particle types, and u is the rapidity variable which parametrizes particle momenta $k_a(u)$. Distinct types of particles in the spectrum are intimately linked to representation theory of the underlying symmetry group. When $|\Delta| > 1$, the thermodynamic spectrum of particles with respect to ferromagnetic vacuum consists of magnons ($a = 1$) and bound states thereof ($a \geq 2$) [44,45]. As explained in Ref. [46], these particle species are in a one-to-one correspondence with quantum transfer matrices composed of (auxiliary) finite-dimensional unitary irreducible representations of quantum group $\mathcal{U}_\mathfrak{q}(\mathfrak{sl}_2)$, see Ref. [37], and also Refs. [47–50]. Spin-reversal invariance of macrostates is a simple corollary of unitarity, in turn implying that $\mathcal{D}^{(s)} = 0$ at $h = 0$, in the entire range of anisotropies $|\Delta| \geq 1$. We note that nonunitary highest-weight representations are of infinite dimension and do not enter into the description of magnonic excitations. In the critical regime $|\Delta| < 1$, however, one finds an intricate situation where the particle content becomes unstable and changes discontinuously upon varying Δ [51]. When $\mathfrak{q}$ is a root of unity, representing a dense set of points in the interval $|\Delta| < 1$, the number of independent unitary transfer matrices and magnonic particles both become finite. The latter represent $N_p = \sum_{i=1}^{l} \nu_i$ bound excitations classified in Ref. [51] with aid of continued fraction representation, $m/\ell = 1/[\nu_1 + (1/\nu_2 + \ldots)] \equiv (\nu_1, \nu_2, \ldots, \nu_l)$ (see Supplemental Material for details [37]), which bijectively correspond to the finite-dimensional irreducible representations of $\mathcal{U}_{\mathfrak{q}}(\mathfrak{sl}_2)$ [46,52]. It is shown in Ref. [52] that the densities of a distinguished pair of particles $\rho_{\bullet,\circ}$ (see Fig. 1) map to the spectrum of the odd-parity charges from Refs. [42,53,54], providing a link to finite-dimensional *nonunitary* representations of $\mathcal{U}_{\mathfrak{q}}(\mathfrak{sl}_2)$. The lack of unitary implies that $\rho_{\bullet,\circ}(u)$ transform nontrivially under the spin-reversal transformation, meaning that a change in the chemical potential h only explicitly influences macrostates via the distributions $\rho_{\bullet,\circ}(u)$, while other densities get affected indirectly via interparticle interactions. The absence of exceptional particles in the $|\Delta| \geq 1$ regime on the other hand signifies that a macrostate is locally equivalent to its spin-reversed counterpart, and, therefore, no ballistic spin transport between two regions with opposite magnetization density takes place.

Drude weights from hydrodynamics.—We now describe a procedure for computing Drude weights using a nonequilibrium "partitioning protocol" developed in Refs. [23,24], drawing on the earlier ideas of Refs. [55,56] and recent studies of CFTs [57–59]. A simple way to implement a thermodynamic gradient is to consider two partitions representing macroscopically distinct semi-infinite equilibrium states joined together at the point contact; see Fig. 2. The imbalance at the junction induces particle flows between the two subsystems, with a local quasistationary state emerging at late times along each ray $\zeta = x/t$. The latter is uniquely specified by the set of distributions $\rho_a(u,\zeta)$, pertaining to all types of particles in the spectrum (labeled by $a = 1, \ldots, N_p$), each obeying a local continuity equation [23,24]

$$\partial_t \rho_a(u,\zeta) + \partial_x[(v_a(u,\zeta)\rho_a(u,\zeta)] = 0. \tag{6}$$

Notice that, in distinction to noninteracting systems, particles' velocities $v_a(u)$ are dressed due to interactions with a nontrivial background (macrostate) [43,60,61], $v_a(u) = \partial\omega_a(u)/\partial p_a(u)$, where $\omega_a(u)$ and $p_a(u)$ are their dressed energy and momenta, respectively (see the Supplemental Material [37]). The solution of Eqs. (6) for each ray ζ gives a family of densities $\rho_a(u,\zeta)$; see Fig. 2.

Computing the Drude weights requires infinitesimal gradients. We thus consider two thermodynamic subsystems prepared in almost identical equilibrium states that differ by a slight amount δq in the charge density q and experience a chemical potential jump $\delta\mu_q$ at the contact. Transforming Eq. (3) to the light cone frame, we find

$$\mathcal{D}^{(q)} = \lim_{\delta\mu_q \to 0} \frac{\beta}{2\delta\mu_q} \int_{-v^{\rm L}_{\rm max}}^{v^{\rm R}_{\rm max}} d\zeta\, j^{(q)}(\zeta;\delta\mu_q), \tag{7}$$

where $j^{(q)}(\zeta;\delta\mu_q)$ designates the quasistationary expectation value of the current density in the direction of ζ emanating from the contact. Using the hydrodynamical approach, we first verified the infinite-temperature results of Eq. (5), and found perfect numerical agreement (with absolute precision $< 10^{-4}$), at $\Delta = \cos(\pi m/\ell)$ for different values of ν_1, ν_2, ν_3. We subsequently confirmed the discontinuous nature of the spin Drude weight as a function of Δ not only at infinite temperature [42], but also at finite temperatures β^{-1}. As temperature is lowered, the discontinuities of $\mathcal{D}^{(s)}$ become less pronounced (see Fig. 4), while as $T \to 0$ we find a power-law behavior $\mathcal{D}^{(s)} - \mathcal{D}^{(s)}(T=0) \sim T^{2/(\ell/m-1)}$ (see Fig. 3 in Ref. [37]). Moreover, the hydrodynamic description remains applicable at finite bias $\delta\mu_q$,

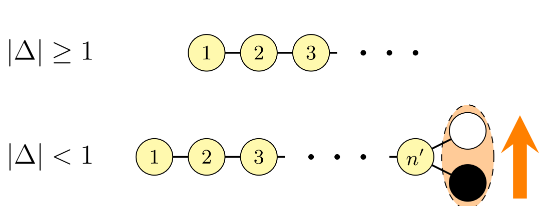

FIG. 1. Particle content of the *XXZ* Heisenberg model for $|\Delta| \geq 1$ (top) and $|\Delta| < 1$ (bottom). While the former consist of infinitely many bound magnons with densities $\rho_a(u)$, $a \in \mathbb{Z}_{\geq 1}$, the latter reduces to N_p particles ($n' = N_p - 2$) whose number depends discontinuously on Δ. Morphology of the graphs reflects how the particles effectively scatter among each other. Black and white end nodes label a distinguished pair that forms a doublet with an effective magnetic moment (orange arrow) being the *only* particles in the spectrum which transform nontrivially under the spin-reversal operation.

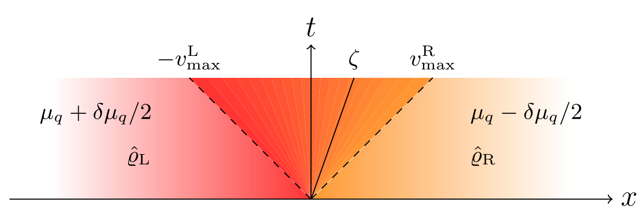

FIG. 2. Partitioning protocol: the initial state is prepared in two nearly identical grand canonical equilibria $\hat{\varrho}_{L,R} \propto \exp(-\beta_{L,R}\hat{H} + h_{L,R}\hat{S}^z)$, representing a q-charged ($q = s$, e) domain wall of size δq, with the corresponding chemical potential drop $\delta\mu_q = \mu_{q,L} - \mu_{q,R}$ (where $\mu_e = \beta$, $\mu_s = h$). The initial defect expands in an inhomogeneous state localized within the light cone $v^L_{\rm max} < \zeta < v^R_{\rm max}$. In the $t \to \infty$ limit, the state along each ray $\zeta = x/t$ relaxes in a quasistationary state, which is uniquely characterized by particle distributions $\rho_a(u,\zeta)$, for $a = 1, \ldots, N_p$. Drude weight $\mathcal{D}^{(q)}$ is proportional to the increment of the total current rate $\lim_{t\to\infty}\mathcal{J}^{(q)}(t;\delta\mu_q)/t$ in the limit $\delta\mu_q \to 0$.

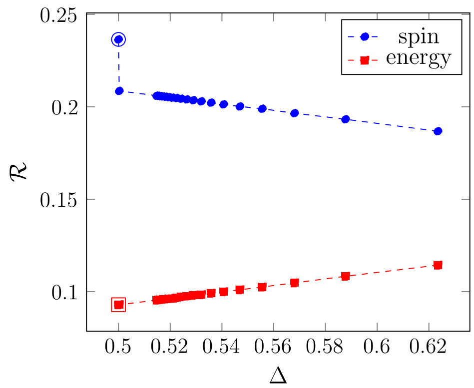

FIG. 3. Asymptotic spin (blue) and energy (red) current rates $\mathcal{R}^{(q)} = \lim_{t\to\infty}\mathcal{J}^{(q)}(t)/t = \int_{-v^L_{\max}}^{v^R_{\max}} d\zeta\, j^{(q)}(\zeta;\delta\beta,\delta h)$, emerging by joining two equilibrium states with chemical potentials $h_{L,R} = \pm 1$ and inverse temperatures $\beta_{L,R} = 1,\ 3$. Considering the sequence $\Delta = \cos[\pi/(3+1/\nu_2)]$ for $\nu_2 = 2,3,\ldots,20,10^3$ (the points at $\nu_2 = 10^3$ are obtained by linear extrapolation of other ν_2 points), we demonstrate that $\lim_{\nu_2\to\infty}\mathcal{D}^{(s)} \neq \mathcal{D}^{(s)}[\gamma = (\pi/3)]$ (open circle). The same holds in general when approaching a value of γ parametrized by $l-1$ integers ν_i as the $\nu_l \to \infty$ limit of the order-l sequence of ν_i. This indicates that spin current is a nowhere-continuous function of Δ within $|\Delta| < 1$ [cf. Eq. (5) for the exact analytic high-temperature results]. Unlike $\mathcal{R}^{(s)}$, the thermal current rate $\mathcal{R}^{(e)}$ depends continuously on Δ, as shown for $\nu_1 = 3$ (open square).

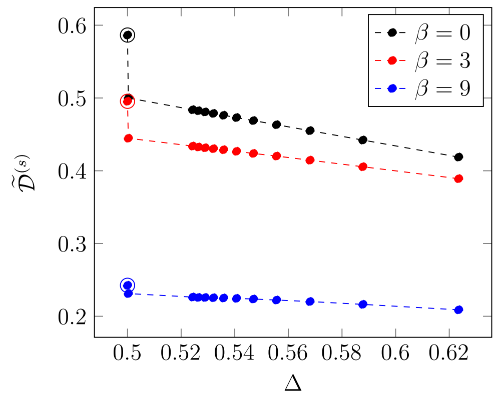

FIG. 4. Rescaled spin Drude weight $\tilde{\mathcal{D}}^{(s)} = (16/\beta)\mathcal{D}^{(s)}$ obtained from Eq. (7) for various temperatures at $\Delta = \cos[\pi/(3+1/\nu_2)]$, for $\nu_2 = \{2,3,\ldots,12,10^3\}$ ($\nu_2 = 10^3$ are obtained by linear extrapolation of other ν_2-points), and for $\Delta = \cos(\pi/3) = 0.5$ (open circles). For a dense set of commensurate anisotropies $\Delta = \cos(\pi m/\ell)$, $\mathcal{D}^{(s)}$ is found to be a discontinuous function of Δ at arbitrary finite temperature (see explanation in the caption of Fig. 3).

hence allowing us to probe quantum transport properties even in the nonlinear regime [23,24,52,62] and revealing that the asymptotic spin-current rate $\mathcal{R}^{(s)} = \lim_{t\to\infty}\mathcal{J}^{(s)}(t)/t$ is (unlike, e.g., energy current rate $\mathcal{R}^{(e)}$) an everywhere discontinuous function of anisotropy Δ; see Fig. 3. Additional figures, showing low-temperature behavior of $\mathcal{D}^{(s)}$ and its dependence on chemical potential h, are given in the Supplemental Material [37].

Hubbard model.—A situation analogous to that of the isotropic Heisenberg model occurs in the (fermionic) Hubbard model [60,63], where in spite of solid evidence in favor of the vanishing finite-temperature spin and charge Drude weights $\mathcal{D}^{(c,s)} = 0$ in the absence of the respective chemical potentials (see Refs. [11,29,64,65]), the definite conclusion is still lacking [9,66]. A possibility of having additional (unknown) odd-parity conservation laws can, however, now be quickly ruled out by invoking group-theoretic arguments along the same lines of the isotropic Heinsenberg model. In the Hubbard model, the entire space macrostates is in a one-to-one correspondence with particle-hole invariant commuting (fused) transfer matrices, pertaining to a discrete family of *unitary* irreducible representations of the underlying quantum symmetry [67]. This readily implies vanishing finite-temperature charge or spin Drude weights $\mathcal{D}^{(c,s)} = 0$ when the corresponding chemical potentials vanish, irrespective of the interaction strength. In the presence of external potentials the Drude weights are known to take finite values by virtue of Mazur bounds, cf. Ref. [2]. As the particle content of the Hubbard model is robust against varying the coupling strength, the Drude weights exhibit a continuous dependence on it.

Conclusions.—We presented a rigorous and intuitive picture for understanding the phenomenon of ideal conductivity in generic integrable quantum models. Dissipationless transport of generic local charges is shown to be directly linked to the interacting particles of a theory. Nonetheless, spin (or charge) Drude weights in particle-hole symmetric models in the half-filled regimes show exceptional behavior and require a careful analysis by examining the particle content of the model.

While our framework is applicable in general, we focused on the interesting case of the anisotropic Heisenberg model. In the gapped phase, $|\Delta| \geq 1$, particles correspond to an infinite hierarchy of magnonic bound states that are robust under varying the anisotropy parameter [44,45]. The fact that the corresponding particle density operators are insensitive to flipping the spins implies that two thermodynamic states that are characterized in terms of mode occupation distributions are (locally) identical, and no ballistic flow of particles across the magnetic domain wall at zero magnetization density can occur. Within the interval $|\Delta| < 1$, however, the particle content for commensurate values of Δ consists of finitely many particles whose number depends discontinuously on Δ [51]. In this case, ballistic *spin* transport is enabled by the appearance of a distinguished pair of particles that are not invariant under the spin reversal and hence allow for chiral (i.e., spin-carrying) states. It should be stressed that the above qualitative picture can be established independently from any quantitative analysis.

By employing a nonequilibrium partitioning protocol, we presented an exact numerical computation of Drude weights and applied it on the anisotropic Heisenberg spin chain. Our results rigorously prove that the formal infinite-temperature bound derived in Ref. [42] is the exact Drude weight at infinite temperature, and, moreover, that the time-asymptotic spin current rate in the XXZ chain is a nowhere-continuous function of Δ for any finite temperature and even in the nonlinear regime. These observations indicate that the physics in the gapless regime $|\Delta| < 1$ depends abruptly on the "commensurability effect," resembling the pattern found in the famous Hofstadter butterfly [68–70] multifractal spectrum. As a future task, it would be valuable to perform high-precision finite-time numerical analysis to determine whether the "fractality" can be detected via anomalously large relaxation times.

A number of intriguing open problems remain. Most notably, understanding the microscopic mechanism underlying normal or anomalous diffusion which typically coexists with the ballistic channel, see, e.g., Refs. [15,71–74]), and recent work [75,76]. Another open question is to explain diffusive behavior in the semiclassical regime of the Heisenberg ferromagnet [77] governed by Landau–Lifshitz action [78], whose solitons are identified as long-wavelength macroscopic bound states [79].

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Note added.—After this work appeared online, an independent work [80], which partially overlaps with this Letter, also shows that the spin Drude weight could be obtained from hydrodynamics.

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